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New goodness of FIT test for Lindley distribution under cumulative entropy

Marwa KH. Hassan
Department of Mathematics
Faculty of Education
Ain Shams University
Cairo 11566
Egypt

Abstract

The Lindley distribution is a significant statistical distribution due to its superior statistical and mathematical properties compared to the exponential distribution. Consequently, it is utilized in various fields. This paper aims to examine different goodness-of-FIT tests for the Lindley distribution using cumulative entropy. The classical estimation method is employed to estimate the distribution parameters. A Monte Carlo simulation study is conducted to analyze the power of the tests across different distributions and sample sizes. To facilitate this comparison, we utilized several established goodness-of-FIT tests based on empirical distribution. Finally, a numerical example is present to demonstrate the validity of our test.

Subject Classification: *62G10, 94A17, 65C05.*

Keywords: *: Lindley distribution (LD), Cumulative residual entropy (CRE), Goodness of FIT test (GFT), Power test, Maximum likelihood estimation method (MLE), Monte carlo simulation study (MC).*

1. Introduction

[1] introduced the Lindley distribution, in a Bayesian context as follow

Definition 1.1: A random variable X is the LD if its probability density function (pdf) and its cumulative distribution function (cdf) are given as follow

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)\exp(-\theta x)\; x > 0, \theta > 0. \quad (1)$$

and,

ORCID-ID: 0000-0003-3730-7869

E-mail: marwa_khalil@edu.asu.edu.eg

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$$F(x;\theta) = 1 - (1 + \frac{\theta}{1+\theta}x)\exp(-\theta x). \tag{2}$$

respectively.

[2] studied the Lindley distribution and proved that it outperforms the exponential distribution. The Lindley distribution has applications in various fields, such as lifetime data models with censored data[3], load-sharing system models, [4] and [5], stress-strength models see, [6] and risk computing model [7], [8] and [9]. This study is constructed: In Section 2, entropies are present and GFT tests using different entropies. In Section 3, we derive the cumulative residual entropy for the Lindley distribution and introduce a new goodness-of-FIT test for the Lindley distribution against another distribution using the measure introduced by [10]. In Section 4, we conduct a numerical study to determine the critical value of our proposed test and perform a power analysis to demonstrate that our test is robust. In Section 5, we utilize real data to assess the validity of our test. Section 6, we present the conclusion of our study.

2. Entropies and Goodness of FIT Test

[11] was suggested entropy, which is used in many fields such as communications theory and econometrics.

Definition 2.1: Shannon entropy is defined for a continuous random variables X as

$$H(x) = -\int_0^\infty f(x)\ln(f(x))dx.$$

where $f(x)$ is (pdf) of X.

In information theory, entropy is defined as the average amount of information of data redundancy is excluded. Many authors have generalized the Shannon entropy such as [12], [13], [14] and [15]. [16] used information theory as a tool for statistical inference in many fields, when used the Shannon entropy to construct a GFT for normal distribution. After that, many authors used this tool for different distributions, such as [17], [18], [19], [20], [12], [21] proposed Kullback-Leibler (KL) divergence of Q and P which measure the lost of information when Q is approximate to P.

Definition 2.2: The KL divergence of two continuous cdf F and G is defined as

$$KL(F:) = \int_0^\infty f(x)\ln\left(\frac{f(x)}{g(x)}\right)dx$$

where $f(x)$ and $g(x)$ are pfd of $F(x)$ and $G(x)$ respectively.

The KL divergence test is developed by [22] and [23]. [24] are shown the following two lemma

Lemma 2.1: *From the inequality* $Ln(x) \leq x - 1$, *then* $KL(F:G) \geq 0$.

Lemma 2.2: $\mathrm{F(x)}$ and $\mathrm{G(x)}$ are equally distribution $\Leftrightarrow$ $\mathrm{KL(F:G = 0)}$.

From lemma (1) and lemma (2), if $G(x)$ is Lindly distribution, then the empirical function $F_n(x)$ can be calculated from the data. Hence, $KL(F:G)$ can be used as a fitting test. [25] present a new measure of information is called the cumulative residual entropy (CRE) and is defined as

Definition 2.3: For the random variable X the CRE based on the survival function $\bar{F}(x) = 1 - F(x)$ is defined as $CRE = -\int_0^\infty \bar{F}(x) ln(\bar{F}(x))dx$.

