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The beta (β_5) distribution of Kind-5

G. S. David Sam Jayakumar *
Jamal Institute of Management
Jamal Mohamed College (Affiliated to Bharathidasan University)
Trichy 620020
Tamil Nadu
India

W. Samuel †
Department of Business Administration
Jamal Mohamed College (Affiliated to Bharathidasan University)
Trichy 620020
Tamil Nadu
India

Abstract

This paper introduces a beta distribution (BD) of Kind-5 with its density function expressed in terms of Gauss hypergeometrric function. Key distributional properties including the cumulative distribution function and the mandatory constants such as central moments, non-central moments, skewness, kurtosis, log-geometric mean, Harmonic mean, Shannon's differential entropy are thoroughly examined. Additionally, the moment, cumulant, characteristic functions are derived. Special cases are explored through transformations of beta kind-5 variable and parameter shifting. The paper also covers parameter estimation using the method of unconstrained maximum likelihood estimation, the construction of Fisher's information matrix and a constrained maximum likelihood approach by using non-linear programming with an illustrative application provided.

Subject Classification: *62H10, 62G07.*

Keywords: *Beta distribution of kind-5, Gauss hyper-geometric function, Central moments, Fisher's information matrix, Constrained maximum likelihood estimation, Sequential quadratic programming.*

* *ORCID-ID:* **0000-0002-6630-8783**
† *ORCID-ID:* **0000-0002-9112-9744**

* *E-mail:* **samjaya77@gmail.com (Corresponding Author)**
† *E-mail:* **wsamuel365@gmail.com;**
wsm@jmc.edu

1. Introduction

The inception of beta distribution had an age before 10 decades ago and it was first identified by the renowned statistician Karl pearson in the year 1857. He introduced the beta distribution, which is a special case of Pearson's Type-I distribution. Today, the distribution is widely used to representing the uncertain magnitude of percentages and proportions defined over the interval [0,1]. By applying an appropriate Jacobian transformation, a beta variable of kind-1 can be transformed into a beta variable of kind-2, whose distribution is known as the beta prime distribution. Apart from this, the distribution is having its application in diverse fields which the random variable is non-negative. Here we show some of the the related reviews of beta prime distribution which shows its application in selected fields. Jambunathan [1] studied and presented two type of beta distribution which is derived from the Gamma distribution and he also discussed its important. Similarly, Mcdonald and Xu [2], Nagar and Gupta [3], Silva et.al [16], Armagan, Clyde, and Dunson [4], Zea et. al [5], Jafari, Tahmasebi, and Alizadeh [6], Insuk, Bodhisuwan, and Jaroengeratikun [7], Saboor, Bakouch, and Khan [8], Loubes, Arias-Serna, and Caro-Lopera [9] introduced a five-parameter beta distribution, matrix variate generalization of Kummer Beta family of distributions, beta generalized exponential distribution, beta modified Weibull distribution introduced the Beta-hyperbolic secant (BHS) distribution, new class of normal scale mixtures through a novel generalized beta distribution. The Beta-Frechet density function, Beta-Exponentiated Pareto distribution, Beta-Gompertz (BG) distribution, Beta-Exponentiated Weibull Poisson (BEWP) distribution, Beta Sarhan–Zaindin Modified Weibull (BSZMW) distribution, and the Alpha-Beta Skew Generalized-t distribution, respectively. Furthermore, Jayakumar, Sulthan, and Samuel [10] proposed a novel form of the bivariate beta distribution of Kind-1, Type-A. The associated density function is formulated using the Generalized Hypergeometric function, and its marginal distributions are univariate beta distributions of Kind-1. Similarly, Jayakumar, Sulthan, and Samuel [11] introduced a new McKay-type bivariate beta distribution of the third Kind, with univariate beta distributions of the third Kind as its marginal. Drawing from these developments and the diverse variants of beta distributions, the authors were motivated to propose a new beta prime distribution, referred to as the Beta Distribution of Kind-5. The characteristics and structural properties of this proposed distribution are thoroughly examined in the subsequent sections.

