Journal of Statistics & Management Systems
ISSN 0972-0510 (Print), ISSN 2169-0014 (Online)
Vol. 29 (2026), No. 2, pp. 151–166
DOI : 10.47974/JSMS-1487

The computation of time-varying long memory Hurst parameter for global stock markets

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Abstract

This research paper focuses on examining the behaviour of stock markets and investigating whether global stocks exhibit the property of long memory. Several methodologies, such as aggregated variance, rescaled-range analysis, and the periodogram, are employed to test this hypothesis. The criteria for determining the presence of long memory involve the point estimation of the Hurst exponent. Beyond point estimation, the return series and volatility series are further analysed using rolling samples to observe the dynamic nature of long memory over the studied period. Our findings suggest that market behaviour evolves over time, demonstrating that a single static Hurst exponent is insufficient to assess the overall efficiency of the market.

Subject Classification: *91B15, 91B84.*

Keywords: *Financial time series, Long memory Hurst exponent, Volatility.*

1. Introduction

Numerous studies have explored whether stock market prices can be predicted using historical data. Early investigations, such as the classical

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studies by Fama [1], primarily examined the short-term dependence of stock returns. Fama later introduced the Efficient Market Hypothesis (EMH), which continues to be one of the most widely accepted frameworks for understanding market behaviour. Bao and Ying [2] studied stock returns by calculating the correlation coefficients between successive returns. Their findings revealed no significant correlations, indicating that stock returns lack linear dependence—a result consistent with the weak form of the EMH.

The random walk model is mathematically rooted in the premise that a market is efficient when prices fully incorporate all available information. This concept, a cornerstone of modern financial theory [1], posits that financial markets are both informationally transparent and aligned with the principles of efficiency. However, prior research [3] has shown that the EMH does not universally hold. Numerous studies have sought to either validate or challenge the hypothesis, yet no definitive consensus has been established [4]. For every piece of evidence supporting the EMH, there exists counter-evidence refuting it. This ongoing debate spans both emerging and developed economies, leaving the question of true market efficiency unresolved.

The Fractal Market Hypothesis (FMH), an extension of the EMH, proposes that time series data of stock market prices exhibit fractal-like properties. These properties are attributed to varying time horizons and information asymmetries among investors. According to the FMH, unexpected market crashes and crises can arise from heightened market uncertainty. More recently, researchers and practitioners have moved their focus to the opposite of short-term dependence—long memory [2]. Long memory describes the correlation structure of a data series across multiple lags. In this context, correlation refers to the relationship between two variables, which can be positive, negative, or neutral (no correlation). When a time series exhibits long memory, it indicates that distant observations over time are related, with autocorrelations decaying gradually. Such series display distinct, non-periodic cyclical patterns is defined as possessing "fractal dimensions" by Mandelbrot [5].

Long memory in asset prices poses a challenge to the weak form of EMH by revealing potential non-linear dependencies in the data structure. This suggests the existence of a predictable component [6] within the series, which could serve as evidence against weak-form efficiency. In this study, we adopts a widely used method for detecting long memory in stock markets, with a particular emphasis on estimating and interpreting the Hurst exponent. We will explore and compare various techniques for

estimating the Hurst parameter, using multiple stock index datasets to illustrate our findings.

Over recent decades, a growing body of literature [7] has investigated long memory stock returns and volatility. The study of asset pricing models and asset returns under long-range dependence has drawn the attention of academics, practitioners, and financial professionals alike. As a key stylized fact, long-range dependence in volatility has been extensively analyzed. This paper seeks to further explore the origins and implications of long memory, which holds a vital place in finance, particularly in areas such as forecasting, trend analysis, and understanding low-frequency variations.

But who discovered the long memory process? And how do these early studies contribute to contemporary research? The concept of long memory [8] emerged over a span of less than three decades during the mid-20th century, sparking numerous debates over its mathematical and philosophical interpretations. Among the prominent contributors, Benoît B. Mandelbrot stands out as a pivotal figure in advancing the theory of long memory processes. His groundbreaking work [5] bridged disciplines such as applied mathematics and economics, utilizing stochastic processes to explain phenomena like heavy tails and long memory. Mandelbrot was the first to distinguish between these two effects [8], leaving an enduring legacy that continues to shape modern financial research.