This definition is valid for continuous and discrete random variable. The cumulative residual entropy is used in many fields such as engineering and computer vision see [26]. In reliability field see [27], for more applications of CRE see [28], [29] and [10]. [30] introduced a new measure called the cumulative residual Kullback-Leibler (CKRKL) divergence.

Definition 2.4: The cumulative residual Kullback-Leibler (CRKL) divergence is defined as

$$CRKL(F:G) = \int_0^\infty \bar{F}(x)\left(-\ln\left(\frac{\bar{F}(x)}{\bar{G}(x)}\right)\right)dx - (E(X) - E(Y))$$

where $\bar{F}(x)$ and $\bar{G}(x)$ are survival function, $E(X)$ and $E(Y)$ are the expected value of the random variables X and Y respectively.

It is easy to show that $CRKL \geq 0$. [31] introduced the cumulative entropy (CE) which defined based on Shannon entropy by replace $f(x)$ by $F(x)$ as follows

$$CE(x) = \int_0^\infty F(x)(-ln(F(x)))dx.$$

Park, Rao, and Dong [32] introduced cumulative Kullback-Leibler (CKL) which is defined as

$$CKL(F:G) = \int_0^\infty F(x)\left(-\ln\left(\frac{F(x)}{G(x)}\right)\right)dx - (E(X) - E(Y)).$$

It easy also to prove that $CKL(F:G) \geq 0 \Leftrightarrow F(x) = G(x)$.

3. GFT Based on CKL

We adopted the technique which was introduced by [10] aand used it to construct a GFT for exponential distribution, also [33] used the same technique to construct a goodness of FIT test for Rayleigh distribution. The main aim of our paper is using the same technique for Lindley distribution. To verify this aim, we estimate the Lindley distribution parameter θ using MLE method. So, let $X_1 \ldots X_n$ be a random sample from the Lindley distribution with unknown parameter θ. It is easy to prove that the maximum likelihood estimator for θ which is denoted by $\hat{\theta}$ is

$$\hat{\theta} = \frac{-(\bar{X}-1)+\sqrt{(\bar{X}-1)^2+8\bar{X}}}{2\bar{X}},\ \bar{X} > 0$$

Definition 3.1: Let X be a random variable with Lindley distribution with parameter θ, $E(X)=\frac{\theta+2}{\theta(\theta+1)}$, $E(X^2)=\frac{2(3+\theta)}{\theta^2(1+\theta)}$, and

$$CRE(X)=-\int_0^\infty \bar{F}(x)ln(\bar{F}(x))dx$$
$$=\frac{1}{\theta(\theta+1)}(4+\theta+\exp(1+\theta)\Gamma(\theta,1+\theta)).$$

Now, to construct the cumulative residual entropy test for Lindly distribution. Let, $X_1 \ldots X_n$ be a random sample with an absolutely continuous cumulative distribution function F with ordre statistics $X_{(1)} \ldots X_{(n)}$ with the maximum likelihood estimator $\hat{\theta}=\frac{-(\bar{X}-1)+\sqrt{(\bar{X}-1)^2+8\bar{X}}}{2\bar{X}}$, $\bar{X}>0$. Let $F_0(x;\theta)=1-\frac{\exp(-\theta x)(1+\theta+\theta x)}{(1+\theta)}$, $\theta>0$, $x>0$, denoted the cumulative distribution function of Lindly distribution where θ is unknown parameter. Our motivation to test $H_0: F(x)=F_0(x;\theta)$ against $H_1: F(x)\neq F_0(x;\theta)$ under H_0 $CKL(F:F_0)=0$ otherwise with value of $CKL(F:F_0)$ means, rejection of H_0. the calculation of $CKL(F:F_0)$ needs F and F_0 are known, then

$$CKL(F:F_0)=-CRE(F)-\int_0^\infty \bar{F}(x)ln(\bar{F}_0(x;\theta))dx-E(X)+\frac{2+\theta}{\theta(1+\theta)}$$
$$=-CRE(F)+\frac{\theta}{2}E(X^2)-(1-ln(1+\theta))E(X)+$$
$$\frac{2+\theta}{\theta(1+\theta)}-\int_0^\infty \bar{F}(x)ln(1+\theta+\theta x)dx$$

