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Efficient estimation of population variance in sample surveys utilizing auxiliary information

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Abstract

This paper addresses the issue of estimating population variance by incorporating auxiliary information within the framework of a simple random sampling without a replacement scheme. We have suggested classes of estimators for population variance (S_y^2) of the study variable (Y) utilizing information available on the population variance (S_x^2) of the auxiliary variable (X). Expressions for the bias and mean squared error (MSE) for the suggested estimators up to the first order of approximation have been obtained. Furthermore, we have established optimal conditions under which these new estimators outperform certain existing ones. An empirical study using three real population datasets has been conducted to substantiate the theoretical results.

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1. Introduction

Incorporating auxiliary information during the estimation process enhances the efficiency of the estimates for population parameters such as the mean, total, variance, proportion, coefficient of variation, etc. The use of auxiliary information for estimating the population mean in sample surveys has been extensively discussed. For instance, see Singh [13], Singh [12], Upadhyaya and Singh [16], Kumar and Siddiqui [8], Mahmud, Pal, and Singh [17] and the references cited therein. However, in practice, estimating the variance of a finite population is a matter of concern, as managing variability in real-world applications often proves difficult. This issue commonly emerges in fields such as biological and agricultural research, where outcomes may deviate from expectations. Das and Tripathi [5] were the first to use the auxiliary information in estimating population variance. Later, Isaki [7] recommended a ratio estimator of population variance (S_y^2) using knowledge of population variance S_x^2 of the auxiliary variable X. Afterwards, various authors, including Upadhyaya and Singh [16], Gupta and Shabbir [6], Sharma and Singh [11], Singh and Chandra [15], Singh and Solanki [14], Yadav et al. [19], Daraz and Khan [2], Daraz et al. [3, 4] and Mishra et al. [10].

The present paper is designed in the succeeding sections. In Section 2, the methodology and notations used are explained. Some prominent estimators, along with their mean squared error (MSE) and minimum MSE expressions, are presented in Section 3. We have suggested classes of estimators and assessed their theoretical properties in Sections 4 and 6. Sections 5 and 7 provide a mathematical comparison. Section 8 provides an empirical illustration of the percent relative efficiency (PRE) of the suggested classes of estimators relative to the usual unbiased estimator using three real population data sets. The paper ends with the result and discussion in Section 9 and the conclusion in Section 10.

2. Methodology And Notations

Consider a finite population consisting of N units, designated by $U=(U_1,U_2,...,U_N)$. For the ith unit, let y_i and x_i indicate values of the study variable (Y) and auxiliary variable (X), respectively. Let us define.

$\bar{Y}=\frac{1}{N}\sum_{i=1}^{N} y_i$: Population mean of Y,

$\bar{X}=\frac{1}{N}\sum_{i=1}^{N} x_i$: Population mean of X,

$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{Y})^2$: Population variance (mean square) of Y,

$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{X})^2$: variance (mean square) of X,

$\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \overline{Y})^r (x_i - \overline{X})^s$,

$\lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}}$, (r, s) being non-negative integers.

It is assumed that the population variance S_x^2 is known. To estimate S_y^2, a sample of size n with SRSWOR is selected from population U of size N. Then

$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$: Sample mean of Y,

$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$: Sample mean of X,

$s_y^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2$: Sample variance (mean square) of Y,

$s_x^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2$: Sample variance (mean square) of X,

Let us consider the error terms as $e_0 = (s_y^2 - S_y^2)/S_y^2$ and $e_1 = (s_x^2 - S_x^2)/S_x^2$. such that the expected values of these error terms are zero and

$$E(e_0^2) = \xi(\lambda_{40} - 1),\ E(e_1^2) = \xi(\lambda_{04} - 1) \text{ and } E(e_0 e_1) = \xi(\lambda_{22} - 1).$$

where $\xi = \left(\frac{1}{n} - \frac{1}{N}\right)$.

3. Reviewing Some Existing Estimators

This section is devoted to some prominent available estimators, which are discussed below, along with their MSE/minimum MSE expressions.

The conventional unbiased estimator of S_y^2 of Y is defined as

$$t_0 = s_y^2 = \frac{1}{\left(n-1\right)}\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2. \tag{3.1}$$

with MSE to the first degree of approximation (FDA) as

$$MSE\left(t_0\right) = \xi\left(\lambda_{40} - 1\right)S_y^4. \tag{3.2}$$

Utilizing information on S_x^2, Isaki [7] proposed a ratio estimator for S_y^2 as

$$t_R = s_y^2\left(S_x^2 / s_x^2\right). \tag{3.3}$$

The MSE of t_R to the FDA is obtained as

$$MSE\left(t_R\right) = S_y^4\xi\left[\lambda_{40} - 2\lambda_{22} + \lambda_{04}\right]. \tag{3.4}$$

The difference estimator for S_y^2 of y due to Das and Tripathi [5] is given by

$$t_d = s_y^2 + d\left(S_x^2 - s_x^2\right). \tag{3.5}$$

where 'd' is a suitably chosen constant, determined in such a way that the MSE of t_d is minimized.

The corresponding minimum MSE of t_d is expressed as

$$MSE_{\min}\left(t_d\right) = S_y^4\,\xi\left[\left(\lambda_{40} - 1\right) - \frac{\left(\lambda_{22} - 1\right)^2}{\left(\lambda_{04} - 1\right)}\right]. \tag{3.6}$$

Das and Tripathi [5] have also defined the classes of estimators of S_y^2, given as follows:

$$t_{DT1} = s_y^2\left(\frac{S_x^2}{s_x^2}\right)^{\alpha}, \tag{3.7}$$

$$t_{DT2} = s_y^2\frac{S_x^2}{\left\{S_x^2 + \delta\left(s_x^2 - S_x^2\right)\right\}}. \tag{3.8}$$

where (α, δ) are appropriately selected constants.

The common minimum MSE of t_{DT1} and t_{DT2} is specified as:

$$MSE_{\min}\left(t_{DTi}\right) = S_y^4\xi\left[\left(\lambda_{40} - 1\right) - \frac{\left(\lambda_{22} - 1\right)^2}{\left(\lambda_{04} - 1\right)}\right]; i = 1,\, 2. \tag{3.9}$$

Kumar et al. [9] introduced a ratio-type exponential estimator for S_y^2 as:

$$t_K = s_y^2\exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right). \tag{3.10}$$

To the FDA, the MSE of t_K is obtained as

$$MSE\left(t_K\right) = S_y^4 \xi \left[\lambda_{40} + \frac{1}{4}\left(\lambda_{04} - 1\right) - \lambda_{22} \right]. \tag{3.11}$$

Singh, Horn, and Yu [1] suggested an improved difference estimator for S_y^2, defined as:

$$t_S = w_1 s_y^2 + w_2 \left(S_x^2 - s_x^2\right). \tag{3.12}$$

which is revisited by Sharma and Singh [11], where (w_1, w_2) are appropriately chosen constants determined to ensure the minimum MSE of t_s.

