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Efficient algorithm for solving some inventory control problems

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Abstract

Some inventory control problems are considered in the paper, in particular, box constrained inventory control problems. The optimal solution is described by a characterization theorem, and an efficient convergent polynomial algorithm is proposed.

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1. Introduction

Consider the following box constrained inventory control problem.

(C)

$$\min\left\{c(x) \equiv c(x_1, \dots, x_n) = \sum_{i=1}^{n} \left(\frac{s_i a_i}{x_i} + \frac{p_i x_i}{2}\right)\right\} \tag{1}$$

subject to

$$\sum_{i=1}^{n} d_i x_i \le b \tag{2}$$

$$x_i \ge 0, \quad i = 1, \dots, n. \tag{3}$$

In problem (C), for the item i, where $i = 1, \dots, n$, s_i is the cost of setup, a_i is the demand rate, x_i is quantity of the order, p_i is the cost for holding a unit of the item, d_i is the storage area per facility unit, and b is the capacity of the store (the maximum storage area).

The objective function describes the total cost and it must be minimized.

The unconstrained minimum of function $c(x)$, defined by (1), is

$$x_i^* = \sqrt{\frac{2 s_i a_i}{p_i}}, \quad \forall\, i. \tag{4}$$

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Taking the positive values of the square roots in (4), then x_i^*'s satisfy constraints (3) of problem (C). If x_i^*'s satisfy constraint (2) as well, then x_i^*'s constitute the optimal solution of problem (C).

Otherwise, if $\sum_{i=1}^n d_i x_i^* > b$, the inequality constraint (2) has to be satisfied with equality

$$\sum_{i=1}^n d_i x_i^* = b.$$

The function of Lagrange for the considered problem (C) (1)-(3) is

$$L(\mathrm{x}, t) = \sum_{i=1}^n \left(\frac{s_i a_i}{x_i} + \frac{p_i x_i}{2}\right) + t\left(\sum_{i=1}^n d_i x_i - b\right),$$

with $t \geq 0$ the dual variable, corresponding to the single inequality constraint (2).

Because $L(x, t)$ is a convex function of t and x, the optimal values t^* and x^* satisfy the optimality conditions

$$\frac{\partial L}{\partial x_i} \equiv -\frac{s_i a_i}{x_i^2} + \frac{p_i}{2} + t d_i = 0, \quad \forall\, i, \tag{5}$$

$$\frac{\partial L}{\partial t} \equiv \sum_{i=1}^n d_i x_i - b = 0. \tag{6}$$

Conditions (5) imply

$$x_i^* = \sqrt{\frac{2 s_i a_i}{p_i + 2 t d_i}}, \quad \forall\, i. \tag{7}$$

If $t = 0$, that is, if there is no limit on the storage, expressions (4) and (7) coincide, and (4) is known as the *Wilson's economic lot size formula*.

Consider problem (CBV), which is a version of the above problem (C) (1)-(3), such that instead of nonnegativity constraints (3), box constraints are included.

(CBV)

$$\min\left\{c(\mathrm{x}) \equiv c(x_1, \dots, x_n) \equiv \sum_{i=1}^n c_i(x_i) = \sum_{i=1}^n \left(\frac{s_i a_i}{x_i} + \frac{p_i x_i}{2}\right)\right\} \tag{8}$$

s.t.

$$\sum_{i=1}^n d_i x_i \leq b \tag{9}$$

$$l_i \leq x_i \leq u_i, \quad \forall\, i, \tag{10}$$

where $l_i \geq 0$, $i = 1, \dots, n$, in accordance with economic interpretation of x_i's; and the values of s_i, a_i, p_i, and d_i are also assumed to be nonnegative (even positive) for each $i = 1, \dots, n$, in accordance with their economic meaning.

Note that if $l_i = 0$ and $u_i = +\infty$ $\forall\, i = 1, \dots, n$, constraints (10) imply constraints (3).

