

The Marshall-Olkin-Type II Topp-Leone-exponentiated half logistic-G family of distributions with applications

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Abstract

This study introduces a new family of continuous distributions. Statistical properties of the distribution are thoroughly explored, and specific cases within this family are presented. The proposed model is applied to situations where the data indicate extreme skewness. We carried out simulation study to explore the estimation of the model parameters through maximum likelihood technique. Two datasets are analysed to demonstrate the utility of the proposed model in practical application.

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1. Introduction

The idea of developing new generalized and extended distributions remains a significant area in statistics. Several classical distributions are often failing to model some real data sets. Generalized families of continuous distributions has recently drew attention of many researchers due to their versatile in capturing different characteristics underlying the data sets. Generalizing and extending of the usual classical distributions have been broadly studied in several areas including physics, health, insurance, just to state few.

The Marshall-Olkin-G (MO-G) family of distribution (FD) plays a significant role in statistical modeling, particularly in the analysis of data arising from reliability engineering, survival analysis, and biological studies. Introduced by Marshall and Olkin [14], this family represents a pivotal method by incorporating an additional shape parameter to a given baseline distribution. This transformation not only extends the flexibility of the original model but also enables the resulting distribution to capture a broader patterns of data behaviours. This transformation enables greater flexibility in modelling, as it broadens the range of distributional behaviours compared to the original baseline model. The MO-G framework has been effectively employed to describe data exhibiting monotonic and non monotonic rate shapes. The MO-G FD has the cumulative density function (cdf) is given by

$$F_{MO-G}(x; \delta, \vartheta) = 1 - \frac{\delta G(x; \vartheta)}{1 - \delta G(x; \vartheta)}, \quad (1.1)$$

where δ is the shape parameter and $G(x; \vartheta)$ is the baseline model, with parameter vector ϑ . Some work from the Marshall and Olkin transformation in the literature include work by [21], [15], [11], [13], [8], [18] and [19]. Type II Topp-Leone generated FD developed by Elgarhy et al. [10] for which the cdf is

$$F(x; b, \vartheta) = 1 - [1 - G^2(x; \vartheta)]^b, \quad (1.2)$$

with b greater than 0 and ϑ representing parameter vector. Extensions of this model includes, the work by [7], [16], [2], [9], [4] and [3]. Rahardja et al. [17] made contribution to the field of binomial data analysis by addressing the challenges posed by misclassification. Their work showed that maximum likelihood estimation (MLE) can be utilised to advance the capabilities of the developed FD. The method of MLE is a powerful tool for fitting distributions to observed data by optimizing parameters to maximize the likelihood of the observed data. The MLE will be employed to estimate parameters that best align the model with observed data, thus enabling more accurate predictions and inferences.

Our study is inspired by the robustness and modeling flexibility provided by generalizations of the MO-G FD. Thus, we propose the new FD named the Marshall-Olkin-Type II Topp-Leone-Exponentiated Half Logistic-G (MO-TIITL-EHL-G). Motivation for proposing the MO-TIITL-EHL-G FD in practice are

- to develop a distribution that offers improved data fitting capabilities and enhances modeling precision,

- to introduce greater flexibility in kurtosis compared to classical distributions,
- to develop model that can handle various skewness and kurtosis,
- to generate models with increasing, decreasing, upside bathtub and bathtub hazard rate functions.

The following is how the paper is set up: Section 2 presents the proposed model and discusses its statistical properties. MLE for the MO-TIITL-EHL-G FD is discussed in Section 3. Section 4 presents several cases for the introduced model. The outcomes of the Monte Carlo simulations for the MO-TIITL-EHL-LLoG distribution are provided in Section 5. Two real data sets are examined in Section 6. The concluding remarks follow thereafter.

2. The Proposed Model

We apply the Marshall-Olkin transformation [14], as given in (1.1), to develop the MO-TIITL-EHL-G model. The probability density function (pdf) and cdf for MO-TIITL-EHL-G FD are defined as follows

$$F_{MO-TIITL-EHL-G}(x; \alpha, \delta, b, \vartheta) = 1 - \frac{\delta[H(x; \vartheta)]^b}{1 - \delta[H(x; \vartheta)]^b} \quad (2.1)$$

and

$$f_{MO-TIITL-EHL-G}(x; \alpha, \delta, b, \vartheta) = \frac{4\alpha\delta b g(x; \vartheta)[G(x; \vartheta)]^{2\alpha-1}}{(1 + \bar{G}(x; \vartheta))^{2\alpha+1}} ([H(x; \vartheta)])^{b-1} \times [1 - \delta[H(x; \vartheta)]^b]^{-2}, \quad (2.2)$$

for $\alpha, b, \delta > 0$, where $\bar{\delta} = 1 - \delta$, $H(x; \vartheta) = \left(1 - \left[\frac{G(x; \vartheta)}{1 + \bar{G}(x; \vartheta)}\right]^{2\alpha}\right)$, and parameter vector ϑ .