To get the estimator of $CKL(F:F_0)$ we replace F by the empirical distribution F_n and θ by its maximum likelihood estimator which donated by $\hat{\theta}$ as follows

$$\hat{C}KL(F_n:F_0)=\sum_{i=1}^{n-1}\frac{n-i}{n}ln\left(\frac{n-i}{n}\right)(x_{i+1}-x_i)+$$
$$\frac{\hat{\theta}}{2n}\sum_{i=1}^{n}x_i^2-\frac{(1-ln(1+\hat{\theta}))}{n}\sum_{i=1}^{n}x_i+$$
$$\frac{2+\hat{\theta}}{\hat{\theta}(1+\hat{\theta})}-\sum_{i=1}^{n-1}\frac{n-i}{n}\int_0^{x_n}ln(1+\hat{\theta}+\hat{\theta}x)dx.$$

Now, our proposed test statistic, which is used as a goodness of FIT test is defined as

$$T_n=\frac{\hat{C}KL(F_n:F_0)}{\bar{X}}. \quad (3)$$

4. Numerical Study

The aim of this section, is to calculate the critical values and perform the power analysis of our proposed test statistic. Since, we can't get the distribution of T_n under the null hypothesis, so the MC simulation study is performed with 100000 replicates for different sample sizes from the standard Lindley distribution ($\theta=1$). All calculations for simulation study are made by Mathematica 11. Table 1 shows the critical values $T_{n,0.95}$ and $T_{n,0.99}$. Table 2 shows the results of the statistical error of type I was controlled using $95\%$of our test statistic for Lindley distribution with different values of θ and different sampling sizes.

Table 1

Table 2: Critical Values of Test Statistic T_n

n	$\alpha = 0.01$	$\alpha = 0.01$	n	$\alpha = 0.01$	$\alpha = 0.05$
1	0.8912	0.7315	19	0.1732	0.1222
2	0.7872	0.7032	20	0.1396	0.1038
3	0.6809	0.5628	25	0.1171	0.0912
4	0.5570	0.4090	30	0.1075	0.0826
5	0.4404	0.3353	35	0.1066	0.0804
6	0.4058	0.3039	40	0.0943	0.0727
7	0.3535	0.2572	45	0.0931	0.0722
8	0.3402	0.2521	50	0.0839	0.0687
9	0.2999	0.2323	55	0.0799	0.0655
10	0.2811	0.2179	60	0.0795	0.0637
11	0.2650	0.1950	65	0.0721	0.0620
12	0.2345	0.1779	70	0.0692	0.0609
13	0.2327	0.1749	75	0.0687	0.0589
14	0.2308	0.1646	80	0.0662	0.0577
15	0.2223	0.1591	85	0.0652	0.0565
16	0.2134	0.1519	90	0.0651	0. 0564
17	0.1976	0.1493	95	0.0638	0.0553
18	0.1741	0.1356	100	0.0613	0.0546

Table 2

Table 2: Type I Error Control of T_n Test $\alpha = 0.05$ Using Simulation study with 100000 Replication

θ	$n = 5$	$n = 15$	$n = 25$
$\theta = 2$	0613	0.0538	0.0519
$\theta = 3$	0611	0.0510	0.0508
$\theta = 4$	0601	0.0485	0.0467
$\theta = 5$	0653	0.0458	0.0431

We observe from Table 1 a logic results critical values at 0.01 is lager than critical values at 0.05. Also, critical values are decreasing as n increasing. Table 2 shows that a good Type I error control from our proposed test statistics T_n. Now, we estimate the power of our proposed test statistic T_n versus another distributions. Monte-Carlo simulation study using 100000 samples of size $n = 5,10,15,20,25,30,35,40,45,50$ from alternative distributions are generated. In this study we suggested some continuous alternative distributions such that

1. Weibull distribution with probability density function

$$f(x; \lambda, \beta) = \frac{\beta x^{\beta-1} \exp(-(\frac{x}{\lambda})^{\beta})}{\lambda^{\beta}}, \ \beta > 0, \lambda > 0, x \geq 0.$$