Beta distribution of Kind-5 and its Constants

Definition 1.1: Let X be the random variable followed BD of kind-5 with two shape parameters (a,b) then its density function is defined as

$$f_X(x;a,b) = (1/2)^{a+b}(B(a,b))^{-1}\frac{x^{a-1}}{(1+x)^{a+b}}\,{}_2F_1(a+b,a+b;a;x/2(1+x)) \qquad (1.1)$$

where $0 \le x < \infty, a,b > 0$ and ${}_2F_1(\,)$ is the Gauss hyper-geometric function respectively.

Result 1.2: Figure 1 illustrates the probability curves of the BD of Kind-5 for selected values of the shape parameters (a, b). The curves exhibit right skewness, and their shapes vary depending on the values of the shape parameters**.**

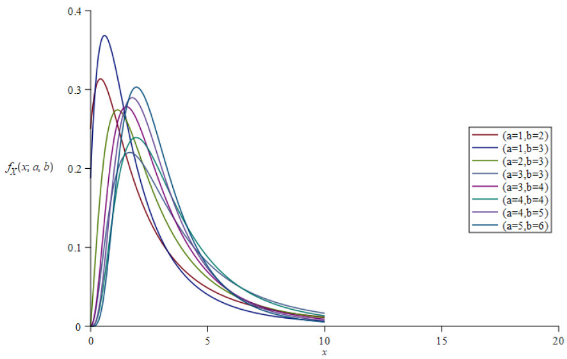

Figure 1

Probability curves of BD of kind-5 for the selected values of shape parameters

Theorem 1.3: *The CDF of BD of kind-5 is*

$$F_X(x;a,b) = \frac{(1/2)^{a+b}}{B(a,b)}\sum_{k=0}^{\infty}\left(\frac{(a+b)_k(a+b)_k}{(a)_k k!}(1/2)^k B(x;a+k,b)\right) \qquad (1.2)$$

Proof: Let the CDF of a distribution is

$$F_X(x;a,b) = \int_0^x f_S(s;a,b)\,ds$$

$$= \frac{(1/2)^{a+b}}{B(a,b)}\int_0^x \frac{s^{a-1}}{(1+s)^{a+b}}\,{}_2F_1(a+b,a+b;a;s/2(1+s))\,ds \qquad (1.3)$$

By expanding the Gauss hypergeometric function in equation (1.3), the CDF can be expressed in its final integral form as follows:

$$F_X(x;a,b)=\frac{(1/2)^{a+b}}{B(a,b)}\sum_{k=0}^{\infty}\left(\frac{(a+b)_k(a+b)_k}{(a)_k k!}(1/2)^k B(x;a+k,b)\right)$$

Where, $B(x;a+k,b)$ is the incomplete beta function.

Theorem 1.4: *The r^{th} non-central moments of the BD is*

$$E_X(x^r)=\frac{(1/2)^{a+b}B(a+r,b)}{B(a,b)}\,{}_3F_2(a+b,a+b,a+r;a,a+b+r;1/2) \qquad (1.4)$$

Proof: The r^{th} non-central moment of the distribution is

$$E_X(x^r)=\frac{(1/2)^{a+b}}{B(a,b)}\int_0^{\infty}\frac{x^{a+r-1}}{(1+x)^{a+b}}\,{}_2F_1(a+b,a+b;a;x/2(1+x))\,dx$$

Now expands the Gauss hyper-geometric function and integrate (1.4), then the r^{th} non-central moment of the distribution is

$$E_X(x^r)=\frac{(1/2)^{a+b}B(a+r,b)}{B(a,b)}\,{}_3F_2(a+b,a+b,a+r;a,a+b+r;1/2) \qquad (1.5)$$