2.1 Density Expansions

This subsection presents the linear representation of the MO-TIITL-EHL-G FD. Thus, for $\delta \in (0, 1)$ it is as follows

$$f(x; \alpha, \delta, b, \vartheta) = \sum_{m,q=0}^{\infty} w_{m,q}^* g_{2\alpha(1+m)+q}(x; \vartheta). \quad (2.3)$$

where $g_{2\alpha(1+m)+q}(x; \vartheta)$ represents the exponentiated-G (expo-G) pdf characterised by the power parameter $2\alpha(m+1) + q$, and the coefficient is

$$w_{m,q}^* = \sum_{j,p,m,q=0}^{\infty} \frac{4\alpha b(-1)^{m+p+q}}{2\alpha(1+m)+q} w_{j,k} \binom{p}{q} \times \binom{-(2\alpha(1+m)+1)}{p} \binom{b(j+1-k)-1}{m}.$$

Eqn (2.4) provides the MO-TIITL-EHL-G FD pdf as a linear combination of expo-G pdf. This formulation makes it possible to compute characteristics of the MO-TIITL-EHL-G FD based on this representation. Furthermore, for δ greater than 1, we get

$$f(x; \alpha, \delta, b, \vartheta) = \sum_{n,q=0}^{\infty} v_{n,q}^* g_{2\alpha(n+1)+q}(x; \vartheta). \quad (2.4)$$

where $g_{2\alpha(n+1)+q}(x; \vartheta)$ denotes expo-G pdf characterized by a power parameter $(2\alpha(n+1) + q)$ and the coefficient is

$$v_{n,q}^* = \sum_{j,p=0}^{\infty} \frac{4ab(-1)^{q+n+p}}{2\alpha(n+1)+q} V_j \binom{b(j+1)-1}{n} \binom{p}{q} \binom{-(2\alpha(n+1)+1)}{p}. \quad (2.5)$$

Various properties of the MO-TIITL-EHL-G FD can be obtained from those properties of the expo-G FD. Visit the appendix for derivations.

2.2 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ denote a random sample of independent and identically distributed (i.i.d.) variables drawn from the distribution specified in Equation (2.2). The pdf of the i^{th} order statistic $X_{i:n}$, for $\delta \in (0,1)$, we get

$$f_{i:n}(x) = \sum_{q,m=0}^{\infty} U_{q,m}^* g_{2\alpha(m+1)+q}(x; \vartheta), \quad (2.6)$$

where

$$\begin{aligned} U_{q,m}^* &= \sum_{l=0}^{n-i} \sum_{k=0}^j \sum_{p=0}^{\infty} U_{j,l,k} \binom{-(2\alpha(m+1)+1)}{p} \binom{b(l-k+i+j)-1}{m} \\ &\times \frac{4ab(-1)^{q+p+m}}{2\alpha(m+1)+q} \binom{p}{q} \end{aligned} \quad (2.7)$$

and $g_{2\alpha(m+1)+q-1}(x; \vartheta)$ corresponds to expo-G pdf characterised by a power parameter $2\alpha(m+1) + q$. Moreover, when δ greater than 1, we get

$$f_{i:n}(x) = \sum_{m,w=0}^{\infty} C_{m,w}^* g_{2\alpha(m+1)+w}(x; \vartheta), \quad (2.8)$$

where $g_{2\alpha(m+1)+w}$ denotes expo-G pdf characterised by power parameter $(2\alpha(m+1) + w)$ and the coefficient is given by

$$\begin{aligned} C_{m,w}^* &= \sum_{j,s=0}^{\infty} \sum_{l=0}^{n-i} \frac{4ab(-1)^{m+s+w}}{2\alpha(m+1)+w} C_{j,l} \binom{b(i+l+j)-1}{m} \binom{s}{w} \\ &\times \binom{-(2\alpha(m+1)+1)}{s}. \end{aligned} \quad (2.9)$$

From Eqn (2.8), the properties of i^{th} can be obtained from those properties of the random variable $g_{2\alpha(m+1)+w}$. Thus, the t^{th} moments of $X_{(i)}$ is given by

$$E(X_{(i)}^t) = \sum_{m,w=0}^{\infty} C_{m,w}^* E(g_{2\alpha(m+1)+w}). \quad (2.10)$$

Refer to the appendix for a detailed derivation of the i^{th} order statistic.

2.3 Rényi Entropy

Given $\delta \in (0,1)$, Rényi entropy [20] is as follows

$$I_R(v) = \ln[\sum_{n,p=0}^{\infty} e_{n,p}^* e^{I_{REG}(1-v)}] \frac{1}{(1-v)}, \quad (2.11)$$

where

$$\begin{aligned} e_{n,p}^* &= \sum_{j,m,w=0}^{\infty} e_j (4ab)^v (-1)^{m+n+w+p} \binom{w}{p} \binom{j}{m} \\ &\times \binom{-(2\alpha(v+n)+v)}{w} \binom{b(v+m)-v}{n} \frac{1}{\left(\frac{2\alpha(v+n)+p}{v}\right)^v} \end{aligned} \quad (2.12)$$