2. Exponential distribution with pdf

$$f(x;\lambda)=\frac{1}{\lambda}\exp(-\lambda x),\ x>0,\lambda>0.$$

3. the Reyleigh distribution with pdf

$$f(x;\sigma)=\frac{x}{\sigma^2}\exp(-\frac{x^2}{2\sigma^2}),\ x>0,\sigma>0$$

4. The Pareto distribution with pdf

$$f(x;\beta)=\beta(1+x)^{-(\beta+1)},\ x\geq 0,\ \beta>0$$

To study the power of our test statistic, we compare between it and another known goodness of FIT test such that

1. Vansoest test: $w=\sum_{i=1}^{n}(F_0(x_{(i)};\alpha)-\frac{2i-1}{2n})^2+\frac{1}{12n}$, for more details see [34].
2. Finkelstrinand and Schafers test :

   $$FS=\sum_{i=1}^{n} max\left(\left|F_0\left(x_{(i)};\alpha\right)-\frac{i}{n}\right|,\left|F_0\left(x_{(i)};\alpha\right)-\frac{i-1}{n}\right|\right),$$

   For more details see [35].
3. Anderson and Darling test:

   $$A=-n-\sum_{i=1}^{n}\frac{2i-1}{n}(ln(F_0(x_{(i)};\alpha)))+ln(1-F_0(x_{(n+1-i)};\alpha)),$$

   for more details see [36].

H_0 is rejected for large values of the previous test statistics. Table 3-6 show the power estimates of 0.01 and 0.05 for different vales of distribution parameters and different sample sizes.

Table 3

Power analysis for the tests T_n, W, FS, A against Weibull at $\alpha=0.01$ and $\alpha=0.05$ with $\lambda=1$ and different values of β

n	β	T_n		W		FS		A	
		$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$
5	2	0.2550	0.3108	0.0391	0.0611	0.2207	0.2615	0.2049	0.3012
	3	0.2878	0.3188	0.0429	0.0596	0.2116	0.2849	0.2269	0.2574
	4	0.2956	0.3191	0.0457	0.0595	0.2322	0.2595	0.2185	0.2817
10	2	0.3487	0.4207	0.1134	0.1617	0.4378	0.4951	0.6237	0.8867
	3	0.3254	0.3722	0.1208	0.1513	0.4415	0.4854	0.6372	0.7391
	4	0.3236	0.3557	0.1265	0.1526	0.4542	0.4864	0.5045	0.7217
15	2	0.3735	0.4343	0.1239	0.1656	0.3859	0.4422	0.3618	0.8867
	3	0.3301	0.4565	0.1304	0.1530	0.3969	0.4348	0.3969	0.7391
	4	0.4180	0.4398	0.1313	0.1418	0.4038	0.4316	0.3085	0.3507
20	2	0.4309	0.4353	0.1834	0.2450	0.5483	0.6067	0.5477	0.7454
	3	0.4664	0.4838	0.1918	0.2176	0.5476	0.5819	0.4797	0.5675

Contd...

	4	0.4847	0.5588	0.1887	0.2100	0.5349	0.5749	0.4452	0.5021
25	2	0.4609	0.5762	0.2642	0.3085	0.6915	0.7475	0.6672	0.8088
	3	0.4214	0.6114	0.2493	0.2848	0.6731	0.7297	0.5937	0.5928
	4	0.5508	0.6865	0.2454	0.2730	0.6817	0.7156	0.4639	0.5348
30	2	0.5072	0.6927	0.3322	0.3874	0.8504	0.9095	0.6566	0.8764
	3	0.5788	0.6193	0.3009	0.3302	0.8184	0.8634	0.4740	0.5383
	4	0.6322	0.7189	0.3182	0.3509	0.8360	0.8883	0.5529	0.6293
35	2	0.6711	0.8536	0.4120	0.4678	0.4978	0.5385	0.7751	0.9478
	3	0.6717	0.8675	0.3773	0.4237	0.4864	0.5159	0.5489	0.6481
	4	0.6037	0.9453	0.3581	0.3875	0.4741	0.5005	0.4866	0.5447
40	2	0.7170	0.9133	0.4750	0.5398	0.5663	0.6124	0.7910	0.9584
	3	0.8246	0.9276	0.4233	0.4865	0.5546	0.5881	0.5753	0.6855
	4	0.8320	0.9802	0.4240	0.4604	0.5454	0.5764	0.5155	1
45	2	0.8103	0.9818	0.2816	0.3048	0.6547	0.6928	0.9233	0.7575
	3	0.8409	0.9898	0.2566	0.2757	0.6231	0.6636	0.6814	0.6443
	4	0.8789	1	0.2367	0.2602	0.6218	0.6477	0.5815	0.5348
50	2	0.8557	0.9247	0.3153	0.3546	0.7329	0.7752	1	1
	3	0.8638	0.9259	0.2798	0.3130	0.7024	0.7374	0.7466	0.8608
	4	0.8884	1	0.2660	0.2903	0.6890	0.7158	0.6540	0.7293