The minimum MSE of t_s is given by

$$MSE_{\min}\left(t_s\right) = \frac{S_y^4 \xi\left(\lambda_{40} - 1\right)\left(1 - \rho^{*2}\right)}{\left[1 + \xi\left(\lambda_{40} - 1\right)\left(1 - \rho^{*2}\right)\right]}.$$

where $\rho^{*2} = \dfrac{\left(\lambda_{22} - 1\right)^2}{\left(\lambda_{40} - 1\right)\left(\lambda_{04} - 1\right)}$.

Yadav and Kadilar [18] suggested a class of ratio-type exponential estimators for S_y^2 as:

$$t_{YK} = s_y^2 \exp\left\{ \frac{S_x^2 - s_x^2}{S_x^2 + (a-1)s_x^2} \right\}. \tag{3.13}$$

where ‘a’ is a real constant.

The minimum MSE of t_{YK} is specified by

$$MSE_{\min}\left(t_{YK}\right) = S_y^4 \xi\left(\lambda_{40} - 1\right)\left(1 - \rho^{*2}\right). \tag{3.14}$$

which is equivalent to the minimum MSE achieved by the difference estimator t_d.

Recently, Mishra et al. [10] suggested two classes of estimators of population variance S_y^2 of y as

$$t_{M1} = s_y^2 + \eta \log\left(\frac{s_x^2}{S_x^2}\right) \tag{3.15}$$

$$t_{M2} = s_y^2\left(w_1+1\right) + w_2 \log\left(\frac{s_x^2}{S_x^2}\right). \tag{3.16}$$

where (η, w_1, w_2) are suitably chosen constants.

The minimum mean squared errors, of t_{M1} and t_{M2} are respectively given by

$$MSE_{\min}\left(t_{M1}\right) = S_Y^4 \xi\left(\lambda_{40}-1\right)\left(1-\rho^{*2}\right), \tag{3.17}$$

$$MSE_{\min}\left(t_{M2}\right) = S_y^4\left[A_0 - \frac{\left(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2\right)}{\left(A_1A_2 - A_3^3\right)}\right]. \tag{3.18}$$

where $A_0 = \xi\left(\lambda_{40}-1\right)$, $A_1 = \left[1+\xi\left(\lambda_{40}-1\right)\right]$,

$$A_2 = \frac{\xi\left(\lambda_{04}-1\right)}{R^2 S_x^4},\ A_3 = \frac{\xi}{RS_x^2}\left(\lambda_{22} - \frac{1}{4}\lambda_{04} - \frac{1}{2}\right),$$

$$A_4 = \xi\left(\lambda_{40}-1\right),\ A_5 = \frac{\xi}{RS_x^2}\left(\lambda_{22}-1\right).$$

It is observed from Eqs. (3.6), (3.9), (3.14) and (3.17) that

$$MSE_{\min}\left(t_d\right) = MSE_{\min}\left(t_{DTi}\right) = MSE_{\min}\left(t_{YK}\right) = MSE_{\min}\left(t_{M1}\right);\ i=1,2. \tag{3.19}$$

which shows that the estimators t_d, $t_{DTi}(i=1,2), t_{YK}$ and t_{M1} are equally efficient.

- From Eqs. (3.2) and (3.4), we have that $MSE(t_R) < MSE(s_y^2)$ if

$$\frac{\left(\lambda_{22}-1\right)}{\left(\lambda_{04}-1\right)} > \frac{1}{2}. \tag{3.20}$$

- From Eqs. (3.2) and (3.11), we note that $MSE(t_K) < MSE(s_y^2)$ if

$$\frac{\left(\lambda_{22}-1\right)}{\left(\lambda_{04}-1\right)} > \frac{1}{4}. \tag{3.21}$$

- From Eqs. (3.4) and (3.11), we have that $MSE(t_K) < MSE(t_R)$ if

$$\frac{\left(\lambda_{22}-1\right)}{\left(\lambda_{04}-1\right)} < \frac{3}{4}. \tag{3.22}$$

- From Eqs. (3.21) and (3.22), the estimator t_K is more efficient than s_y^2 and t_R if

$$\frac{1}{4} < \frac{(\lambda_{22}-1)}{(\lambda_{04}-1)} < \frac{3}{4}. \tag{3.23}$$

Further from Eqs. (3.2), (3.4), (3.11) and (3.19), we have

$$MSE(s_y^2) - MSE_{\min}(t_z) = S_y^4\xi(\lambda_{40}-1)\rho^{*2} > 0, \tag{3.24}$$

$$MSE(t_R) - MSE_{\min}(t_z) = S_y^4\xi(\lambda_{22}-\lambda_{04})^2 > 0 \tag{3.25}$$

provided $\lambda_{22} \neq \lambda_{04}$,

$$MSE(t_K) - MSE_{\min}(t_z) = S_y^4\xi\left\{(\lambda_{22}-1) - \frac{1}{2}(\lambda_{04}-1)\right\}^2 > 0 \tag{3.26}$$

provided $(\lambda_{22}-1) \neq \frac{1}{2}(\lambda_{04}-1)$.
where $z = d,\ DTi\ \ (i = 1,2),\ YK,\ M1$.

From Eqs. (3.24), (2.25) and (3.26), it follows that the estimators $t_d, t_{DTi}(i=1,2)$, t_{YK} and t_{M1} are more efficient than the estimators s_y^2, t_R and t_K.

From Eqs. (3.12) and (3.19), we have

$$MSE_{\min}(t_z) - MSE_{\min}(t_s) = \frac{S_y^4\xi^2(\lambda_{40}-1)^2(1-\rho^{*2})^2}{\left[1+\xi(\lambda_{40}-1)(1-\rho^{*2})\right]} > 0. \tag{3.27}$$

where $z = d,\ DTi(i=1,2),\ YK,\ M1$.
which follows that the estimators t_s due to Singh et al. [1] is more efficient than the estimators $t_d, t_{DT_i}(i=1,2), t_{YK}$ and t_{M1}, and hence the estimators s_y^2, t_R and t_K.

From Eqs. (3.12) and (3.18), it is evident that the estimator t_{M2} by Mishra et al. [10] outperforms that of t_s due to Singh et al. [1] if the condition:

$$A_0 < \left[\frac{\xi(\lambda_{40}-1)(1-\rho^{*2})}{\left\{1+\xi(\lambda_{40}-1)(1-\rho^{*2})\right\}} + \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right] \tag{3.28}$$

is satisfied.

4. Suggested Classes of Estimators

We introduce the following classes of estimators aimed at estimating population variance (S_y^2):

$$\hat{\Sigma}_1 = \left[(w_1+1)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right), \tag{4.1}$$

$$\hat{\Sigma}_2 = \left[(\alpha_1+1)s_y^2 + \alpha_2 \log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right), \tag{4.2}$$

$$\hat{\Sigma}_3 = \left[(\delta_1+1)s_y^2 + \delta_2\left(S_x^2 - s_x^2\right)\right]\exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right). \tag{4.3}$$

where (w_1, w_2), (α_1, α_2) and (δ_1, δ_2) are appropriately chosen constants so as to ensure the minimum MSEs of $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$.