and I_{REG} corresponds to the Rényi entropy of the expo-G pdf defined by the power parameter $((\frac{2\alpha(\nu+n)+p}{\nu}))$. Also, when $\delta > 1$, we have

$$I_R(\nu) = \ln[\sum_{m,p=0}^{\infty} h_{m,p}^* e^{I_{REG}(1-\nu)}] \frac{1}{(1-\nu)}, \quad (2.13)$$

where

$$h_{m,p}^* = \sum_{j,w=0}^{\infty} (4\alpha b)^{\nu} (-1)^{m+w+p} \binom{W}{p} \binom{b(\nu+j)-\nu}{p} \\ \times \binom{-(2\alpha(\nu+m)+\nu)}{w} \frac{1}{\left(\left(\frac{2\alpha(\nu+m)+p}{\nu}+1\right)\right)^{\nu}}$$

and I_{REG} denotes Rényi entropy of the expo-G pdf with power parameter $\left(\frac{2\alpha(\nu+m)+p}{\nu}\right)$. Access the appendix for derivations.

2.4 Moment Generating Function

For $\delta \in (0,1)$, the mgf for X is expressed as:

$$M_X(t) = \sum_{m,q=0}^{\infty} w_{m,q}^* E(e^{tW(2\alpha(m+1)+q)}),$$

where $E(e^{tW(2\alpha(m+1)+q)})$ represents the mgf for the expo-G pdf with power parameter $((2\alpha(m+1)+q))$. Given $\delta > 1$,

$$M_X(t) = \sum_{n,q=0}^{\infty} v_{n,q}^* E(e^{tW(2\alpha(n+1)+q)}),$$

where $E(e^{tW(2\alpha(n+1)+q)})$ denotes the expo-G mgf governed by power parameter $(2\alpha(1+n)+q)$.

2.5 Quantile Function

The function that calculates quantiles for the MO-TIITL-EHL-G FD is given as

$$x(t) = G^{-1} \left[\frac{2 \left(1 - \left(\frac{1-t}{\delta + \delta(1-t)} \right)^{\frac{1}{b}} \right)^{\frac{1}{2\alpha}}}{1 + \left(1 - \left(\frac{1-t}{\delta + \delta(1-t)} \right)^{\frac{1}{b}} \right)^{\frac{1}{2\alpha}}} \right]. \quad (2.14)$$

If we specify the baseline cdf G , random numbers from the MO-TIITL-EHL-G FD can be generated using equation (2.14). Software like R, SAS or MATLAB can be utilised to solve the equation. Visit appendix for derivations.

3. Estimation

Suppose X_i : MO – TIITL – EHL – G($b, \alpha, \delta; \vartheta$) denote a rv following the MO-TIITL-EHL-G model, parameterized by the vector $\Delta = (b, \alpha, \delta; \vartheta)^T$. Then, the total log-likelihood function, denoted by $\ell = \ell(\Delta)$, is expressed as:

$$\begin{aligned}
\ell = & n \log(4b\alpha\delta) + \sum_{i=1}^n \log[g(x_i; \vartheta)] + (2\alpha - 1) \sum_{i=1}^n \log[G(x_i; \vartheta)] \\
& + (2\alpha + 1) \times \sum_{i=1}^n \log[1 + \bar{G}(x_i; \vartheta)] \\
& + (b - 1) \sum_{i=1}^n \log \left[1 - \left(\frac{G(x_i; \vartheta)}{1 + \bar{G}(x_i; \vartheta)} \right)^{2\alpha} \right] \\
& - 2 \sum_{i=1}^n \log \left[1 - \delta \left(1 - \left(\frac{G(x_i; \vartheta)}{1 + \bar{G}(x_i; \vartheta)} \right)^{2\alpha} \right) \right].
\end{aligned} \tag{3.1}$$

Consequently, the MLEs of the parameters b, α, δ , and ϑ are obtained by maximizing the log-likelihood function given in equation (3.1). To perform this optimization, we employed iterative numerical methods available in the R software, based on the specified baseline cdf $G(x; \vartheta)$.

4. Special Cases

We introduce several specific cases for the MO-TIITL-EHL-G distribution. These are constructed using baseline distributions such as log-logistic, Weibull, Uniform, and Kumaraswamy. Thus, we have the MO-TIITL-EHL-LLoG, MO-TIITL-EHL-W, MO-TIITL-EHL-U, and MO-TIITL-EHL-K distributions, respectively.