Mandelbrot's development of long memory theory was deeply influenced by the pioneering work of Hurst [9]. Interestingly, the concept of long-range dependence did not originate within mathematics but was first introduced by Hurst, a British hydrologist who worked extensively in Egypt from 1906 to 1968. Hurst's [9] research on determining the long-term storage capacity of reservoirs—specifically, his analysis of fluctuations in the Nile River's water levels—revealed the existence of long memory in hydrology. In his study of river flows, Hurst meticulously examined a statistical measure of cumulative river flow over time, which is now widely recognized as the "adjusted range" in R programming. His findings laid the groundwork for later research into long memory processes, extending their relevance beyond hydrology to other fields.

In modern finance, substantial evidence supports the presence of long memory in various financial assets and in volatility [7]. This paper seeks to further investigate the claim within financial literature that markets are becoming unpredictable over time, using the Hurst exponent as a tool to analyze and quantify long memory in financial markets. Empirical studies offer strong support for long memory in financial time

series. For instance, Flemming and Kirby [10] identified long memory patterns in financial data. Additionally, Bollerslev and Jubinski [11] examined the extent to which volatility and trading volume exhibit a common fractional integration order, based on evidence of long memory in volatility. Using highly precise estimates from high-frequency returns, they concluded that both volume and volatility display long memory and that changes in volatility and volume are strongly correlated.

2. Methodology

Long memory in financial time series is characterized by slowly decaying autocorrelations, which signify that observations separated by long time intervals remain correlated. This phenomenon is prevalent across numerous natural and artificial processes, such as climate change, financial markets, and network traffic. In this research, we analyse long memory in the stock market using three widely adopted methods: aggregated variance, rescaled range analysis, and the periodogram. The objective to compute the Hurst exponent (H) of a given financial time series.

2.1 *Aggregated Variance*

This method involves partitioning a time series into non-overlapping segments, or "blocks," of a fixed size, commonly referred to as "lags" or "windows." The variance is then calculated for each block, and these variances are aggregated—typically summed—across all blocks. The key idea is that if the aggregated variance exceeds the expected value under the assumption of short-range dependence, this provides evidence for the presence of long memory.

Denote Y_t as a time series that exhibits long-range dependence. The return series $X_t = Y_{t+1} - Y_t$ is used to ensure the time series data is stationary and follows a Gaussian process.

Step 1: Divide the time series $\{X_t\}$, $t = 1, 2, \ldots, N$, into $[N/m]$ units, for an integer m between 2 and $N/2$, constructing a new series using the means give:

$$\bar{X}_k^{(m)} = \frac{1}{m} \sum_{t=(k-1)m+1}^{km} X_t, \qquad k = 1, 2, \ldots, [N/m].$$

Step 2: $\overline{X}$ will be the average of the series $\{X_t\}$. According to Park et. al. [12], m^{2H-2} describes the relation of stationary long-range dependence time series of length m and finite variance. Thus, when the sample is large enough, it gives

$$Var(\overline{X}_k^{(m)}) = C \cdot m^{2H-2},$$

where C is a constant that depends on the distribution of the increments. Further deducing gives

$$\log Var(\overline{X}_k^{(m)}) = (2H-2)\log m + C.$$

with C being a constant.

Step 3: Plotting $\log s_m$ against $\log m$ implies the slope

$$\beta = 2H - 2 \Rightarrow H = \frac{\beta+2}{2}.$$

However, the aggregated variance method has some limitations. One major drawback is the absence of an explicit variance estimator, which complicates its application. Additionally, the method lacks sufficient robustness [12] when deviations from standard Gaussian assumptions occur, further limiting its effectiveness.

2.2 *Rescaled Range analysis (R/S)*

The H can also be evaluated using one of the most popular and widely applied techniques: rescaled range (R/S) analysis [9]. This method is employed on return time series data to estimate and interpret the H, providing insights into the presence of long-range dependence. Consider a sample of N returns $\{X_t\}$ on a stock over period t and denote $\overline{X}_n$ as the sample mean $\frac{1}{N}\sum_{i=1}^{N} X_i$.

Step 1: The R/S statistics is given by

$$\left(\frac{R}{S}\right)_N = \frac{1}{s_N}\left[\max_{1\le k\le N}\sum_{j=1}^{k}(X_j - \overline{X}_N) - \min_{1\le k\le N}\sum_{j=1}^{k}(X_j - \overline{X}_N)\right]$$

where s_n here stands for the standard deviation estimate

$$s_N = \left[\frac{1}{N}\sum_{j=1}^{N}(X_j - \overline{X}_N)^2\right]^{1/2}.$$

Hurst [9] then discovered that the following empirical relation provides an excellent description of the rescaled range with numerous recordings across time.