Since $c_i'(x_i) = -\frac{s_i a_i}{x_i^2} + \frac{p_i}{2}$, and therefore $c_i''(x_i) = \frac{2 s_i a_i}{x_i^3} > 0$ by assumption, then functions $c_i(x_i)$ turn out to be strictly convex, and consequently, $c(x)$ is also a strictly convex function.

Problems (C), (CBV) and related problems arise in inventory control, facility location, production planning, scheduling theory, allocation of resources, etc. Related topics are considered, e.g., in reference titles [1-34].

A characterization theorem for the optimal solution of problem (CBV) is established in Section 2. An efficient algorithm for solving this problem is proposed, the convergence is proved, and polynomial complexity of the algorithm is established in Section 3. In Section 4, content of the paper is summarized, and an open problem for future work is formulated.

2. Characterization theorem for problem (CBV)

Assume that

(A.1) $l_i \leq u_i$ for each $i = 1, \ldots, n$. If $l_k = u_k$ for an index k, where $1 \leq k \leq n$, the value of $x_k := l_k = u_k$ is known in advance.

(A.2)

$$\sum_{i=1}^{n} d_i l_i \leq b, \tag{11}$$

else constraints (9)-(10) are incompatible and therefore the feasible set X (9)-(10) would be empty. We also assume that $b \leq \sum_{i=1}^{n} d_i u_i$ in certain cases, which will be described below.

The function of Lagrange for (CBV) is

$$L(\mathrm{x}, \mathrm{v}, \mathrm{w}, t) = \sum_{i=1}^{n} \left(\frac{s_i a_i}{x_i} + \frac{p_i x_i}{2}\right) + t\left(\sum_{i=1}^{n} d_i x_i - b\right) + \sum_{i=1}^{n} v_i (l_i - x_i)$$
$$+ \sum_{i=1}^{n} w_i (x_i - u_i), \tag{12}$$

where $t \in \mathbb{R}_+^1$, $v = (v_1, \ldots, v_n) \in \mathbb{R}_+^n$, $w = (w_1, \ldots, w_n) \in \mathbb{R}_+^n$.

The minimization conditions of Karush-Kuhn-Tucker (in short KKT) for problem (CBV) are as follows

$$-\frac{s_i a_i}{(x_i^*)^2} + \frac{p_i}{2} + t d_i - v_i + w_i = 0, \quad \forall\, i, \tag{13}$$

$$v_i (l_i - x_i^*) = 0, \quad \forall\, i, \tag{14}$$

$$w_i (x_i^* - u_i) = 0, \quad \forall\, i, \tag{15}$$

$$t\left(\sum_{i=1}^{n} d_i x_i^* - b\right) = 0, \quad t \in \mathbb{R}_+^1, \tag{16}$$

$$\sum_{i=1}^{n} d_i x_i^* \leq b, \tag{17}$$

$$l_i \leq x_i^* \leq u_i, \quad \forall\, i, \tag{18}$$

$$v_i \in \mathbb{R}_+^1, w_i \in \mathbb{R}_+^1, \quad \forall\, i, \tag{19}$$

where t, v_i, and w_i, are the dual variables (Lagrange multipliers) for the constraints (9), $l_i \leq x_i$, and $x_i \leq u_i$ for each $i = 1, \ldots, n$, respectively. If $l_i = -\infty$ or $u_i = +\infty$ for an index i, $1 \leq i \leq n$, then the corresponding KKT condition (14) (condition (15), respectively) and the dual variable v_i (dual variable w_i, respectively) are not taken into account.