4.1 MO-TIITL-EHL-LLoG Distribution

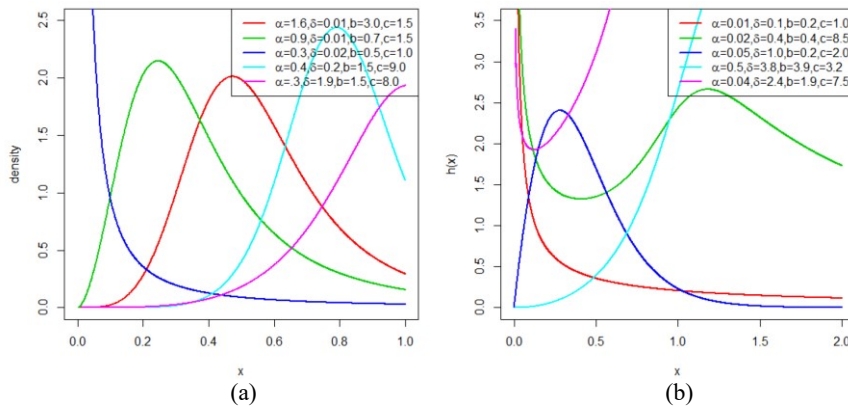


Figure 1
MO-TIITL-EHL-LLoG Distribution: pdf and hrf Plots

From Figures 1(a) and 1(b) the pdf demonstrates a wide range of shapes, such as left- and right-skewed and reverse-J. The corresponding hazard rate functions exhibit diverse behaviors such as monotonic and non monotonic patterns. Additionally, Figures 2(a) & 2(b) graphss reveal that by fixing certain parameters, the distribution is capable of producing various combinations of skewness and kurtosis, thereby demonstrating its flexibility in modeling data with a wide range of distributional characteristics.

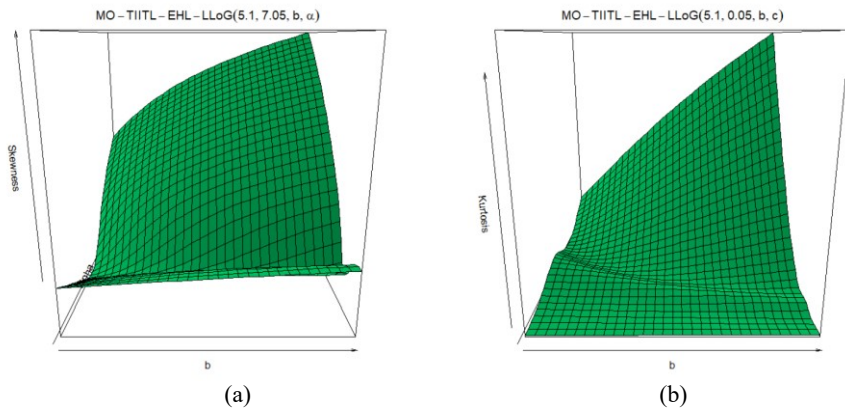


Figure 2
MO-TIITL-EHL-LLoG: Skewness and Kurtosis Plots

4.2 MO-TIITL-EHL-Weibull Distribution

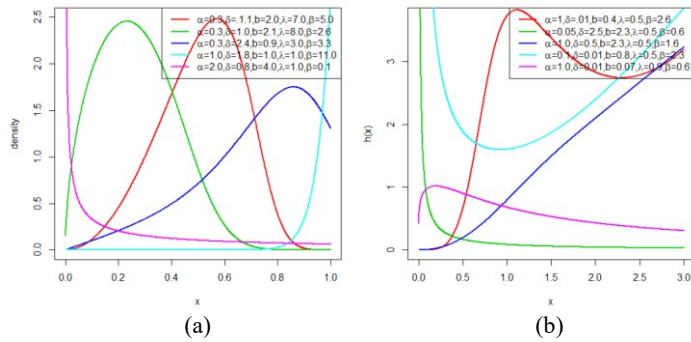


Figure 3
MO-TIITL-EHL-W Distribution: pdf and hrf Plots

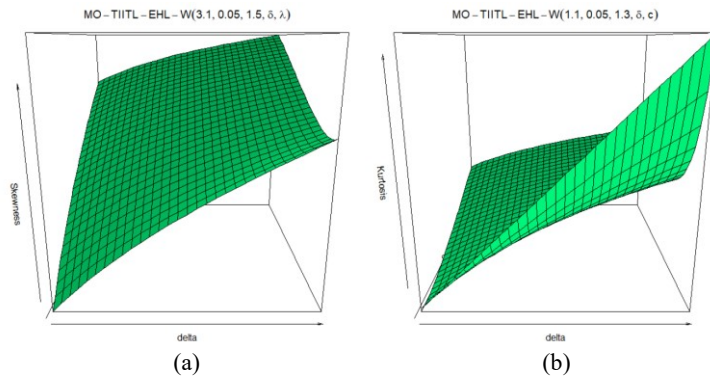


Figure 4
MO-TIITL-EHL-W Distribution: Skewness and Kurtosis Plots

Figures 3(a) and 3(b) display the plots of the MO-TIITL-EHL-W distribution. The pdf demonstrates considerable flexibility, taking shapes such as J-shaped, reverse-J, nearly symmetric, as well as left- and right-skewed. The corresponding hrfs exhibit shapes such as increasing trends and various bathtub-shaped forms. Furthermore, Figures 4(a) and 4(b) present 3D surface plots for skewness and kurtosis of the MO-TIITL-EHL-W distribution. These plots show that, for fixed values of certain parameters, the distribution produces a wide variety of skewness and kurtosis profiles.