Step 2: Similarly to the aggregated variance method, by taking the logarithm, we can have $\log(R/S)_N = H\log(N/2)$. The H, can be estimated as the slope β of the log-log scale of $(R/S)_N$ over $N/2$.

2.3 *Periodogram*

The periodogram [12] characterizes the power-law behaviour of the spectrum. This approach estimates the power spectral density of signals across different frequencies.

Step 1: The periodogram of a time series $\{X_t, 1 \le t \le N\}$ where is the length of time series is defined as

$$I_N(\lambda) = \frac{1}{2\pi N}\left|\sum_{t=1}^{N} X_t e^{i\lambda t}\right|^2,$$

where i represents imaginary number and X_t represents the return series used. In order to estimate the H parameter, it is required to calculate the periodogram. With $I_N(\lambda)$ being the estimator of the spectral density, a series that exhibits long memory should have periodogram which is proportional to $|\lambda|^{1-2H}$.

Step 2: A coefficient of $1 - 2H$ should then be measured by regressing the logarithm of periodogram and frequency, it then gives the slope as $H = \frac{1-\beta}{2}$.

2.4 *Interpretation on Hurst Exponent*

The H, is a key metric in time series analysis that quantifies long-range dependence. It measures the degree to which autocorrelations persist and how they decay as the lag between data points increases. Different values of the H signify different conclusions:

- $0 < H < 0.5$, anti-persistence behaviour. In this range, long memory does not exist, and the overall trend is anti-persistent. If the time series shows an upward trend initially, it is likely to reverse and trend downward, and vice versa. This indicates that future trends are negatively correlated with past trends. The closer the value of H to 0, the stronger this negative correlation becomes. As a result,

trading in the stock market under these conditions is considered risky.

- $H = 0.5$ indicated random behaviour. The time series exhibits random, uncorrelated behaviour. Future values are not related to past values, representing a random walk process. In this state, the market aligns with the Efficient Market Hypothesis (EMH), where traders cannot outperform the market using any strategy. As such, trading is not recommended.
- $0.5 < H < 1$, persistence behaviour. In this range, the time series exhibits long-range dependence and persistence. The higher the value of H, the stronger the positive correlation between future and past data. This means that trends in the market tend to persist over time, creating long-lasting cycles. A higher H suggests a stronger trend in the market, providing opportunities for traders to identify and follow these patterns.

3. Empirical Study

In our study, we focus on global stock indices to investigate whether they exhibit long memory properties. Data from multiple emerging markets and global indices are collected for comparative analysis. Daily closing prices form the basis of our analysis. The indices collected are from the following countries: United States: S&P 500 (GSPC), Dow Jones Industrial Average (DJI), Malaysia (KLSE), Japan (N225), China (399001. SZ), NASDAQ Composite (IXIC), Hong Kong (HSI), France (FCHI), New Zealand (NZ50), and Australia (AXJO). The data covers the period from June 2004 to June 2024 and is obtained from Yahoo Finance. We will be using the mathematical software MATLAB for our analysis due to its powerful numerical computing capability. With the methodologies introduced in previous section, the calculated returns Hurst exponents (H) are shown as follows:

Table 1

Hurst Exponent of the world indices returns with different methodologies.

Hurst Exponent of returns			
	Methodology		
Index	Aggregated Variance	Rescaled Range	Periodogram
GSPC	0.4163	0.5362	0.4687
DJI	0.4082	0.5361	0.4736
IXIC	0.4005	0.5338	0.4741
KLSE	0.4284	0.6015	0.5228
N225	0.4688	0.5607	0.4938
399001.SZ	0.5259	0.6339	0.5065
HSI	0.4488	0.5723	0.4991
FCHI	0.4294	0.5301	0.4905
NZ50	0.5155	0.5730	0.5219
AXJO	0.4375	0.5321	0.4830

As shown in Table 1, the three different methodologies yield varying H, ranging from 0 to 1, which aligns with the definition that H [3] fall within the interval [0, 1]. The H of returns tend to fluctuate around 0.5, typically falling between 0.4 and 0.7. This range suggests a mixed behaviour in the stock market, incorporating elements of persistence, anti-persistence, and randomness over time. The markets do not exhibit perfect efficiency, as both mean-reverting and trending behaviours are present. However, it can be argued that a single H is not sufficient to fully characterize the behaviour of an entire time series. Since market efficiency tends to evolve over time, we proceed by calculating a time-varying H, as introduced in reference [13,14]. Our results aim to align with the objectives of Costa and Vasconcelos [15], emphasizing the importance of computing a time-varying H to better understand market efficiency over time.