Expressing $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$ in terms of (e_0, e_1), we have

$$\hat{\Sigma}_1 = S_y^2\left[(w_1+1)(1+e_0) + \frac{w_2}{RS_x^2}(1+e_1)^{-1}\right]\exp\left\{-\frac{e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\}, \tag{4.4}$$

$$\hat{\Sigma}_2 = S_y^2\left[(\alpha_1+1)(1+e_0) + \alpha_2\left(\frac{1}{RS_x^2}\right)\log(1+e_1)\right]\exp\left\{-\frac{e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\} \tag{4.5}$$

$$\hat{\Sigma}_3 = S_y^2\left[(\delta_1+1)(1+e_0) - \frac{\delta_2}{R}e_1\right]\exp\left\{-\frac{e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\}. \tag{4.6}$$

where $R = S_y^2 / S_x^2$.

We determine the bias of $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$ to the FDA, respectively, as:

$$B(\hat{\Sigma}_1) = S_y^2\left[\xi\left\{\frac{3}{8}(\lambda_{04}-1) - \frac{1}{2}(\lambda_{22}-1)\right\} + w_1\left\{1+\xi\left(\frac{3}{8}(\lambda_{04}-1) - \frac{1}{2}(\lambda_{22}-1)\right)\right\}\right.$$
$$\left. + w_2\left(\frac{1}{RS_x^2}\right)\left\{1+\frac{15}{8}\xi(\lambda_{04}-1)\right\}\right], \tag{4.7}$$

$$B(\hat{\Sigma}_2) = S_y^2\left[\xi\left\{\frac{3}{8}(\lambda_{04}-1) - \frac{1}{2}(\lambda_{22}-1)\right\} + \alpha_1\left\{1+\xi\left(\frac{3}{8}(\lambda_{04}-1) - \frac{1}{2}(\lambda_{22}-1)\right)\right\}\right.$$
$$\left. - \alpha_2\left(\frac{1}{RS_x^2}\right)\xi(\lambda_{04}-1)\right], \tag{4.8}$$

$$B\left(\hat{\Sigma}_3\right)=S_y^2\left[\xi\left\{\frac{3}{8}\left(\lambda_{04}-1\right)-\frac{1}{2}\left(\lambda_{22}-1\right)\right\}+\delta_1\left\{1+\xi\left(\frac{3}{8}\left(\lambda_{04}-1\right)-\frac{1}{2}\left(\lambda_{22}-1\right)\right)\right\}\right.$$
$$\left.+\delta_2\left(\frac{1}{2R}\right)\xi\left(\lambda_{04}-1\right)\right]. \tag{4.9}$$

We obtain the MSEs of $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$ to the FDA, respectively, as:

$$MSE\left(\hat{\Sigma}_1\right)=S_y^4\left[B_0+w_1^2B_1+w_2^2B_2+2w_1w_2B_3+2w_1B_4+2w_2B_5\right], \tag{4.10}$$

$$MSE\left(\hat{\Sigma}_2\right)=S_y^4\left[C_0+\alpha_1^2C_1+\alpha_2^2C_2+2\alpha_1\alpha_2C_3+2\alpha_1C_4+2\alpha_2C_5\right], \tag{4.11}$$

$$MSE\left(\hat{\Sigma}_3\right)=S_y^4\left[D_0+\delta_1^2D_1+\delta_2^2D_2+2\delta_1\delta_2D_3+2\delta_1D_4+2\delta_2D_5\right]. \tag{4.12}$$

where

$$B_0=C_0=D_0=\xi\left[\lambda_{40}+\frac{1}{4}\left(\lambda_{04}-1\right)-\lambda_{22}\right],$$
$$B_1=C_1=D_1=\left[1+\xi\left(\lambda_{40}-2\lambda_{22}+\lambda_{04}\right)\right]$$

$$B_2=\left(\frac{1}{R^2S_x^4}\right)\left\{1+6\xi\left(\lambda_{04}-1\right)\right\}, B_3=\left(\frac{1}{RS_x^2}\right)\left[1+\xi\left(3\lambda_{04}-2\lambda_{22}-1\right)\right],$$

$$B_4=\xi\left[\lambda_{40}-\frac{3}{2}\lambda_{22}+\frac{5}{8}\lambda_{04}-\frac{1}{8}\right]=C_4=D_4,$$

$$B_5=\left(\frac{\xi}{RS_x^2}\right)\left(\frac{9}{8}\lambda_{04}-2\lambda_{22}+\frac{7}{8}\right), C_2=\left(\frac{1}{R^2S_x^4}\right)\xi\left(\lambda_{04}-1\right),$$

$$C_3=\left(\frac{1}{RS_x^2}\right)\xi\left\{\left(\lambda_{22}-1\right)-\frac{3}{2}\left(\lambda_{04}-1\right)\right\},$$

$$C_5=\left(\frac{1}{RS_x^2}\right)\xi\left\{\left(\lambda_{22}-1\right)-\frac{1}{2}\left(\lambda_{04}-1\right)\right\}, D_2=\left(\frac{\xi}{R^2}\right)\left(\lambda_{04}-1\right),$$

$$D_3=\left(\frac{\xi}{R}\right)\left\{\lambda_{04}-\lambda_{22}\right\}, D_5=\frac{\xi}{R}\left\{\frac{1}{2}\left(\lambda_{04}-1\right)-\left(\lambda_{22}-1\right)\right\}.$$

The MSEs of $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$ are, respectively, minimized for

$$\left.\begin{aligned} w_1 &= \frac{\left(B_3B_5 - B_2B_4\right)}{\left(B_1B_2 - B_3^2\right)} = w_{10}(say) \\ w_2 &= \frac{\left(B_3B_4 - B_1B_5\right)}{\left(B_1B_2 - B_3^2\right)} = w_{20}(say) \end{aligned}\right\}, \tag{4.13}$$

$$\left.\begin{aligned} \alpha_1 &= \frac{\left(C_3C_5 - C_2C_4\right)}{\left(C_1C_2 - C_3^2\right)} = \alpha_{10}(say) \\ \alpha_2 &= \frac{\left(C_3C_4 - C_1C_5\right)}{\left(C_1C_2 - C_3^2\right)} = \alpha_{20}(say) \end{aligned}\right\}, \tag{4.14}$$

$$\left.\begin{aligned} \delta_1 &= \frac{\left(D_3D_5 - D_2D_4\right)}{\left(D_1D_2 - D_3^2\right)} = \delta_{10}(say) \\ \delta_2 &= \frac{\left(D_3D_4 - D_1D_5\right)}{\left(D_1D_2 - D_3^2\right)} = \delta_{20}(say) \end{aligned}\right\}. \tag{4.15}$$