4.3 MO-TIITL-EHL-Uniform Distribution

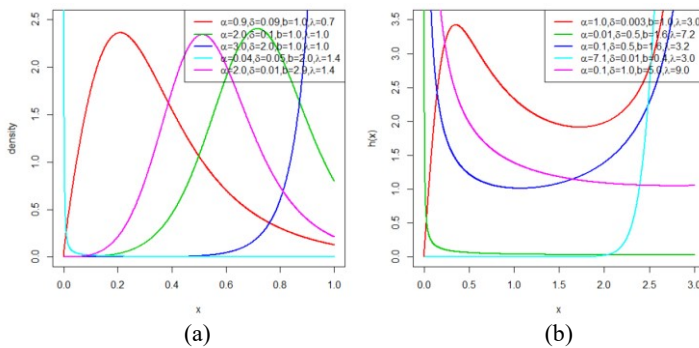


Figure 5
MO-TIITL-EHL-U Distribution: pdf and hrf Plots

Figures 5(a) and 5(b) depict the behaviour of the plots associated with the MO-TIITL-EHL-U distribution under various parameter selections. The density curves reveal diverse structural shapes, ranging from reverse-J shaped to nearly symmetric, as well as exhibiting left or right tail shape. Notably, the corresponding hazard functions are able to accommodate several shapes, underscoring the model's versatility in describing a broad array of data patterns.

4.4 MO-TIITL-EHL-Kumaraswamy Distribution

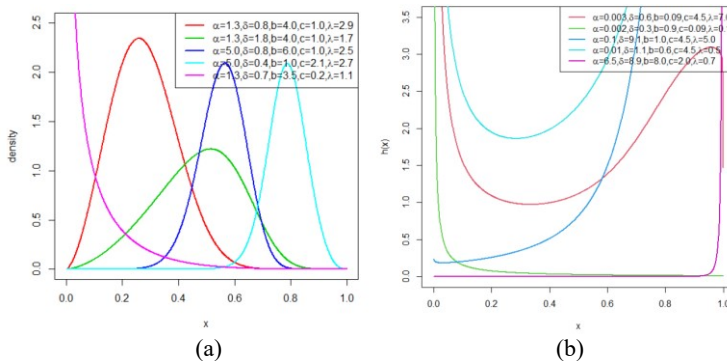


Figure 6
MO-TIITL-EHL-K distribution: pdf and hrf graphs

The MO-TIITL-EHL-K distribution can have left-skewed, right-skewed, or nearly symmetrical features for the pdf, and its hrf can have both monotonic and non-monotonic forms. The plots are shown in Figures 6 (a) and 6 (b).

5. Simulation Study

Simulation study was done using the R software to assess the performance of the MO-TIITL-EHL-LLoG distribution. The outcomes of the simulations are presented in Table 1. Our simulations encompassed $N = 1000$ independent samples, each drawn from the MO-TIITL-EHL-LLoG distribution, with sample sizes $n = 50, 100, 200, 400$ and 800 . To evaluate the performance of the estimated parameters, we employed the average bias (AvBias) and root mean square error (RMSe) metrics.

Table 1
Simulation Results

	I	$\alpha = 1.5, \delta = 0.5, \equiv 0.5, c = 1.0$			II	$\alpha = 2.0, \delta = 0.5, \equiv 0.5, c = 0.5$	
Parameter	n	mean	RMSe	AvBias	mean	RMSe	AvBias
α	50	1.419570	0.670155	-0.080431	2.224823	1.385724	0.224823
	100	1.442207	0.567731	-0.057793	2.045567	0.642051	0.045567
	200	1.406313	0.528609	-0.093687	1.887189	0.483078	-0.112811
	400	1.443140	0.438139	-0.056860	1.886206	0.364585	-0.113794
	800	1.466433	0.338314	-0.033567	1.885156	0.282245	-0.114844
δ	50	1.302918	1.756655	0.802918	1.421010	2.095473	0.921010
	100	0.980116	1.070045	0.480116	1.060907	1.218317	0.560907
	200	0.911810	0.926651	0.411810	1.027745	1.045322	0.527745
	400	0.746231	0.643892	0.246231	0.859386	0.679996	0.359386
	800	0.637454	0.434248	0.137454	0.738887	0.456150	0.238887
b	50	0.881232	3.754679	0.381232	0.765625	3.139030	0.265625
	100	0.469118	0.284681	-0.030882	0.499639	0.415571	-0.000361
	200	0.468525	0.209201	-0.031475	0.481098	0.173105	-0.018902
	400	0.471689	0.139246	-0.028311	0.484600	0.120710	-0.015400
	800	0.487893	0.103278	-0.012107	0.501676	0.085858	0.001676
c	50	1.593470	1.092593	0.593470	0.812913	0.573552	0.312912
	100	1.393144	0.832117	0.393144	0.691281	0.394491	0.191280
	200	1.300325	0.659729	0.300325	0.651887	0.301328	0.151887
	400	1.180381	0.474583	0.180381	0.602428	0.208020	0.102428
	800	1.094440	0.321771	0.094440	0.560087	0.131056	0.060087
	III	$\alpha = 1.0, \delta = 0.5, \equiv 0.5, c = 0.5$			IV	$\alpha = 2.0, \delta = 2.0, \equiv 0.5, c = 0.5$	
Parameter	n	mean	RMSe	AvBias	mean	RMSe	AvBias
α	50	1.045947	0.631911	0.045947	3.263559	3.493692	1.263559
	100	0.993614	0.573936	-0.006386	2.553344	1.656418	0.553344
	200	0.960411	0.515963	-0.039589	2.061112	0.844767	0.061112
	400	0.922963	0.492763	-0.077037	2.014795	0.422616	0.014795
	800	0.903582	0.409759	-0.096418	1.937300	0.253983	-0.062701