To compute the time-varying H, the H is calculated over successive, overlapping, or non-overlapping windows of the time series data. This approach allows for analysis of how long memory in the data evolves over time. A key question arises: "How should one determine the correct window size?" There is an inherent trade-off between choosing the largest possible window size, where the H is expected to remain stable, and the minimum number of data points required to estimate the H accurately. While identifying the ideal time window is an important task, it is beyond the scope of this study and will not be discussed here. For our analysis, we

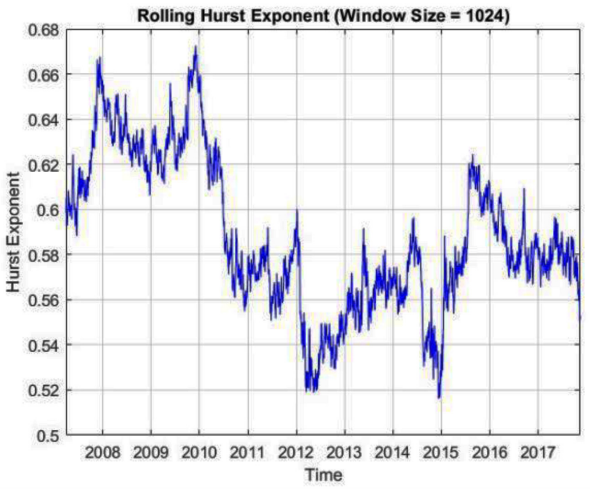


Figure 1

Rolling Hurst of the KLSE index returns.

use a time window size of 1024 observations. The most common method, rescaled range analysis, will be employed to estimate the dynamic H.

Figure 1 shows the dynamic H of the KLSE index return. The H values fluctuate between approximately 0.5166 to 0.6725 over time. The changes across maximum and minimum H of the stock market indicate the time

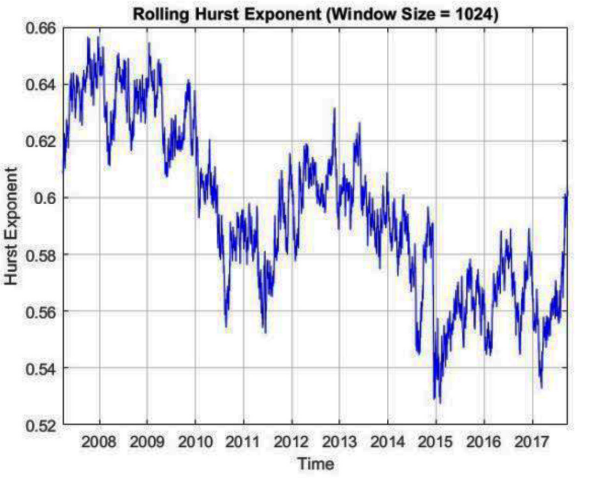


Figure 2

Rolling Hurst of the 399001.SZ (China) index returns.

series is persistent. As observed from Figure 1, the H values seem to oscillate at around 0.6 to 0.67 from around 2008 to 2010, the market is less efficient at this period of time and is moving towards fractal. As time goes by, the H values tend to decrease and is moving closer to 0.5, which indicates that the market is getting much more efficient.