From (1.5), If r =1, 2, 3, 4 then the following order of non-central moments are given as

$$E_X(x)=\frac{a}{a+b}(1/2)^{a+b}\,{}_3F_2(a+b,a+b,a+1;a,a+b+1;1/2),$$

$$E_X(x^2)=\frac{a(a+1)}{(a+b)(a+b+1)}(1/2)^{a+b}\,{}_3F_2(a+b,a+b,a+2;a,a+b+2;1/2),$$

$$E_X(x^3)=\frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)}(1/2)^{a+b}\,{}_3F_2(a+b,a+b,a+3;a,a+b+3;1/2),$$

$$E_X(x^4)=\frac{a(a+1)(a+2)(a+3)}{(a+b)(a+b+1)(a+b+2)(a+b+3)}(1/2)^{a+b}\,{}_3F_2(a+b,a+b,a+4;a,a+b+4;1/2).$$

Theorem 1.5: *The Skewness and Kurtosis of the BD is given as*

$$Skew(X)=\frac{K_3}{K_2^{3/2}} \qquad (1.6)$$

$$Kurtosis(X) = \frac{K_4}{K_2^2} \tag{1.7}$$

Where K_2, K_3, K_4 are the Central moments of the distribution respectively.

Proof: The r^{th} central moments of the distribution is given as

$$K_r = E_X(x - E_X(x))^r \tag{1.8}$$

From (1.8), if r =1, 2, 3, 4 then the following central moments are given as

$$K_1 = E_X(x - E_X(x)) = 0$$

$$K_2 = E_X(x - E_X(x))^2 = E_X(x^2) - (E_X(x))^2$$

$$K_3 = E_X(x - E_X(x))^3 = E_X(x^3) + 4(E_X(x))^3 + 3E_X(x^2)E_X(x)$$

$$K_4 = E_X(x - E_X(x))^4 = 2E_X(x^4) + 7E(x^2) - 8E(x^3)E(x)$$

Then substitute the non-central moments derived from (1.5) and substitute in the derived central moments from (1.8) we get the exact structure of the Central moments and then Substitute these in (1.6) and (1.7), the skewness and Kurtosis are found.

Theorem 1.6: *The log-Geometric mean of the BD is given as*

$$\log G_X = (1/2)^{a+b} \sum_{k=0}^{\infty} \left(\frac{(a+b)_k (\Psi(a+k) - \Psi(a+b+k))}{k!} (1/2)^k \right) \tag{1.9}$$

Where $\Psi(\,)$ is the di-gamma function.

Proof: The log-Geometric mean of a distribution is the expected value of $\log x$ is

$$\log G_X = E_{\log X}(\log x) = \frac{(1/2)^{a+b}}{B(a,b)} \int_0^{\infty} \log x \; \frac{x^{a-1}}{(1+x)^{a+b}} \, {}_2F_1(a+b, a+b; a; x/2(1+x)) dx \tag{1.10}$$

From (1.10) expands the Gauss hyper-geometric function and integrate it, then the result follows on writing the log-geometric mean as

$$\log G_X = (1/2)^{a+b} \sum_{k=0}^{\infty} \left(\frac{(a+b)_k (\Psi(a+k) - \Psi(a+b+k))}{k!} (1/2)^k \right)$$

Theorem 1.7: *The Harmonic mean of the BD is expressed as follows*

$$H_X = \frac{B(a,b)}{(1/2)^{a+b} B(a-1,b)\,{}_3F_2(a+b,a+b,a-1;a,a+b-1;1/2)} \tag{1.11}$$

Proof: The Harmonic mean of a distribution is the inverse of the expected value of $1/x$ is

$$E_{1/X}(1/x) = \frac{(1/2)^{a+b}}{B(a,b)} \int_0^\infty (1/x)\,\frac{x^{a-1}}{(1+x)^{a+b}}\,{}_2F_1(a+b,a+b;a;x/2(1+x))dx \tag{1.12}$$