Thus, the minimum MSEs of the estimators $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$ are, respectively, given by

$$MSE_{\min}\left(\hat{\Sigma}_1\right) = S_y^4\left(B_0 - L_1\right), \tag{4.16}$$

$$MSE_{\min}\left(\hat{\Sigma}_2\right) = S_y^4\left(C_0 - L_2\right), \tag{4.17}$$

$$MSE_{\min}\left(\hat{\Sigma}_3\right) = S_y^4\left(D_0 - L_3\right). \tag{4.18}$$

where

$$L_1 = \frac{\left(B_2B_4^2 - 2B_3B_4B_5 + B_1B_5^2\right)}{\left(B_1B_2 - B_3^2\right)}, L_2 = \frac{\left(C_2C_4^2 - 2C_3C_4C_5 + C_1C_5^2\right)}{\left(C_1C_2 - C_3^2\right)}$$

and $L_3 = \dfrac{\left(D_2D_4^2 - 2D_3D_4D_5 + D_1D_5^2\right)}{\left(D_1D_2 - D_3^2\right)}$.

The $MSE_{\min}(\hat{\Sigma}_1), MSE_{\min}(\hat{\Sigma}_2)$ and $MSE_{\min}(\hat{\Sigma}_3)$ are non-negative respectively if

$$0 < L_1 < B_0 \text{ and } \left(B_1B_2 - B_3^2\right) > 0, \tag{4.19}$$

$$0 < L_2 < C_0 \text{ and} \left(C_1C_2 - C_3^2\right) > 0, \tag{4.20}$$

$$0 < L_3 < D_0 \text{ and} \left(D_1D_2 - D_3^2\right) > 0. \tag{4.21}$$

5. Comparisons of Suggested Estimators $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$ with t_{M_2}

From Eqs. (3.18), (4.16), (4.17) and (4.18), we have that

- $MSE_{\min}(\hat{\Sigma}_1) < MSE_{\min}(t_{M_2})$ if $\left(B_0 + L_0\right) < \left(A_0 + L_1\right)$, (5.1)
- $MSE_{\min}\left(\hat{\Sigma}_2\right) < MSE_{\min}\left(t_{M_2}\right)$ if $\left(C_0 + L_0\right) < \left(A_0 + L_2\right)$, (5.2)
- $MSE_{\min}\left(\hat{\Sigma}_3\right) < MSE_{\min}\left(t_{M_2}\right)$ if $\left(D_0 + L_0\right) < \left(A_0 + L_3\right)$. (5.3)

where $L_0 = \dfrac{\left(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2\right)}{\left(A_1A_2 - A_3^2\right)}$.

Thus, the developed estimators $\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3$ exhibit greater efficiency compared to Mishra et al. [10] estimator t_{M_2} as long as the conditions (5.1), (5.2) and (5.3), respectively, hold good.

6. Generalized Estimators

Generalized versions of the developed estimators $(\hat{\Sigma}_1, \hat{\Sigma}_2$ and $\hat{\Sigma}_3)$ are given by

$$\hat{\Sigma}_{g1} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{u\left(S_x^2 - s_x^2\right)}{u\left(S_x^2 + s_x^2\right) + 2v}\right\}, \tag{6.1}$$

$$\hat{\Sigma}_{g2} = \left[\left(\alpha_1 + 1\right)s_y^2 + \alpha_2 \log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{u\left(S_x^2 - s_x^2\right)}{u\left(S_x^2 + s_x^2\right) + 2v}\right\}, \tag{6.2}$$

$$\hat{\Sigma}_{g3} = \left[\left(\delta_1 + 1\right)s_y^2 + \delta_2\left(S_x^2 - s_x^2\right)\right]\exp\left\{\frac{u\left(S_x^2 - s_x^2\right)}{u\left(S_x^2 + s_x^2\right) + 2v}\right\}. \tag{6.3}$$

where u and v are either real amounts or some available population parametric values of auxiliary variable x and study variable y. Some sub-classes of estimators $\hat{\Sigma}_{g1}, \hat{\Sigma}_{g2}$ and $\hat{\Sigma}_{g3}$ are listed in Tables 1, 2, and 3.

Table 1

Some individuals from the proposed class $\hat{\Sigma}_{g1}$

Unknown Members	U	v
$\hat{\Sigma}_{g1}^{(1)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right\}$	1	0
$\hat{\Sigma}_{g1}^{(2)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{S_x^2 + s_x^2}{S_x^2 + s_x^2 + 2C_x}\right\}$	1	C_x
$\hat{\Sigma}_{g1}^{(3)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2 + 2\beta_2(x)}\right\}$	1	$\beta_2(x)$
$\hat{\Sigma}_{g1}^{(4)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2 + 2\rho}\right\}$	1	ρ
$\hat{\Sigma}_{g1}^{(5)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{\beta_2(x)\left(S_x^2 - s_x^2\right)}{\beta_2(x)\left(S_x^2 + s_x^2\right) + 2C_x}\right\}$	$\beta_2(x)$	C_x
$\hat{\Sigma}_{g1}^{(6)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2 - s_x^2\right)}{C_x\left(S_x^2 + s_x^2\right) + 2\beta_2(x)}\right\}$	C_x	$\beta_2(x)$
$\hat{\Sigma}_{g1}^{(7)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{\rho\left(S_x^2 - s_x^2\right)}{\rho\left(S_x^2 + s_x^2\right) + 2C_x}\right\}$	ρ	C_x
$\hat{\Sigma}_{g1}^{(8)} = \left[\left(w_1 + 1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2 - s_x^2\right)}{C_x\left(S_x^2 + s_x^2\right) + 2\rho}\right\}$	C_x	ρ

Contd...

$\hat{\Sigma}_{g1}^{(9)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{\beta_2(x)\left(S_x^2-s_x^2\right)}{\beta_2(x)\left(S_x^2+s_x^2\right)+2\rho}\right\}$	$\beta_2(x)$	ρ
$\hat{\Sigma}_{g1}^{(10)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{\rho\left(S_x^2-s_x^2\right)}{\rho\left(S_x^2+s_x^2\right)+2\beta_2(x)}\right\}$	ρ	$\beta_2(x)$
$\hat{\Sigma}_{g1}^{(11)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2N}\right\}$	1	N
$\hat{\Sigma}_{g1}^{(12)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2n}\right\}$	1	n
$\hat{\Sigma}_{g1}^{(13)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{N\left(S_x^2-s_x^2\right)}{N\left(S_x^2+s_x^2\right)+2n}\right\}$	N	n
$\hat{\Sigma}_{g1}^{(14)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{n\left(S_x^2-s_x^2\right)}{n\left(S_x^2+s_x^2\right)+2N}\right\}$	N	N
$\hat{\Sigma}_{g1}^{(15)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{N\left(S_x^2-s_x^2\right)}{N\left(S_x^2+s_x^2\right)+2C_x}\right\}$	N	C_x
$\hat{\Sigma}_{g1}^{(16)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2N}\right\}$	C_x	N
$\hat{\Sigma}_{g1}^{(17)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{n\left(S_x^2-s_x^2\right)}{n\left(S_x^2+s_x^2\right)+2C_x}\right\}$	N	C_x
$\hat{\Sigma}_{g1}^{(18)} = \left[\left(w_1+1\right)s_y^2 + w_2\left(\frac{S_x^2}{s_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2n}\right\}$	C_x	n