Theorem 1 (Characterization theorem for problem (CBV))

Denote $I = \{1, \ldots, n\}$. A feasible point $x^* = (x_i^*)_{i \in I} \in X$ (9)-(10) is optimal solution of minimization problem (CBV) iff $\exists$ a nonnegative number $t \in \mathbb{R}_+^1$ with

$$x_i^* = l_i, \quad i \in I_l^t \stackrel{\text{def}}{=} \left\{ i \in I : t \geq \frac{s_i a_i}{d_i l_i^2} - \frac{p_i}{2 d_i} \right\}, \tag{20}$$

$$x_i^* = u_i, \quad i \in I_u^t \stackrel{\text{def}}{=} \left\{ i \in I : t \le \frac{s_i a_i}{d_i u_i^2} - \frac{p_i}{2d_i} \right\}, \tag{21}$$

$$x_i^* = \sqrt{\frac{2s_i a_i}{p_i + 2t d_i}}, \quad i \in I^t \stackrel{\text{def}}{=} \left\{ i \in I : \frac{s_i a_i}{d_i u_i^2} - \frac{p_i}{2d_i} < t < \frac{s_i a_i}{d_i l_i^2} - \frac{p_i}{2d_i} \right\}. \tag{22}$$

Note that the expressions of x_i^* in (22) and in (7) coincide.

Proof: *Necessity.* Suppose that $x^* = (x_i^*)_{i \in I}$ is the optimal solution to problem (CBV). Then there exist scalars (dual variables) t, v_i, w_i, $i \in I$, satisfying the KKT optimality conditions (13)-(19).

(1) Let $t > 0$. Then system (13)-(19) of the KKT optimality conditions is of the form $(13), (14), (15), (18), (19)$ and $\sum_{i \in I} d_i x_i^* = b$.

(a) If $x_i^* = l_i$, then (19) implies $v_i \ge 0$ and (15) implies $w_i = 0$. Then (13) implies

$$-\frac{s_i a_i}{(x_i^*)^2} + \frac{p_i}{2} = v_i - t d_i \ge -t d_i.$$

By assumption $d_i > 0$, hence

$$t \ge \frac{s_i a_i}{d_i (x_i^*)^2} - \frac{p_i}{2d_i} \equiv \frac{s_i a_i}{d_i l_i^2} - \frac{p_i}{2d_i}.$$

(b) If $x_i^* = u_i$, then (14) implies $v_i = 0$ and (19) implies $w_i \ge 0$. Then from (13) it follows that

$$-\frac{s_i a_i}{(x_i^*)^2} + \frac{p_i}{2} = -w_i - t d_i \le -t d_i.$$

Hence,

$$t \le \frac{s_i a_i}{d_i (x_i^*)^2} - \frac{p_i}{2d_i} \equiv \frac{s_i a_i}{d_i u_i^2} - \frac{p_i}{2d_i}.$$

(c) If $l_i < x_i^* < u_i$, then (14) and (15) imply $v_i = w_i = 0$. Then from (13) it follows that

$$\frac{s_i a_i}{(x_i^*)^2} - \frac{p_i}{2} = t d_i.$$

Hence,

$$t = \frac{s_i a_i}{d_i (x_i^*)^2} - \frac{p_i}{2d_i} \quad \text{and} \quad x_i^* = \sqrt{\frac{2s_i a_i}{p_i + 2t d_i}}.$$

By assumption $d_i > 0$, $i \in I$; $t > 0$, and $l_i < x_i^* < u_i$, therefore

$$t = \frac{s_i a_i}{d_i (x_i^*)^2} - \frac{p_i}{2d_i} < \frac{s_i a_i}{d_i l_i^2} - \frac{p_i}{2d_i}, \quad t = \frac{s_i a_i}{d_i (x_i^*)^2} - \frac{p_i}{2d_i} > \frac{s_i a_i}{d_i u_i^2} - \frac{p_i}{2d_i},$$

hence,

$$\frac{s_i a_i}{d_i u_i^2} - \frac{p_i}{2d_i} < t < \frac{s_i a_i}{d_i l_i^2} - \frac{p_i}{2d_i}.$$