From (1.12) expands the Gauss hyper-geometric function and integrate it, then the result follows on writing (2.17) is

$$E_{1/X}(1/x) = \frac{(1/2)^{a+b} B(a-1,b)}{B(a,b)}\,{}_3F_2(a+b,a+b,a-1;a,a+b-1;1/2) \tag{1.13}$$

Then the inverse of (1.13) is the harmonic mean of the BD is

$$H_X = \frac{B(a,b)}{(1/2)^{a+b} B(a-1,b)\,{}_3F_2(a+b,a+b,a-1;a,a+b-1;1/2)}$$

Theorem 2.3: *The differential entropy of the BD, in the sense of Shannon, is given by:*

$$h = \sum_{j=1}^{4} \omega_j(a,b) \tag{1.14}$$

where

$$\omega_1(a,b) = -\log((1/2)^{a+b}(B(a,b))^{-1})$$

$$\omega_2(a,b) = -(1/2)^{a+b}(a-1)\sum_{k=0}^{\infty}\left(\frac{(a+b)_k(\Psi(a+k)-\Psi(a+b+k))}{k!}(1/2)^k\right)$$

$$\omega_3(a,b) = -(1/2)^{a+b}(a+b)\sum_{k=0}^{\infty}\left(\frac{(a+b)_k(\Psi(a+b+k)-\Psi(b))}{k!}(1/2)^k\right)$$

$$\omega_4(a,b) = -\frac{(1/2)^{a+b}}{B(a,b)}\int_0^\infty \left(\begin{array}{l}\log({}_2F_1(a+b,a+b;a;x/2(1+x)))\dfrac{x^{a-1}}{(1+x)^{a+b}} \\ \times\ {}_2F_1(a+b,a+b;a;x/2(1+x))\end{array}\right)dx$$

and $\Gamma(\ )$, $\Psi(\ )$ *are the Gamma, Di-gamma functions respectively.*

Proof: It is found from

$$h = -\int_0^\infty f_X(x)\log(f_X(x))\,dx$$

$$= -\int_0^\infty (1/2)^{a+b}(B(a,b))^{-1}\frac{x^{a-1}}{(1+x)^{a+b}}\,{}_2F_1(a+b,a+b;a;x/2(1+x))\log\left(\begin{array}{l}(1/2)^{a+b}(B(a,b))^{-1}\dfrac{x^{a-1}}{(1+x)^{a+b}}\\ \times {}_2F_1(a+b,a+b;a;x/2(1+x))\end{array}\right)dx$$

(1.15)

By integrating (1.15), we will get the final form of Shannon's differential entropy is

$$h = \sum_{j=1}^{4}\omega_j(a,b) \tag{1.16}$$

From (1.16), for the auxiliary function $\omega_4(a,b)$, it is not possible to derive an explicit expression to the integral, due to its non-convergence.

2. Generating Functions

Theorem 2.1: *The Moment generating function (MGF) of the BD is defined as*

$$M_X(t) = \frac{(1/2)^{a+b}\Gamma(a)}{B(a,b)}\sum_{k=0}^{\infty}\frac{(a+b)_k(a+b)_k U(a+k,1-b,-t)}{k!}(1/2)^k \tag{2.1}$$

where $U(\;)$ *is the Kummer's U function.*

Proof: Let the MGF of the distribution be given by

$$M_X(t) = (1/2)^{a+b}(B(a,b))^{-1}\int_0^\infty e^{tx}\frac{x^{a-1}}{(1+x)^{a+b}}\,{}_2F_1(a+b,a+b;a;x/2(1+x))\,dx \tag{2.2}$$

By expanding the Gauss hypergeometric function in equation (2.2), the resulting integral can be expressed as follows:

$$M_X(t) = \frac{(1/2)^{a+b}\Gamma(a)}{B(a,b)}\sum_{k=0}^{\infty}\frac{(a+b)_k(a+b)_k U(a+k,1-b,-t)}{k!}(1/2)^k$$