Table 2

Some members from the suggested class $\hat{\Sigma}_{g2}$

Unknown Members	U	v
$\hat{\Sigma}_{g2}^{(1)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right\}$	1	0
$\hat{\Sigma}_{g2}^{(2)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{S_x^2+s_x^2}{S_x^2+s_x^2+2C_x}\right\}$	1	C_x
$\hat{\Sigma}_{g2}^{(3)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2\beta_2(x)}\right\}$	1	$\beta_2(x)$
$\hat{\Sigma}_{g2}^{(4)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2\rho}\right\}$	1	ρ
$\hat{\Sigma}_{g2}^{(5)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{\beta_2(x)\left(S_x^2-s_x^2\right)}{\beta_2(x)\left(S_x^2+s_x^2\right)+2C_x}\right\}$	$\beta_2(x)$	C_x
$\hat{\Sigma}_{g2}^{(6)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2\beta_2(x)}\right\}$	C_x	$\beta_2(x)$
$\hat{\Sigma}_{g2}^{(7)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{\rho\left(S_x^2-s_x^2\right)}{\rho\left(S_x^2+s_x^2\right)+2C_x}\right\}$	ρ	C_x
$\hat{\Sigma}_{g2}^{(8)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2\rho}\right\}$	C_x	ρ
$\hat{\Sigma}_{g2}^{(9)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{\beta_2(x)\left(S_x^2-s_x^2\right)}{\beta_2(x)\left(S_x^2+s_x^2\right)+2\rho}\right\}$	$\beta_2(x)$	ρ
$\hat{\Sigma}_{g2}^{(10)}=\left[(\alpha_1+1)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{\rho\left(S_x^2-s_x^2\right)}{\rho\left(S_x^2+s_x^2\right)+2\beta_2(x)}\right\}$	ρ	$\beta_2(x)$

Contd...

$\hat{\Sigma}_{g2}^{(11)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2N}\right\}$	1	N
$\hat{\Sigma}_{g2}^{(12)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2n}\right\}$	1	n
$\hat{\Sigma}_{g2}^{(13)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{N\left(S_x^2-s_x^2\right)}{N\left(S_x^2+s_x^2\right)+2n}\right\}$	N	n
$\hat{\Sigma}_{g2}^{(14)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{n\left(S_x^2-s_x^2\right)}{n\left(S_x^2+s_x^2\right)+2N}\right\}$	N	N
$\hat{\Sigma}_{g2}^{(15)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{N\left(S_x^2-s_x^2\right)}{N\left(S_x^2+s_x^2\right)+2C_x}\right\}$	N	C_x
$\hat{\Sigma}_{g2}^{(16)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2N}\right\}$	C_x	N
$\hat{\Sigma}_{g2}^{(17)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{n\left(S_x^2-s_x^2\right)}{n\left(S_x^2+s_x^2\right)+2C_x}\right\}$	N	C_x
$\hat{\Sigma}_{g2}^{(18)} = \left[\left(\alpha_1+1\right)s_y^2+\alpha_2\log\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2n}\right\}$	C_x	n

Table 3

Some individuals from the proposed class $\hat{\Sigma}_{g3}$

Unknown Members	U	v
$\hat{\Sigma}_{g3}^{(1)} = \left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right\}$	1	0
$\hat{\Sigma}_{g3}^{(2)} = \left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2C_x}\right\}$	1	C_x

Contd...

$\hat{\Sigma}_{g3}^{(3)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2\beta_2(x)}\right\}$	1	$\beta_2(x)$
$\hat{\Sigma}_{g3}^{(4)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2\rho}\right\}$	1	ρ
$\hat{\Sigma}_{g3}^{(5)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{\beta_2(x)\left(S_x^2-s_x^2\right)}{\beta_2(x)\left(S_x^2+s_x^2\right)+2C_x}\right\}$	$\beta_2(x)$	C_x
$\hat{\Sigma}_{g3}^{(6)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2\beta_2(x)}\right\}$	C_x	$\beta_2(x)$
$\hat{\Sigma}_{g3}^{(7)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{\rho\left(S_x^2-s_x^2\right)}{\rho\left(S_x^2+s_x^2\right)+2C_x}\right\}$	ρ	C_x
$\hat{\Sigma}_{g3}^{(8)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2\rho}\right\}$	C_x	ρ
$\hat{\Sigma}_{g3}^{(9)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{\beta_2(x)\left(S_x^2-s_x^2\right)}{\beta_2(x)\left(S_x^2+s_x^2\right)+2\rho}\right\}$	$\beta_2(x)$	ρ
$\hat{\Sigma}_{g3}^{(10)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{\rho\left(S_x^2-s_x^2\right)}{\rho\left(S_x^2+s_x^2\right)+2\beta_2(x)}\right\}$	ρ	$\beta_2(x)$
$\hat{\Sigma}_{g3}^{(11)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2N}\right\}$	1	N
$\hat{\Sigma}_{g3}^{(12)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{S_x^2-s_x^2}{S_x^2+s_x^2+2n}\right\}$	1	n
$\hat{\Sigma}_{g3}^{(13)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{N\left(S_x^2-s_x^2\right)}{N\left(S_x^2+s_x^2\right)+2n}\right\}$	N	n

Contd...

$\hat{\Sigma}_{g3}^{(14)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{n\left(S_x^2-s_x^2\right)}{n\left(S_x^2+s_x^2\right)+2N}\right\}$	N	N
$\hat{\Sigma}_{g3}^{(15)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{N\left(S_x^2-s_x^2\right)}{N\left(S_x^2+s_x^2\right)+2C_x}\right\}$	N	C_x
$\hat{\Sigma}_{g3}^{(16)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2N}\right\}$	C_x	N
$\hat{\Sigma}_{g3}^{(17)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{n\left(S_x^2-s_x^2\right)}{n\left(S_x^2+s_x^2\right)+2C_x}\right\}$	N	C_x
$\hat{\Sigma}_{g3}^{(18)}=\left[\left(\delta_1+1\right)s_y^2+\delta_2\left(S_x^2-s_x^2\right)\right]\exp\left\{\frac{C_x\left(S_x^2-s_x^2\right)}{C_x\left(S_x^2+s_x^2\right)+2n}\right\}$	C_x	n