(2) Let $t = 0$. Then system (13)-(19) of the KKT optimality conditions is of the form

$$-\frac{s_i a_i}{(x_i^*)^2} + \frac{p_i}{2} - v_i + w_i = 0, i \in I, \quad and \quad (14), (15), (17), (18), (19).$$

(a) If $x_i^* = l_i$, then (19) implies $v_i \geq 0$ and (15) implies $w_i = 0$. Then from (13) it follows that

$$-\frac{s_i a_i}{l_i^2} + \frac{p_i}{2} \equiv -\frac{s_i a_i}{(x_i^*)^2} + \frac{p_i}{2} = v_i \geq 0.$$

Multiply both sides of the above inequality by $-\frac{1}{d_i}$ (< 0 by the assumption), hence

$$\frac{s_i a_i}{d_i l_i^2} - \frac{p_i}{2d_i} \leq 0 \equiv t.$$

(b) If $x_i^* = u_i$, then (14) implies $v_i = 0$ and (19) implies $w_i \geq 0$. Then from (13) it follows that

$$-\frac{s_i a_i}{u_i^2} + \frac{p_i}{2} \equiv -\frac{s_i a_i}{(x_i^*)^2} + \frac{p_i}{2} = -w_i \leq 0.$$

Multiply the above inequality by $-\frac{1}{d_i} < 0$, hence

$$\frac{s_i a_i}{d_i u_i^2} - \frac{p_i}{2d_i} \geq 0 \equiv t.$$

(c) If $l_i < x_i^* < u_i$, then (14) and (15) imply $v_i = w_i = 0$. Then (13) implies

$$\frac{s_i a_i}{(x_i^*)^2} - \frac{p_i}{2} = 0.$$

By assumption, $l_i < x_i^* < u_i$, $i \in I$. Then

$$0 = \frac{s_i a_i}{(x_i^*)^2} - \frac{p_i}{2} < \frac{s_i a_i}{l_i^2} - \frac{p_i}{2}, \quad 0 = \frac{s_i a_i}{(x_i^*)^2} - \frac{p_i}{2} > \frac{s_i a_i}{u_i^2} - \frac{p_i}{2}.$$

Multiply the above two inequalities by $\frac{1}{d_i} > 0$ and take into consideration that $t = 0$ in the considered case (2). Then in the subcase (c) we get

$$\frac{s_i a_i}{d_i u_i^2} - \frac{p_i}{2d_i} < 0 \equiv t < \frac{s_i a_i}{d_i l_i^2} - \frac{p_i}{2d_i}.$$

Introduce the index sets I_l^t, I_u^t, and I^t, defined by (20), (21) and (22), respectively. Evidently $I_l^t \cup I_u^t \cup I^t = I$. Using these three index sets, subcases (a), (b), and (c) of cases (1)-(2) are described by (20), (21), and (22), respectively.

Sufficiency. Let $x^* \in X$, and let elements of the vector (point) x^* satisfy (20), (21), and (22) with $t \in \mathbb{R}_+^1$.

(1) Let $t > 0$. Set:

$$t = \frac{s_i a_i}{d_i (x_i^*)^2} - \frac{p_i}{2d_i} (> 0), \quad v_i = w_i = 0, \quad i \in I^t;$$

$$v_i = -\frac{s_i a_i}{l_i^2} + \frac{p_i}{2} + t d_i \quad (\geq 0), \quad w_i = 0, \quad i \in I_l^t;$$

$$v_i = 0, \quad w_i = \frac{s_i a_i}{u_i^2} - \frac{p_i}{2} - t d_i \quad (\geq 0), \quad i \in I_u^t.$$

Obviously the KKT optimality conditions (13), (14), (15), (16), (19) are fulfilled, and the KKT conditions (17)-(18) are fulfilled as well in accordance with the assumption of Theorem 1 that $x^* \in X$.