δ	50	0.851753	0.937623	0.351753	3.508439	3.160846	1.508439
	100	0.831202	0.772511	0.331202	3.160785	2.687719	1.160785
	200	0.791150	0.644562	0.291150	3.654924	2.747743	1.654924
	400	0.796151	0.615247	0.296151	3.057332	1.883003	1.057332
	800	0.742213	0.489334	0.242213	2.934108	1.251423	0.934107
b	50	1.407241	7.630477	0.907241	0.665165	1.100236	0.165165
	100	0.867657	7.127432	0.367657	0.656359	1.596407	0.156358
	200	0.542406	0.356175	0.042406	0.475769	0.192811	-0.024231
	400	0.494200	0.220644	-0.005800	0.499682	0.139908	-0.000318
	800	0.487894	0.173251	-0.012106	0.513024	0.100949	0.013024
c	50	0.681936	0.455663	0.181936	0.746199	0.465893	0.246199
	100	0.669619	0.418316	0.169619	0.636484	0.335215	0.136484
	200	0.640936	0.347315	0.140936	0.670942	0.301333	0.170942
	400	0.641657	0.316219	0.141657	0.593192	0.197262	0.093192
	800	0.614217	0.255443	0.114217	0.562853	0.111427	0.062853

6. Applications

The usefulness of the proposed model is shown by fitting its special case MO-TIITL-EHL-LLoG model to two real datasets. We presented several selection criteria goodness-of-fit statistics (GoFs), such as Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), and negative 2 log-likelihood ($-2 \log(L)$). We also included other GoFs, as stated by Chen and Balakrishnan [6], specifically Andersen-Darling and Cramér-von Mises.

These statistics serve the purpose of evaluating the GoFs of different models to the provided dataset. Smaller these statistics' values indicate a better fit of the distribution. Moreover, the Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value were employed to further assess GoFs.

We employed the R `nlm` function to perform parameter estimation for the models. The resulting estimated parameters, with their respective standard errors, and the GoFs can be found in Tables 2 and 3. In our analysis, we have included a graphical representation of the MO-TIITL-EHL-LLoG distribution, including fitted density, probability plots, estimated cumulative distribution function (ECDF), Kaplan-Meier (K-M) curves, estimated hazard rate function (HRF) plots, and total time on test (TTT) plots. We compared the MO-TIITL-EHL-LLoG distribution with some competitive distributions. The competing models' names and pdfs are in the appendix.

6.1 Waiting Times Data

The study utilised a data set containing the waiting times (in minutes) before service for 100 bank customers. Ghitany, Atieh, and Nadarajah [12] have also analyzed this data. Visit the appendix for the observations.

Table 2 shows that the proposed model gives the least figures of the GoFs, highest p-value for the Kolmogorov-Smirnov (KS) statistic and provides the overall best fit on waiting times data set. As well Figures 7(a) and 7(b), 8(a) and 8(b) and 9(a) and 9(b) displays clearly that the MO-TIITL-EHL-LLoG has better fit and captures the waiting times data set well. The histogram shows that the introduced model can well accommodate the extremely tailed data sets. The K-M and ECDF curves shows that MO-TIITL-EHL-LLoG model performs well. The fitted hrf is exhibiting an increasing shape as it aligns with what the TTT plot indicates.