After expressing Eqs. (6.1), (6.2) and (6.3) in terms of e's, we determine the bias of $\hat{\Sigma}_{gi}(i=1,2,3)$ to the FDA, respectively, as:

$$B\left(\hat{\Sigma}_{g1}\right)=S_y^2\left[\xi\left(\frac{3}{8}\lambda^2\left(\lambda_{04}-1\right)-\frac{\lambda}{2}\left(\lambda_{22}-1\right)\right)+w_1\left\{1+\xi\left(\frac{3\lambda^2}{8}\left(\lambda_{04}-1\right)-\frac{\lambda}{2}\left(\lambda_{22}-1\right)\right)\right\}\right.$$

$$\left.+w_2\left(\frac{1}{RS_x^2}\right)\left\{1+\xi\left(1+\frac{\lambda}{2}+\frac{3\lambda^2}{8}\right)\left(\lambda_{04}-1\right)\right\}\right], \tag{6.4}$$

$$B\left(\hat{\Sigma}_{g2}\right)=S_y^2\left[\left(\xi\left(\frac{3}{8}\lambda^2\left(\lambda_{04}-1\right)-\frac{\lambda}{2}\left(\lambda_{22}-1\right)\right)\right)+\alpha_1\left\{1+\xi\left(\frac{3}{8}\lambda^2\left(\lambda_{04}-1\right)-\frac{\lambda}{2}\left(\lambda_{22}-1\right)\right)\right\}\right.$$

$$\left.-\alpha_2\left(\frac{\xi}{RS_x^2}\right)\frac{\left(1+\lambda\right)}{2}\left(\lambda_{04}-1\right)\right], \tag{6.5}$$

$$B\left(\hat{\Sigma}_{g3}\right)=S_y^2\left[\xi\left(\frac{3}{8}\lambda^2\left(\lambda_{04}-1\right)-\frac{\lambda}{2}\left(\lambda_{22}-1\right)\right)+\delta_1\left\{1+\xi\left(\frac{3}{8}\lambda^2\left(\lambda_{04}-1\right)-\frac{\lambda}{2}\left(\lambda_{22}-1\right)\right)\right\}\right.$$

$$\left.+\delta_2\frac{\lambda\xi}{2R}\left(\lambda_{04}-1\right)\right]. \tag{6.6}$$

where $R = S_y^2 / S_x^2$ and $\lambda = \dfrac{uS_x^2}{\left(uS_x^2 + v\right)}$.

We get the MSEs of $\hat{\Sigma}_{g1}, \hat{\Sigma}_{g2}$ and $\hat{\Sigma}_{g3}$ to the FDA, respectively as:

$$MSE\left(\hat{\Sigma}_{g1}\right) = S_y^4\left[B_0^* + w_1^{*2}B_1^* + w_2^{*2}B_2^* + 2w_1^*w_2^*B_3^* + 2w_1^*B_4^* + 2w_2^*B_5^*\right]. \tag{6.7}$$

$$MSE\left(\hat{\Sigma}_{g2}\right) = S_y^4\left[C_0^* + \alpha_1^{*2}C_1^* + \alpha_2^{*2}C_2^* + 2\alpha_1^*\alpha_2^*C_3^* + 2\alpha_1^*C_4^* + 2\alpha_2^*C_5^*\right] \tag{6.8}$$

$$MSE\left(\hat{\Sigma}_{g3}\right) = S_y^4\left[D_0^* + \delta_1^{*2}D_1^* + \delta_2^{*2}D_2^* + 2\delta_1\delta_2D_3^* + 2\delta_1^*D_4^* + 2\delta_2^*D_5^*\right] \tag{6.9}$$

where $B_0^* = C_0^* = D_0^* = \xi\left[\left(\lambda_{40}-1\right) + \frac{1}{4}\lambda^2\left(\lambda_{04}-1\right) - \lambda\left(\lambda_{22}-1\right)\right]$,

$$B_1^* = \left[1 + \xi\left\{\left(\lambda_{40}-1\right) - 2\lambda\left(\lambda_{22}-1\right) + \lambda^2\left(\lambda_{04}-1\right)\right\}\right] = C_1^* = D_1^*,$$

$$B_2^* = \frac{1}{R^2S_x^4}\left[1 + \xi\left(3 + 2\lambda + \lambda^2\right)\left(\lambda_{04}-1\right)\right],\ C_2^* = \frac{\xi}{R^2S_x^4}\left(\lambda_{04}-1\right),$$

$$D_2^* = \frac{\xi}{R^2}\left(\lambda_{04}-1\right), B_3^* = \left(\frac{1}{RS_x^2}\right)\left[1 + \xi\left\{\left(1+\lambda+\lambda^2\right)\left(\lambda_{04}-1\right) - \left(1+\lambda\right)\left(\lambda_{22}-1\right)\right\}\right]$$

$$C_3^* = \frac{\xi}{RS_x^2}\left[\left(\lambda_{22}-1\right) - \left(\lambda + \frac{1}{2}\right)\left(\lambda_{04}-1\right)\right], D_3^* = \frac{\xi}{R}\left[\lambda\left(\lambda_{04}-1\right) - \left(\lambda_{22}-1\right)\right]$$

$$B_4^* = \xi\left\{\left(\lambda_{40}-1\right) + \frac{5\lambda^2}{8}\left(\lambda_{04}-1\right) - \frac{3\lambda}{2}\left(\lambda_{22}-1\right)\right\},$$

$$C_4^* = \xi\left[\left(\lambda_{40}-1\right) - \frac{3}{2}\lambda\left(\lambda_{22}-1\right) + \frac{5}{8}\lambda^2\left(\lambda_{04}-1\right)\right] = D_4^*,$$

$$B_5^* = \left(\frac{\xi}{RS_x^2}\right)\left[\frac{\lambda}{2}\left(1 + \frac{5}{4}\lambda\right)\left(\lambda_{04}-1\right) - \left(1+\lambda\right)\left(\lambda_{22}-1\right)\right],$$

$$C_5^* = \left(\frac{\xi}{RS_x^2}\right)\left[\left(\lambda_{22}-1\right) - \frac{\lambda}{2}\left(\lambda_{04}-1\right)\right], D_5^* = \left(\frac{\xi}{R}\right)\left[\frac{\lambda}{2}\left(\lambda_{04}-1\right) - \left(\lambda_{22}-1\right)\right]$$