Table 4

Power analysis for the tests T_n, W, FS, A against exponential $\alpha = 0.01$ and $\alpha = 0.05$ with different values of λ

n	λ	T_n		W		FS		A	
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
5	2	0.0624	0.0673	0.0020	0.0043	0.0203	0.0257	0.0133	0.0255
	3	0.0575	0.0623	0.0019	0.0041	0.0199	0.0259	0.0121	0.0245
	4	0.0559	0.0618	0.0019	0.0044	0.0191	0.0260	0.0117	0.0241
10	2	0.1453	0.1628	0.0107	0.0162	0.0468	0.0526	0.0620	0.0929
	3	0.1372	0.1506	0.0110	0.0156	0.0461	0.0533	0.0598	0.0879
	4	0.1333	0.1455	0.0091	0.0150	0.0420	0.0531	0.0525	0.0855
15	2	0.2371	0.2604	0.0213	0.0290	0.0722	0.0840	0.1397	0.1694
	3	0.2187	0.2384	0.0216	0.0289	0.0740	0.0818	0.1287	0.1603
	4	0.2171	0.2346	0.0201	0.0269	0.0684	0.0809	0.1163	0.1533
20	2	0.3366	0.3651	0.0338	0.0442	0.0995	0.1153	0.1949	0.2566
	3	0.3080	0.3348	0.0339	0.0431	0.1017	0.1115	0.2034	0.2517
	4	0.3067	0.3257	0.0326	0.0417	0.0972	0.1119	0.2887	0.3621
25	2	0.4418	0.4760	0.0482	0.0595	0.1286	0.1454	0.2682	0.3390
	3	0.4015	0.4327	0.0476	0.0582	0.1280	0.1443	0.2584	0.3203
	4	0.3854	0.4217	0.0470	0.0558	0.1317	0.1423	0.4639	0.4389
30	2	0.5423	0.5794	0.0612	0.0726	0.1606	0.1750	0.3716	0.4036
	3	0.4882	0.5313	0.0576	0.0709	0.1577	0.1715	0.3253	0.3885
	4	0.4827	0.5147	0.0580	0.0671	0.1541	0.2045	0.4515	0.5300
35	2	0.6366	0.6907	0.0751	0.0879	0.1180	0.2033	0.4151	0.4960
	3	0.5859	0.6327	0.0731	0.0861	0.1855	0.2030	0.4377	0.5001

Contd...

	4	0.5788	0.6146	0.0756	0.0847	0.1855	0.2373	0.5522	0.5447
40	2	0.7529	0.8048	0.0918	0.1047	0.2235	0.2323	0.5222	0.6359
	3	0.6844	0.7385	0.0867	0.0990	0.2121	0.3216	0.4797	0.5935
	4	0.6639	0.7064	0.0840	0.0996	0.2122	0.5764	0.5155	0.5801
45	2	0.8645	0.9110	0.1030	0.1174	0.2498	0.2693	0.6329	0.7145
	3	0.7839	0.8419	0.0946	0.1146	0.2432	0.2644	0.5694	0.6866
	4	0.7661	0.8189	0.0989	0.1127	0.2400	0.2637	0.5768	0.6561
50	2	0.9632	1	0.1178	0.1323	0.7205	0.8087	0.2830	0.3012
	3	0.8832	0.9482	0.1132	0.1313	0.2782	0.2982	0.6677	0.7635
	4	0.8493	0.9079	0.1134	0.1279	0.2756	0.2964	0.6411	0.7425