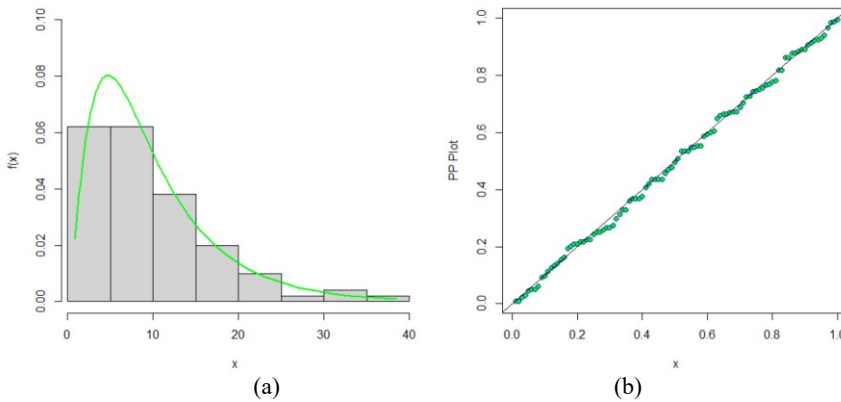


Figure 7
Fitted pdf and PP graphs: waiting times dataset

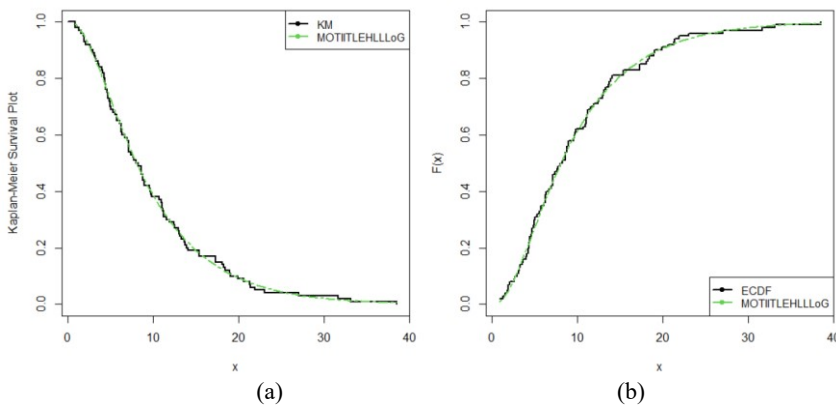


Figure 8
Fitted K-M and CDF graphs: waiting times dataset

Table 3
MLEs and Gof's for several distributions applied to kevlar dataset

Model	MLEs				GoFs							
	α	δ	b	c	$-2\log L$	AIC	$AICc$	BIC	W^*	A^*	KS	$P-value$
MO-TIITL-EHL-LLoG	0.0603	0.9731	0.4475	4.8621	200.1	208.1	208.6	218.6	0.0737	0.5116	0.0691	0.7206
	(0.0307)	(0.7039)	(0.2713)	(2.0660)								
	b	δ	θ	γ								
	1.0922	2.0489	1.0613	0.6613								
MO-TL-HL-W	(0.6325)	(2.0543)	(0.5921)	(0.2560)	203.7	211.7	212.1	222.1	0.1260	0.7663	0.0723	0.6657
	δ	λ	c									
	1.9581	1.9955	0.6932	-								
	(1.9281)	(0.8384)	(0.1701)									
MO-OEHL-LLoG	α	β	δ	c	204.5	212.5	212.9	223.0	0.1350	0.8153	0.0743	0.6326
	0.8575	0.9914	1.1002	0.6497								
	(0.4838)	(0.2497)	(0.5500)	(0.4484)								
	α	β	λ	k								
MOEW	3.9163	1.9955	0.0000	0.6932	203.7	211.7	212.1	222.1	0.1190	0.7369	0.0704	0.6985
	(3.9460)	(1.6095)	(1.1588)	(0.2201)								
	α	b	β	c								
	2.3083	0.7711	0.3558	1.3972								
EOW-TL-LLoG	(1.1203)	(0.3688)	(0.1214)	(0.6383)	204.0	212.0	212.4	222.5	0.1143	0.7241	0.0763	0.5992
	γ	δ	λ	θ								
	8.0942×10^{-9}	1.5212	0.8560	1.2855								
	(0.0217)	(0.9189)	(0.1270)	(0.4307)								
MOGW	α	β	δ	c	204.8	212.8	213.2	223.2	0.1623	0.9359	0.0834	0.4842
	0.8575	0.9914	1.1002	0.6497								
	(0.4838)	(0.2497)	(0.5500)	(0.4484)								
	α	β	λ	k								

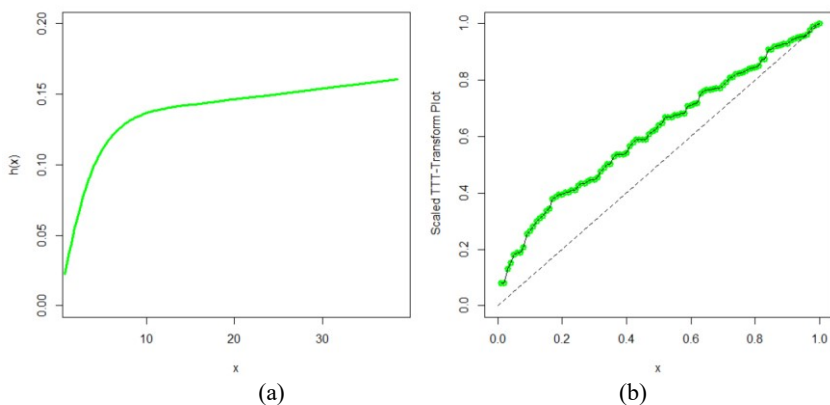


Figure 9
Fitted HRF and TTT graphs: waiting times dataset

6.2 Kevlar Data

The MO-TIITL-EHL-LLoG distribution was applied to the dataset reported by Andrews and Herzberg [1] and Chamunorwa et al. [5]. This dataset comprises time-to-failure, measured in hrs, for Kevlar epoxy composite strands subjected to a constant load corresponding to 90% of their ultimate stress capacity. The complete set of observations utilized in this analysis is provided in the appendix.