The mean squared errors of $\hat{\Sigma}_{g1}, \hat{\Sigma}_{g2}$ and $\hat{\Sigma}_{g3}$ are respectively minimized for

$$\left.\begin{aligned} w_1^* &= \frac{\left(B_3^* B_5^* - B_2^* B_4^*\right)}{\left(B_1^* B_2^* - B_3^{*2}\right)} = w_{10}^*, (say) \\ w_2^* &= \frac{\left(B_3^* B_4^* - B_1^* B_5^*\right)}{\left(B_1^* B_2^* - B_3^{*2}\right)} = w_{20}^*, (say) \end{aligned}\right\}, \tag{6.10}$$

$$\left.\begin{aligned} \alpha_1^* &= \frac{\left(C_3^* C_5^* - C_2^* C_4^*\right)}{\left(C_1^* C_2^* - C_3^{*2}\right)} = \alpha_{10}^*, (say) \\ \alpha_2^* &= \frac{\left(C_3^* C_4^* - C_1^* C_5^*\right)}{\left(C_1^* C_2^* - C_3^{*2}\right)} = \alpha_{20}^*, (say) \end{aligned}\right\}, \tag{6.11}$$

$$\left.\begin{aligned} \delta_1^* &= \frac{\left(D_3^* D_5^* - D_2^* D_4^*\right)}{\left(D_1^* D_2^* - D_3^{*2}\right)} = \delta_{10}^*, (say) \\ \delta_2^* &= \frac{\left(D_3^* D_4^* - D_1^* D_5^*\right)}{\left(D_1^* D_2^* - D_3^{*2}\right)} = \delta_{20}^*, (say) \end{aligned}\right\}. \tag{6.12}$$

Thus, the resulting minimum mean squared errors of $\hat{\Sigma}_{g1}, \hat{\Sigma}_{g2}$ and $\hat{\Sigma}_{g3}$ are respectively given by

$$MSE_{\min}\left(\hat{\Sigma}_{g1}\right) = S_y^4\left(B_0^* - M_1\right), \tag{6.13}$$

$$MSE_{\min}\left(\hat{\Sigma}_{g2}\right) = S_y^4\left(C_0^* - M_2\right), \tag{6.14}$$

$$MSE_{\min}\left(\hat{\Sigma}_{g3}\right) = S_y^4\left(D_0^* - M_3\right). \tag{6.15}$$

where

$$M_1 = \frac{\left(B_2^* B_4^{*2} - 2B_3^* B_4^* B_5^* + B_1^* B_5^{*2}\right)}{\left(B_1^* B_2^* - B_3^{*2}\right)}, M_2 = \frac{\left(C_2^* C_4^{*2} - 2C_3^* C_4^* C_5^* + C_1^* C_5^{*2}\right)}{\left(C_1^* C_2^* - C_3^{*2}\right)}$$

and $M_3 = \dfrac{\left(D_2^* D_4^{*2} - 2D_3^* D_4^* D_5^* + D_1^* D_5^{*2}\right)}{\left(D_1^* D_2^* - D_3^{*2}\right)}$.

The $MSE_{\min}(\hat{\Sigma}_{g1}), MSE_{\min}(\hat{\Sigma}_{g2})$ and $MSE_{\min}(\hat{\Sigma}_{g3})$ are, respectively, non-negative if the following conditions hold good:

$$0 < M_1 < B_0^* \text{ and } \left(B_1^* B_2^* - B_3^{*2}\right) > 0, \tag{6.16}$$

$$0 < M_2 < C_0^* \text{ and } \left(C_1^* C_2^* - C_3^{*2}\right) > 0, \tag{6.17}$$

$$0 < M_3 < D_0^* \text{ and } \left(D_1^* D_2^* - D_3^{*2}\right) > 0. \tag{6.18}$$

7. Comparisons of Generalized Estimators with t_{M2}

We denote $L_0 = \dfrac{\left(A_2 A_4^2 - 2A_3 A_4 A_5 + A_1 A_5^2\right)}{\left(A_1 A_2 - A_3^2\right)}$

Then, the minimum MSE of the estimator t_{M2} due to Mishra et al. [10] is given by

$$MSE_{\min}\left(t_{M_2}\right) = S_y^4\left(A_0 - L_0\right). \tag{7.1}$$

From Eqs. (6.13), (6.14), (6.15) and (7.1), we note that the generalized estimators $\hat{\Sigma}_{g1}, \hat{\Sigma}_{g2}$ and $\hat{\Sigma}_{g3}$ are respectively better than the estimator t_{M2} due to Mishra et al. [10] if the conditions:

$$\left(B_0^* + L_0\right) < \left(A_0 + M_1\right), \tag{7.2}$$

$$\left(C_0^* + L_0\right) < \left(A_0 + M_2\right), \tag{7.3}$$

$$\left(D_0^* + L_0\right) < \left(A_0 + M_3\right). \tag{7.4}$$

hold good.

8. An Empirical Study

In this section, we have considered three natural population data sets earlier considered by Mishra et al. [10] to evaluate how the developed estimators are performed relative to the usual unbiased estimator s_y^2.

We use the following formulae for computing % relative efficiency (PRE) with reference to the conventional unbiased estimator (s_y^2):

$$PRE\left(t_R, s_y^2\right) = \frac{\left(\lambda_{40} - 1\right)}{\left[\lambda_{40} - 2\lambda_{22} + \lambda_{04}\right]} \times 100,$$

$$PRE\left(t_K, s_y^2\right) = \frac{\left(\lambda_{40} - 1\right)}{\left[\lambda_{40} - \lambda_{22} + \frac{1}{4}\left(\lambda_{04} - 1\right)\right]} \times 100.$$

$$PRE\left(t_z, s_y^2\right) = \frac{\left(\lambda_{40} - 1\right)}{\left[\left(\lambda_{40} - 1\right) - \frac{\left(\lambda_{22} - 1\right)^2}{\left(\lambda_{04} - 1\right)}\right]} \times 100,$$

where $z = d$, $DT1$, $DT2$, YK, $M1$.

$$PRE\left(t_s, s_y^2\right) = \left[\xi\left(\lambda_{40} - 1\right) + \frac{1}{\left(1 - \rho^{*2}\right)}\right] \times 100,$$

where $\rho^{*2} = \dfrac{\left(\lambda_{22} - 1\right)^2}{\left(\lambda_{40} - 1\right)\left(\lambda_{04} - 1\right)}$.

$$PRE\left(t_{M2}, s_y^2\right) = \frac{\xi\left(\lambda_{40} - 1\right)}{\left(1 - L_0\right)} \times 100,$$

where $L_0 = \dfrac{\left(A_2 A_4^2 - 2A_3 A_4 A_5 + A_1 A_5^2\right)}{\left(A_1 A_2 - A_3^2\right)}$.

$$PRE\left(\hat{\Sigma}_1, s_y^2\right) = \frac{\xi\left(\lambda_{40} - 1\right)}{\left(B_0 - L_1\right)} \times 100, \; PRE\left(\hat{\Sigma}_2, s_y^2\right) = \frac{\xi\left(\lambda_{40} - 1\right)}{\left(C_0 - L_2\right)} \times 100,$$

$$PRE\left(\hat{\Sigma}_3, s_y^2\right) = \frac{\xi\left(\lambda_{40} - 1\right)}{\left(D_0 - L_3\right)} \times 100,$$

where $\left(L_1, L_2, L_3\right)$ are the same as defined earlier.