Theorem 2.2: *The cumulant of the BD is given by:*

$$C_X(t) = \log\left(\frac{(1/2)^{a+b}\Gamma(a)}{B(a,b)}\right) + \log\left(\sum_{k=0}^{\infty}\frac{(a+b)_k(a+b)_k U(a+k,1-b,-t)}{k!}(1/2)^k\right) \tag{2.3}$$

Proof: This is obtained from

$$C_X(t) = \log(M_X(t))$$

Theorem 2.3: *The characteristic function (CF) of the BD is expressed as:*

$$\phi_X(t) = \frac{(1/2)^{a+b}\Gamma(a)}{B(a,b)} \sum_{k=0}^{\infty} \frac{(a+b)_k (a+b)_k U(a+k,1-b,-it)}{k!}(1/2)^k \tag{2.4}$$

where $U(\,)$ *is the Kummer's U function.*

Proof: Let the CF of a distribution is given as

$$\phi_X(t) = \frac{(1/2)^{a+b}}{B(a,b)} \int_0^{\infty} e^{tx} \frac{x^{a-1}}{(1+x)^{a+b}} \, {}_2F_1(a+b,a+b;a;x/2(1+x))\, dx \tag{2.5}$$

For equation (2.5), it follows directly from Theorem 2.1 that the integration is straightforward, and the final expression for the CF is given by:

$$\phi_X(t) = \frac{(1/2)^{a+b}\Gamma(a)}{B(a,b)} \sum_{k=0}^{\infty} \frac{(a+b)_k (a+b)_k U(a+k,1-b,-it)}{k!}(1/2)^k$$

Theorem 2.4: *The survival function of the BD is*

$$S_X(x;a,b) = 1 - \frac{(1/2)^{a+b}}{\Gamma(b)\Gamma(a+b)} \sum_{k=0}^{\infty} \left(\frac{\Gamma(a+b+k)\Gamma(a+b+k)}{\Gamma(a+k)k!} (1/2)^k B(x;a+k,b) \right) \tag{2.6}$$

Proof: It is found from the following fact

$$S_X(x;a,b) = 1 - F_X(x;a,b)$$

Theorem 2.5: *The hazard function of the BD is*

$$h_X(x;a,b) = \frac{(1/2)^{a+b}(B(a,b))^{-1} \dfrac{x^{a-1}}{(1+x)^{a+b}} \, {}_2F_1(a+b,a+b;a;x/2(1+x))}{1 - \dfrac{(1/2)^{a+b}}{\Gamma(b)\Gamma(a+b)} \displaystyle\sum_{k=0}^{\infty} \left(\frac{\Gamma(a+b+k)\Gamma(a+b+k)}{\Gamma(a+k)k!} (1/2)^k B(x;a+k,b) \right)} \tag{2.3}$$

Proof: It is found from

$$h_X(x;a,b) = \frac{f_X(x;a,b)}{S_X(x;a,b)} \text{ and } S_X(x;a,b) = 1 - F_X(x;a,b)$$

Theorem 2.6: *The CHF of the BD is defined as*

$$H_X(x;a,b) = -\log\left(1 - \frac{(1/2)^{a+b}}{\Gamma(b)\Gamma(a+b)}\sum_{k=0}^{\infty}\left(\frac{\Gamma(a+b+k)\Gamma(a+b+k)}{\Gamma(a+k)k!}(1/2)^k B(x;a+k,b)\right)\right) \tag{2.8}$$

Proof: It is derived from the fact

$$\begin{aligned} H_X(x;a,b) &= -\log(1 - F_X(x;a,b)) \\ &= -\log(\mathrm{S}_X(x;a,b)) \end{aligned}$$

Result 2.7: Table-1 shows the special cases of (1.1) for different settings of parameters are given.