Table 3 indicates that the MO-TIITL-EHL-LLoG model provides a better fit among others considered in this article. Figures 10(a) and 10(b), 11(a) and 11(b) and 12(a) and 12(b) provide sufficient information about the closest fit of the MO-TIITL-EHL-LLoG distribution to the data set under consideration. The fitted hrf shows a non-monotonic pattern, characterized by an initial decrease followed by an increase then decrease again.

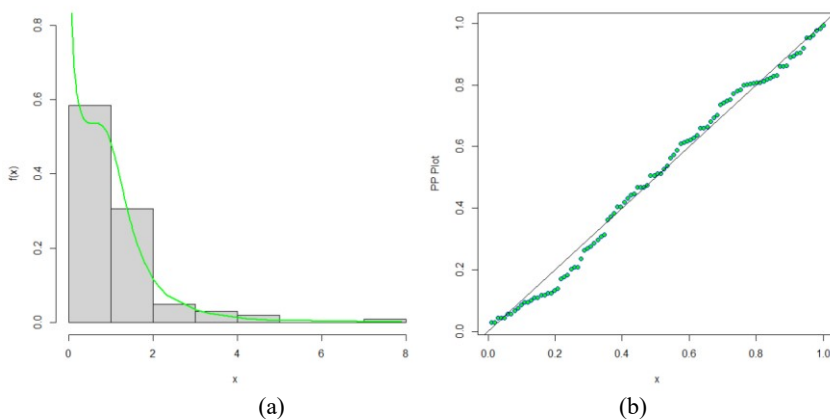


Figure 10
Fitted pdf and PP graphs for kevlar data

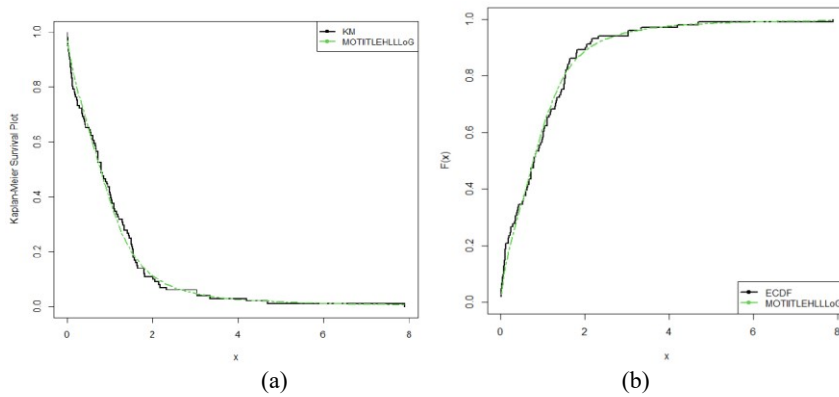


Figure 11
Fitted Kaplan-Meier and CDF plots for kevlar data

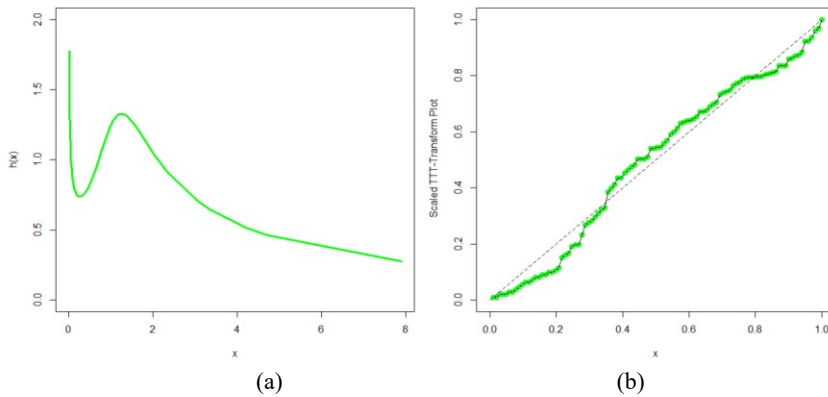


Figure 12
Fitted HRF and TTT graphs: kevlar dataset

7. Conclusion

We introduced a new FD, namely the MO-TIITL-EHL-G FD. Numerous well-established distributions arise as special cases within the MO-TIITL-EHL-G FD. The proposed model is can model heavy-tailed data. Some structural properties of the developed distribution were obtained. To demonstrate its practical utility, two real-world data sets were used. As evidenced in Tables 2 and 3, the model outperformed several competing non-nested alternatives in terms of goodness-of-fit.

Appendix

The URL provided includes detailed derivations of the statistical properties, the components of the score vector, names and pdfs of the competing models.

https://drive.google.com/file/d/1bTdBWdZj4y05NR8FE3xm_01P8REV_JuS/view?usp=sharing

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