Table 4

PREs of various estimators for S_y^2 relative to $t_0 = s_y^2$

Estimators	t_0	t_R	t_K	t_z **(z = d, DT1, DT2, YK, M1)**	t_s	t_{YK}	t_{M2}	$\hat{\Sigma}_1$	$\hat{\Sigma}_2$	$\hat{\Sigma}_3$
Population I	100	88.19	122.82	123.49	133.15	123.49	142.12	206.88	139.12	138.86
Population II	100	320.81	444.02	639.15	575.06	639.15	912.01	1228.13	778.04	692.15
Population III	100	815.61	236.07	1347.98	1324.43	1347.98	2471.13	3853.14	1482.12	1367.28

Table 5

PRE of estimators $\hat{\Sigma}_{g1}$, $\hat{\Sigma}_{g2}$ and $\hat{\Sigma}_{g3}$ relative to $t_0 = s_y^2$ for different values of u and v

	Population- I $C_x = 0.13$, $\rho=0.65$, n=10, $\beta_2(x)=2.24$			Population- II $C_x = 0.16$, $\rho=0.97$, n=10, $\beta_2(x)=2.59$			Population- III $C_x = 0.34$, $\rho=0.93$, n=34, $\beta_2(x)= 3.28$		
(u, v)	$\hat{\Sigma}_{g1}$	$\hat{\Sigma}_{g2}$	$\hat{\Sigma}_{g3}$	$\hat{\Sigma}_{g1}$	$\hat{\Sigma}_{g2}$	$\hat{\Sigma}_{g3}$	$\hat{\Sigma}_{g1}$	$\hat{\Sigma}_{g2}$	$\hat{\Sigma}_{g3}$
1 , 0	206.88	139.12	138.86	1228.13	778.04	692.15	3853.14	1482.12	1367.28
1 , C_x	207.12	139.12	138.83	1229.98	777.99	691.9	3853.31	1482.12	1367.28
1 , ρ	208.1	139.1	138.75	1239.24	777.75	690.66	3853.61	1482.12	1367.28

Contd...

1 , n	224.59	138.94	137.6	1342.79	776.66	679.89	3870.42	1482.10	1367.05
1 , N	317	139.87	135.14	3468.62	828.73	650.26	3938.85	1482.05	1366.18
1 , $\beta_2(x)$	211.03	139.05	138.51	1257.75	777.35	688.36	3854.81	1482.12	1367.26
C_x , ρ	216.15	138.99	138.13	1296.86	776.83	684.08	3854.53	1482.12	1367.26
C_x , n	301.93	139.63	135.32	1936.82	790.32	658.24	3904.02	1482.07	1366.61
C_x , N	419.75	141.6	134.66	12833.15	888.82	648.04	4101.62	1482.17	1364.36
C_x , $\beta_2(x)$	236.47	138.93	137	1411.9	777.09	674.85	3858.07	1482.12	1367.22
ρ , C_x	207.25	139.12	138.82	1230.03	777.99	691.89	3853.32	1482.12	1367.28
ρ , $\beta_2(x)$	213.21	139.03	138.34	1258.55	777.34	688.26	3854.93	1482.12	1367.26
n , C_x	206.91	139.12	138.85	1228.32	778.04	692.12	3853.14	1482.12	1367.28
n , N	224.59	138.94	137.6	1458.1	777.7	672.13	3855.68	1482.12	1367.25
N , C_x	206.89	139.12	138.86	1228.14	778.04	692.15	3853.14	1482.12	1367.28
N , n	207.07	139.12	138.84	1228.7	778.03	692.07	3853.24	1482.12	1367.28
$\beta_2(x)$, C_x	206.99	139.12	138.85	1228.85	778.02	692.05	3853.19	1482.12	1367.28
$\beta_2(x)$, ρ	207.43	139.11	138.81	1232.41	777.92	691.57	3853.28	1482.12	1367.28

The population data sets and required parametric values are described below.

Population-I (Cochran, 1977, p.325):

$$\mathrm{N} = 100,\ \mathrm{n} = 10,\ S_y^2 = 214.69, S_x^2 = 56.76,$$

$$\lambda_{40} = 2.2387, \lambda_{04} = 2.2523, \lambda_{22} = 1.5432, \rho = 0.6515.$$

Population-II (Cochran, 1977, p.203):

$$N = 200, n = 10, S_y^2 = 99.18, S_x^2 = 85.09, \lambda_{40} = 1.9249,$$

$$\lambda_{04} = 2.5932, \lambda_{22} = 2.1149,\ \rho = 0.9736.$$

Population- III: (Sukhatme and Sukhatme 1970, p.185):

$$N = 170,\ n = 10,\ S_y^2 = 26456.89,\ \rho = 0.9770,\ S_x^2 = 22355.76,$$

$$\lambda_{40} = 3.1842,\ \lambda_{04} = 2.2030,\ \lambda_{22} = 2.5597.$$

Findings are depicted in Tables 4 and 5.

9. Result And Discussion

From Tables 4 and 5, the following illustrations can be drawn:

- It is evident from Table 4 that %REs of the recommended estimator $\hat{\Sigma}_1$ are 206.88 %, 1228.13 % and 3853.14 %, respectively, for Populations I, II and III, which exhibit greater efficiency than the Mishra et al. [10] and other existing estimators.
- The %REs of the proposed estimator $\hat{\Sigma}_2$ are 139.12%, 778.04% and 1482.12%, while for the proposed estimator $\hat{\Sigma}_3$ are 138.86%, 692.15% and 1367.28%, respectively, for Populations I, II and III, that are less than t_{M2} estimator but more than the other existing estimators.
- Table 5 depicts that the percent relative efficiencies of the developed class of estimators $\hat{\Sigma}_{g1}$ lies between 206.88% to 419.75 %, 1228.13% to 12833.15 % and 3853.14% to 4101.62 % using different combinations of (u, v), respectively, for Populations I, II and III, demonstrating better performance than the Mishra et al. [10] and other existing estimators.

- Similarly, it can be easily seen that the relative efficiencies of other proposed classes of estimators $(\hat{\Sigma}_{g2}$ and $\hat{\Sigma}_{g3})$ are less than the t_{M2} estimator but more than the other existing estimators under the three population datasets considered in this paper.

10. Conclusion

This paper developed classes of estimators $(\hat{\Sigma}_i, \hat{\Sigma}_{gi};\ i = 1,2,3)$ for S_y^2 incorporating available information on S_x^2. The suggested estimators $(\hat{\Sigma}_1$ and $\hat{\Sigma}_{g1})$ are performing better than Mishra et al. [10] estimators t_{M1} and t_{M2} as well as other existing estimators. While the proposed estimators $(\hat{\Sigma}_2, \hat{\Sigma}_{g2})$ and $(\hat{\Sigma}_3, \hat{\Sigma}_{g3})$ do not perform better than the t_{M2} estimator. Still, they perform better than the difference estimator t_d and other existing estimators for the considered data sets. However, in practice, we may come across such population data sets where all suggested estimators $(\hat{\Sigma}_i, \hat{\Sigma}_{gi};\ i = 1,2,3)$ are better than t_{M2} and other existing estimators.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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