Table 5

Power analysis for the tests T_n, W, FS, A against Rayleigh at $\alpha = 0.01$ and $\alpha = 0.05$with different values of σ

n	σ	T_n		W		FS		A	
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
5	1	0.8419	1	0.0272	0.0588	0.2284	0.2864	0.1445	0.3018
	1.5	0.6116	0.9936	0.0291	0.0578	0.2303	0.2840	0.1881	0.3183
	2	0.4880	0.8477	0.0348	0.0800	0.2349	0.3218	0.2082	0.5001
10	1	0.2700	0.3926	0.1157	0.1746	0.4641	0.5672	0.5820	0.9304
	1.5	0.4637	0.7305	0.1253	0.1701	0.4706	0.5585	0.7057	0.9583
	2	0.5610	0.9358	0.2292	0.2552	0.5472	0.6384	0.1431	0.7481
15	1	0.2227	0.2931	0.1762	0.1936	0.7259	0.8327	0.4837	0.5628
	1.5	0.4111	0.6303	0.2655	0.3822	0.7130	0.8246	0.5608	0.7047
	2	0.7283	0.8261	0.3515	0.4004	0.8280	0.9613	0.7387	0.8807
20	1	0.2523	0.3152	0.2445	0.2699	0.1007	0.2399	0.8610	0.9135
	1.5	0.5220	0.7747	0.3784	0.4231	0.9874	1	0.9674	1
	2	0.2587	0.3784	0.2536	0.2784	0.0585	0.0656	0.1724	0.2221
25	1	0.8440	1	0.1560	0.1709	0.0665	0.0723	0.1265	0.1559
	1.5	0.1700	0.2470	0.2479	0.2709	0.0657	0.0719	0.1390	0.1694
	2	0.3208	0.4920	0.3243	0.3496	0.0751	0.0824	0.2347	0.2879
30	1	0.0956	0.1345	0.1952	0.2104	0.0779	0.0869	0.1592	0.1949
	1.5	0.2099	0.2500	0.3002	0.3310	0.0841	0.0868	0.1830	0.2126
	2	0.3644	0.5071	0.3935	0.4273	0.0909	0.1002	0.2945	0.3838
35	1	0.1063	0.1414	0.2321	0.2496	0.0934	0.1023	0.1912	0.2382
	1.5	0.2232	0.2922	0.3619	0.3972	0.0948	0.1031	0.2174	0.2678
	2	0.4052	0.5496	0.4780	0.509	0.1075	0.1178	0.3726	0.4674
40	1	0.1157	0.1368	0.2761	0.2902	0.1054	0.1164	0.2133	0.2672
	1.5	0.2544	0.3312	0.4326	0.4608	0.1075	0.1179	0.2583	0.3050
	2	0.4572	0.6111	0.5520	0.5845	0.1250	0.1354	0.4717	0.5512
45	1	0.1209	0.1424	0.3069	0.3320	0.1223	0.1322	0.2761	0.3106
	1.5	0.2601	0.3422	0.4779	0.5196	0.1243	0.1366	0.3095	0.3588
	2	0.5110	0.6567	0.6301	0.6607	0.1411	0.1530	0.5138	0.6273
50	1	0.1382	0.1588	0.3459	0.3731	0.1393	0.1480	0.3023	0.3575
	1.5	0.2833	0.93520	0.5325	0.5752	0.1392	0.1501	0.3452	0.4043
	2	0.5667	0.6981	0.6989	0.7414	0.1600	0.1718	0.5999	0.7121

Table 6

Power analysis for the tests T_n, W, FS, A against Pareto at $\alpha = 0.01$ and $\alpha = 0.05$ with different values of β