(2) Let $t = 0$. Therefore (22) with $t = 0$ implies $p_i (x_i^*)^2 - 2 s_i a_i = 0$. Set:

$$t = \frac{s_i a_i}{d_i (x_i^*)^2} - \frac{p_i}{2d_i} \quad (= 0), \quad v_i = w_i = 0, \quad i \in I^{t=0};$$

$$v_i = -\frac{s_i a_i}{l_i^2} + \frac{p_i}{2} + t d_i = -\frac{s_i a_i}{l_i^2} + \frac{p_i}{2} \quad (\geq 0), \quad w_i = 0, \quad i \in I_l^{t=0};$$

$$v_i = 0, \quad w_i = \frac{s_i a_i}{u_i^2} - \frac{p_i}{2} - t d_i = \frac{s_i a_i}{u_i^2} - \frac{p_i}{2} \quad (\geq 0), \quad i \in I_u^{t=0}.$$

Evidently the KKT optimality conditions (13), (14), (15), (19) are fulfilled, the KKT conditions (17)-(18) are fulfilled as well in accordance with the assumption of Theorem 1 that $x^* \in X$, and the KKT condition (16) evidently is satisfied with $t = 0$.

From cases (1) and (2) it follows that the KKT sufficient optimality conditions (13)-(19) for problem (CBV) are satisfied. Therefore x^* is an optimal solution to problem (CBV). This optimal solution is unique due to the strict convexity of the objective function $c(x)$.

3. The Algorithm

The KKT optimality condition (16), with the use of expressions (20), (21), and (22) of the statement of characterization Theorem 1, can be written as follows.

$$t\left(\sum_{i\in I_l^t} d_i l_i + \sum_{i\in I_u^t} d_i u_i + \sum_{i\in I^t} d_i \sqrt{\frac{2s_i a_i}{p_i + 2t d_i}} - b\right) = 0, \quad t \in \mathbb{R}_+^1. \qquad (23)$$

Define

$$r(t) \stackrel{\text{def}}{=} \sum_{i\in I_l^t} d_i l_i + \sum_{i\in I_u^t} d_i u_i + \sum_{i\in I^t} d_i \sqrt{\frac{2s_i a_i}{p_i + 2t d_i}} - b. \qquad (24)$$

Introduce the following notation for iteration No. k of algorithm performance: $t^{(k)}$ is value of the dual variable t for the inequality constraint (9), $b^{(k)}$ is the right-hand side of the inequality constraint (9), $I^{(k)}$, $I_l^{t(k)}$, $I_u^{t(k)}$, and $I^{t(k)}$, respectively, are the current index sets I, I_l^t, I_u^t, and I^t.

Algorithm 1 for problem (CBV) is based on the characterization Theorem 1.

Algorithm 1

0. (Initialization) $n^{(0)} := n$, $k := 0$, $I := \{1, \dots, n\}$, $I^{(0)} := I$, $I_l^t := \emptyset$, $I_u^t := \emptyset$, $b^{(0)} := b$.
   If $\sum_{i\in I} d_i l_i \leq b$, to Step 1, else to Step 9.
1. Construct the initial index sets I_l^0, I_u^0, I^0 with $t = 0$. Calculate the value

   $$r(0) := \sum_{i\in I_l^0} d_i l_i + \sum_{i\in I_u^0} d_i u_i + \sum_{i\in I^0} d_i \sqrt{\frac{2s_i a_i}{p_i}} - b,$$

   using expression (24) with $t = 0$.
   If $r(0) \leq 0$, then $t := 0$, to Step 8
   else if $r(0) > 0$, then:
   if $b \leq \sum_{i\in I} d_i u_i$, then to Step 2

else if $b > \sum_{i \in I} d_i u_i$, then to Step 9.