Table 1

Special Cases

Case No.	Distribution	Transformation	Parameters	
			a	b
1	Beta distribution of Kind-5	$1/x$	b	a
2	Beta distribution of Kind-4	$x/(1+x)$	a	b
3	F- distribution of Kind-2	mx/n	$m/2$	$n/2$
4	Logistic distribution of kind-2	$\log x$	a	b
5	Fisher's z distribution of Kind-2	$m\log x/2n$	$m/2$	$n/2$
6	Lomax distribution of Kind-2	λx	1	b
7	Burr distribution of Type XII of kind-2	$x^{1/c}$	1	b
8	Dagum distribution of Kind-2	$\lambda x^{1/c}$	1	b
9	Generalized Beta distribution of Kind-5	$\lambda x^{1/c}$	a	b
10	Generalized Beta distribution of Kind-4	$\lambda x^{1/c}/(1+\lambda x^{1/c})$	a	b
11	Log-logistic distribution of Kind-2	$\lambda x^{1/c}$	1	1

3. Parameter Estimation

Result 3.1: As a first approach, we consider the estimation by the method of maximum likelihood. The log-likelihood function for a random sample $x_1, x_2, x_3, \dots, x_{n-1}, x_n$ from (1.1) is

$$\log L(a,b) = \left(\begin{array}{l} n(a+b)\log(1/2) - n\log(B(a,b)) + (a-1)\sum_{i=1}^{n}(\log x_i) - (a+b)\sum_{i=1}^{n}(\log(1+x_i)) \\ +\sum_{i=1}^{n}(\log {}_2F_1(a+b, a+b; a; x_i/2(1+x_i))) \end{array} \right) \tag{3.1}$$

The first order derivatives of (3.1) with respect to two parameters are given by

$$\frac{\partial \log L}{\partial a} = n\log(1/2) - n(\Psi(a) - \Psi(a+b)) + \sum_{i=1}^{n}\left(\log\left(\frac{x_i}{1+x_i}\right)\right) + \sum_{i=1}^{n}\left(\frac{{}_2F_{1(a)}(a+b, a+b; a; x_i/2(1+x_i))}{{}_2F_1(a+b, a+b; a; x_i/2(1+x_i))}\right) \tag{3.2}$$

$$\frac{\partial \log L}{\partial b} = n\log(1/2) + n(\Psi(a+b) - \Psi(b)) - \sum_{i=1}^{n}(\log(1+x_i)) + \sum_{i=1}^{n}\left(\frac{{}_2F_{1(b)}(a+b, a+b; a; x_i/2(1+x_i))}{{}_2F_1(a+b, a+b; a; x_i/2(1+x_i))}\right) \tag{3.3}$$

By setting these expressions to zero and solving them simultaneously, the maximum likelihood estimates of the two parameters are obtained. To construct the Fisher Information Matrix, the second-order and cross-partial derivatives of equation (3.1) are required.

$$\frac{\partial^2 \log L}{\partial a^2} = -n(\Psi(1,a) - \Psi(1,a+b)) + A \tag{3.4}$$

where

$$A = \sum_{i=1}^{n}\left(\frac{{}_2F_1(a+b, a+b; a; x_i/2(1+x_i))\,{}_2F_{1(a^2)}(a+b, a+b; a; x_i/2(1+x_i)) - ({}_2F_{1(a)}(a+b, a+b; a; x_i/2(1+x_i)))^2}{({}_2F_1(a+b, a+b; a; x_i/2(1+x_i)))^2}\right)$$

$$\frac{\partial^2 \log L}{\partial a \partial b} = -n\Psi(1,a+b) + B \tag{3.5}$$

where

$$B=\sum_{i=1}^{n}\left(\frac{{}_2F_1(a+b,a+b;a;x_i/2(1+x_i))\,{}_2F_{1(a^2)}(a+b,a+b;a;x_i/2(1+x_i))-({}_2F_{1(a)}(a+b,a+b;a;x_i/2(1+x_i)))^2}{({}_2F_1(a+b,a+b;a;x_i/2(1+x_i)))^2}\right)$$