n	β	T_n		W		FS		A	
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
5	5	0.0090	0.0099	0.0001	0.0003	0.0010	0.0024	0.0263	0.0368
	7	0.0102	0.0117	0.0002	0.0004	0.0021	0.0026	0.0374	0.0460
	10	0.0122	0.0151	0.0001	0.0038	0.0019	0.0025	0.0499	0.0656
10	5	0.0100	0.0112	0.0009	0.0015	0.0046	0.0052	0.0697	0.0825
	7	0.0118	0.0132	0.0009	0.0015	0.0045	0.0053	0.0923	0.1080
	10	0.0147	0.0171	0.0010	0.0015	0.0044	0.0054	0.1241	0.1499
15	5	0.0106	0.0117	0.0020	0.0027	0.0070	0.0081	0.1130	0.1296
	7	0.0129	0.0142	0.0018	0.0026	0.0067	0.0080	0.1475	0.1747
	10	0.0158	0.0180	0.0020	0.0027	0.0068	0.0080	0.2004	0.2330
20	5	0.0112	0.0123	0.0031	0.0039	0.0096	0.0109	0.1524	0.1796
	7	0.0134	0.0146	0.0032	0.0040	0.0101	0.0111	0.2089	0.2334
	10	0.0166	0.0185	0.0031	0.0040	0.0100	0.0110	0.2817	0.3207
25	5	0.0114	0.0123	0.0046	0.0054	0.0128	0.0140	0.2119	0.2275
	7	0.0135	0.0150	0.0042	0.0053	0.0123	0.0138	0.2621	0.3044
	10	0.0174	0.0190	0.0042	0.0052	0.0122	0.0138	0.3754	0.4124
30	5	0.0116	0.0125	0.0055	0.0066	0.0152	0.0168	0.2456	0.2795
	7	0.0139	0.0152	0.0053	0.0067	0.0153	0.0168	0.3362	0.3758
	10	0.0183	0.0198	0.0054	0.0067	0.0153	0.0170	0.4790	0.5098
35	5	0.0121	0.0129	0.0066	0.0081	0.0181	0.0199	0.3070	0.3383
	7	0.0141	0.0155	0.0071	0.0081	0.0187	0.0199	0.3982	0.4390
	10	0.0180	0.0198	0.0072	0.0083	0.0182	0.0200	0.5478	0.6836
40	5	0.0121	0.0129	0.0082	0.0097	0.0212	0.0232	0.3488	0.3866
	7	0.0144	0.0157	0.0083	0.0096	0.0210	0.0229	0.4545	0.5179
	10	0.0190	0.0202	0.0084	0.0097	0.0211	0.0230	0.6557	0.7027
45	5	0.0122	0.0131	0.0093	0.0111	0.0238	0.0260	0.3977	0.4364
	7	0.0148	0.0159	0.0098	0.0109	0.0240	0.0260	0.5452	0.5869
	10	0.0189	0.0205	0.0096	0.0109	0.0238	0.0260	0.7343	0.8074
50	5	0.0125	0.0130	0.0106	0.0125	0.0270	0.0289	0.4426	0.4864
	7	0.0149	0.0159	0.0107	0.0123	0.0271	0.0287	0.5942	0.6554
	10	0.0194	0.0207	0.0105	0.0122	0.0266	0.0289	0.8456	0.8946

From Table 3-6, we get T_n test more powerful than w, FS, A test in many cases. Also, we get T_n and other test, the power is increasing as well as the sample size is increasing.

5. Real Application:Head and Neck Cancer

Our aim in this section is applying our test statistic on real data to show that the Lindley distribution is a good FIT to this real data. We use the two data sets which represented by [37]. First data set represented the survival time of patients suffusing from Head and Neck cancer and were treated using radiotherapy.Second data set represented the survival time of patients suffusing from Head and Neck cancer and were treated using combined radiotherapy and chemotherapy. The main question, why Lindley distribution, because it has many applications in survival analysis. Table 7 Shows results of applying our goodness of FIT test on two data sets

Table 7

The Results of Goodness of FIT test T_n

Data Set	$\hat{\theta}$	T_n	α	Critical Value From Table 1
First Data Set $n = 58$	0.0088	0.0001	0.01	[0.0795,0.0799]
			0.05	[0.0637,0.0655]
Second Data Set $n = 44$	0.009	0.0003	0.01	[0.0931,0.0943]
			0.05	[0.0722,0.0727]

Table 7 shows that LD is a good FIT for two data sets.

6. Conclusions

In this study, we propose a new GFT for the Lindley distribution based on cumulative residual entropy. Critical values for various sample sizes at $\alpha = 0.01$ and $\alpha = 0.05$ are calculated. Additionally, a power analysis is conducted against other distributions. To compute both the critical values and the power of our test, a MC simulation study is performed with 100,000 replications. Finally, we apply our test statistic to real data to demonstrate that the Lindley distribution is a good FIT for this dataset.

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