2. Set $I^{t(k)} := I^{(k)}$. Determine $t^{(k)}$ via the closed form expression for t. To Step 3.
3. Construct the current index sets $I_l^{t(k)}$, $I_u^{t(k)}$, $I^{t(k)}$ via (20), (21), (22) (with $i \in I^{(k)}$) and find $|I_l^{t(k)}|$, $|I_u^{t(k)}|$, and $|I^{t(k)}|$.
   To Step 4.
4. Compute the value
   $$r(t^{(k)}) := \sum_{i \in I_l^{t(k)}} d_i l_i + \sum_{i \in I_u^{t(k)}} d_i u_i + \sum_{i \in I^{t(k)}} d_i \sqrt{\frac{2 s_i a_i}{p_i + 2t^{(k)} d_i}} - b^{(k)}.$$
   To Step 5.
5. If $r(t^{(k)}) = 0$ or $I^{t(k)} = \emptyset$, then:
   $t := t^{(k)}$, $I_l^t := I_l^t \cup I_l^{t(k)}$, $I_u^t := I_u^t \cup I_u^{t(k)}$, $I^t := I^{t(k)}$, to Step 8
   else if $r(t^{(k)}) > 0$, then to Step 6
   else if $r(t^{(k)}) < 0$, then to Step 7.
6. $x_i^* := l_i$ for $i \in I_l^{t(k)}$; $I^{(k+1)} := I^{(k)} \setminus I_l^{t(k)}$; and $b^{(k+1)} := b^{(k)} - \sum_{i \in I_l^{t(k)}} d_i l_i$;
   $n^{(k+1)} := n^{(k)} - |I_l^{t(k)}|$; $I_l^t := I_l^t \cup I_l^{t(k)}$; $k := k + 1$. To Step 2.
7. $x_i^* := u_i$ for $i \in I_u^{t(k)}$; $I^{(k+1)} := I^{(k)} \setminus I_u^{t(k)}$; and $b^{(k+1)} := b^{(k)} - \sum_{i \in I_u^{t(k)}} d_i u_i$;
   $n^{(k+1)} := n^{(k)} - |I_u^{t(k)}|$; $I_u^t := I_u^t \cup I_u^{t(k)}$; $k := k + 1$. To Step 2.
8. $x_i^* := l_i$ for $i \in I_l^t$, $x_i^* := u_i$ for $i \in I_u^t$, $x_i^* := \sqrt{\frac{2 s_i a_i}{p_i + 2t d_i}}$ for $i \in I^t$. To Step 10.
9. Problem (CBV) has no optimal solution because the feasible region X is empty or because there is no $t^* > 0$ which satisfies the characterization Theorem 1.
10. End.

Theorem 2 (Convergence of the proposed Algorithm 1)

If $\{t^{(k)}\}$ *is obtained by Algorithm 1, then:*

(i) $r(t^{(k)}) > 0$ *implies* $t^{(k)} \leq t^{(k+1)}$;

(ii) $r(t^{(k)}) < 0$ *implies* $t^{(k)} \geq t^{(k+1)}$.

The proof of the above Theorem 2 is omitted.

The relations (i) and (ii) of Theorem 2, combined with the expressions (20), (21), and (22) of the formulation of Theorem 1, state that the calculation of the optimal value of t and construction of the optimal index sets I_l^t, I_u^t, and I^t, are in accordance with the statement of the characterization Theorem 1.

Analysis of the proposed Algorithm 1 shows that the value of at least one variable of problem (CBV) is calculated at each iteration of the algorithm by solving problem of the type (CBV) with *less* variables at Steps 2 to 7. Hence, Algorithm 1 is convergent for at most $n = |I|$ iterations. Furthermore, since each step of Algorithm 1 takes time, bounded from above by $O(n)$, then Algorithm 1 is a strictly polynomial algorithm with $O(n^2)$ running time.

4. Conclusions

Some inventory control problems are considered in this paper, in particular, box constrained inventory control problems. The optimal solution is described by a characterization theorem, and an effective convergent polynomial algorithm is suggested.

Open problems for future research are to study other inventory control problems and to develop convergent polynomial algorithms for the corresponding optimization problems.

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