$$\frac{\partial^2\log L}{\partial b^2}=n(\Psi(1,a+b)-\Psi(1,b))+C \tag{3.6}$$

where

$$C=\sum_{i=1}^{n}\left(\frac{{}_2F_1(a+b,a+b;a;x_i/2(1+x_i))\,{}_2F_{1(b^2)}(a+b,a+b;a;x_i/2(1+x_i))-({}_2F_{1(b)}(a+b,a+b;a;x_i/2(1+x_i)))^2}{({}_2F_1(a+b,a+b;a;x_i/2(1+x_i)))^2}\right)$$

and

$$\frac{\partial^2\log L}{\partial b\partial a}=n\Psi(1,a+b)+D \tag{3.7}$$

where

$$D=\sum_{i=1}^{n}\left(\frac{\begin{pmatrix}{}_2F_1(a+b,a+b;a;x_i/2(1+x_i))\,{}_2F_{1(b,a)}(a+b,a+b;a;x_i/2(1+x_i))\\-{}_2F_{1(b)}(a+b,a+b;a;x_i/2(1+x_i))\,{}_2F_{1(a)}(a+b,a+b;a;x_i/2(1+x_i))\end{pmatrix}}{\left({}_2F_1(a+b,a+b;a;x_i/2(1+x_i))\right)^2}\right)$$

one can exhibit the elements of the Fisher Information matrix are

$$E\left(-\frac{\partial^2\log L}{\partial a^2}\right)=n((\Psi(1,a)-\Psi(1,a+b))-E(A)) \tag{3.8}$$

$$E\left(-\frac{\partial^2\log L}{\partial a\partial b}\right)=n(\Psi(1,a+b)-E(B)) \tag{3.9}$$

$$E\left(-\frac{\partial^2\log L}{\partial b^2}\right)=-(n(\Psi(1,a+b)-\Psi(1,b))+E(C)) \tag{3.10}$$

and

$$E\left(-\frac{\partial^2\log L}{\partial b\partial a}\right)=-n(\Psi(1,a+b)+E(D)) \tag{3.11}$$

where $\Psi(\,),\Psi(1,(\,))$ and ${}_2F_1(\,)$ are the di-gamma, Poly-Gamma and Gauss-hyper geometric functions respectively.

Result 2.2: As a second approach, the authors realized that the computational complexity of the maximum likelihood estimation of the proposed distribution is Painstaking due to the high non-linearity of parameters and the involvement of special functions. Hence, by adopting Constrained Maximum likelihood method to estimate the parameters of the Beta of distribution of Kind-5 and the idea is to maximize the log-likelihood function from (4.1) under some restriction and parameter constraints and it is given as follows:

$$Max(\log L(a,b))$$
$$=\left(\begin{array}{l} n(a+b)\log(1/2)-n\log(B(a,b))+(a-1)\sum_{i=1}^{n}(\log x_i)-(a+b)\sum_{i=1}^{n}(\log(1+x_i)) \\ +\sum_{i=1}^{n}(\log {}_2F_1(a+b,a+b;a;x_i/2(1+x_i))) \end{array}\right) \tag{3.12}$$

Subjct to the constraints

$$\left(\begin{array}{l} n(a+b)\log(1/2)-n\log(B(a,b))+(a-1)\sum_{i=1}^{n}(\log x_i)-(a+b)\sum_{i=1}^{n}(\log(1+x_i)) \\ +\sum_{i=1}^{n}(\log {}_2F_1(a+b,a+b;a;x_i/2(1+x_i))) \end{array}\right)\le 0$$

where $a,b\ge 0$

4. Application

In this section, the authors demonstrate the application of the BD of Kind-5 by fitting it to two datasets. The first dataset comprises four variables—Sepal Length, Sepal Width, Petal Length, and Petal Width (measured in centimeters) from Iris plant species: Setosa, Versicolour, and Virginica. The second dataset includes six variables: Plasma Glucose (plasma glucose concentration two hours after an oral glucose tolerance test), Diastolic Blood Pressure (mm Hg), Triceps Skin Fold Thickness (mm), 2-Hour Serum Insulin (μU/mL), Body Mass Index (weight in kg/m^2), and Diabetes Pedigree Function, based on a randomly selected sample of size (n=30).Variables are collected from the Databases such as **Iris**

Plants data [4] **and Pima Indians Diabetes** [12] respectively. The information regarding the databases are clearly given in the references. The parameters of the distribution are estimated using the method of constrained maximum likelihood estimation, implemented through Sequential Quadratic Programming with the aid of Maple Optimization Routines (version 18). The estimated values, rounded to an appropriate decimal approximation, are presented in Table 2 along with the Probability Curve of fitted density Function displayed in Figures 2–4 and the fitted pdf of BD of Kind-5 are visualized as follows:

Table 2

Parameter estimates of Beta distribution of Kind-5

Database	**Variables**	**Constrained Maximum Likelihood estimates**		
		Parameters		**Maximized Log likelihood**
		$\hat{a}$	$\hat{b}$	
Iris Plants	Sepal length*	-	-	**-**
	Sepal Width*	-	-	**-**
	Petal Length	12.235	51.749	**-0.00004728143**
	Petal Width*	-	-	**-**
Pima Indians Diabetes	Plasma glucose*	-	-	-
	Diastolic BP*	-	-	-
	Skin fold thickness	115.472	8.6883	**-112.9310**
	2-Hour serum insulin	93.620	1.8282	**-184.4285**
	Body mass index*	-	-	-
	Diabetes pedigree*	-	-	-

* Optimal Solutions not attained

Probability curve of fitted density function

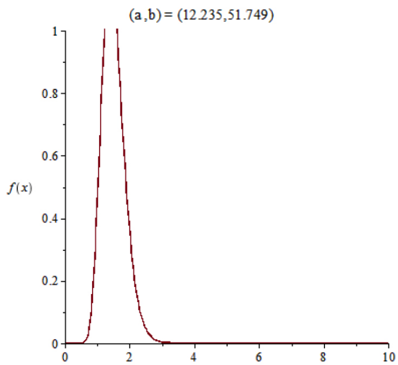

Figure 2

Probability curve of fitted density function of Petal length

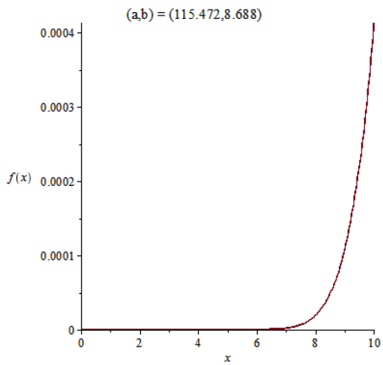

Figure 3

Probability curve of fitted density function of skin fold thickness

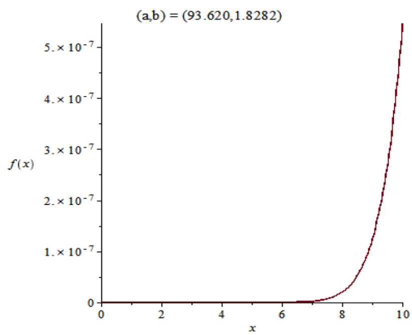

Figure 4

Probability curve of fitted density function of 2 hour serum insulin

5. Conclusion

A new BD of Kind-5 is introduced which it represents an alternative form of the Beta Prime distribution. The features of the distribution were thoroughly examined, supported by a numerical example. An important aspect of this distribution is its potential for similar applications as the Beta Prime distribution, making it a suitable proxy in contexts where the Beta Prime model is traditionally employed. Finally, the authors left the computational complexity of parameters in the unconstrained maximum likelihood estimation of the proposed distribution for future work.

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