Similarly, from Figure 2, the H is decreasing from around 0.65 to 0.55. If the plots are zoomed out and looked at the plot, it shows a gradually

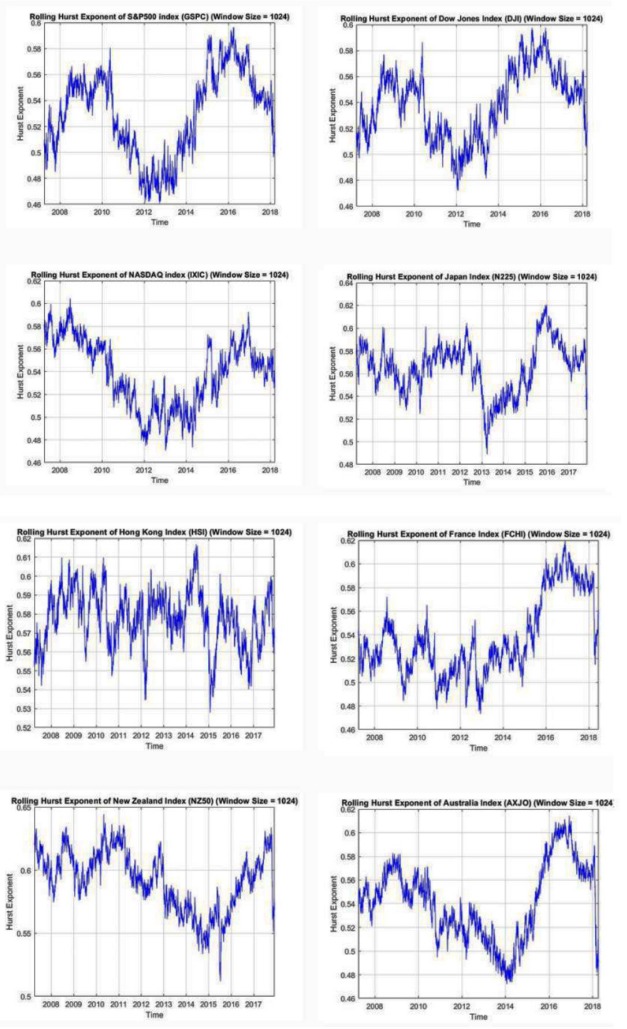


Figure 3

Rolling Hurst of the remaining world indices returns.

Table 2

Range of rolling Hurst exponents for the returns of different world indices.

Rolling Hurst Exponent for returns		
Index	Minimum	Maximum
GSPC	0.4607	0.5964
DJI	0.4721	0.5975
IXIC	0.4707	0.6046
KLSE	0.5162	0.6725
N225	0.4891	0.6207
399001.SZ	0.5275	0.6567
HSI	0.5280	0.6168
FCHI	0.4733	0.62
NZ50	0.5112	0.6443
AXJO	0.4818	0.6144

decreasing H values as time goes by. Thus, this indicates the China index is getting much more efficient. The rest of the world indices show similar pattern as Figure 3

Table 2 shows the fluctuations of the H over time. These fluctuations indicate that the market operates in different forms of efficiency at different periods. Several factors can explain the time-varying nature of the H.

Firstly, capital flow—particularly in emerging stock markets—has seen significant growth in recent years. As more investors enter the market, the shareholder base expands, generating greater liquidity. This increased participation facilitates faster dissemination of market information [7], which should, in theory, enhance market efficiency. However, it is important to note that when capital withdrawals exceed inflows, this effect can sometimes cause the market to move in the opposite direction, temporarily reducing efficiency. Another explanation lies in the growth and development of derivative markets in certain countries. These markets allow investors to manage risk more effectively, contributing to the faster flow of information and, consequently, enhancing market efficiency.

Volatility also plays a crucial role in financial market modelling. It measures how quickly the price of a security changes over time, providing insights into price risk—the level of risk associated with fluctuating prices. As volatility increases, so does risk. In financial markets, volatility is often linked to significant price fluctuations. By estimating the dynamic H, we can further explore its relationship with volatility. Various methods [16]

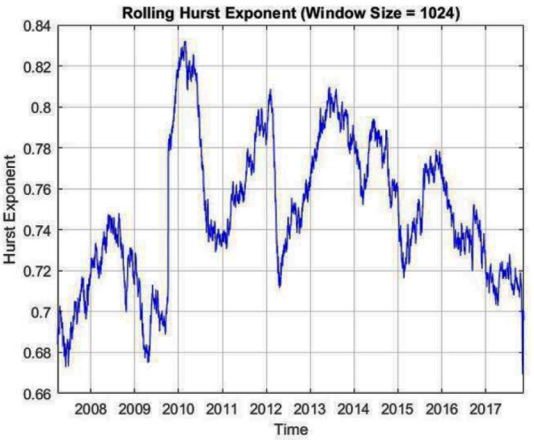


Figure 4

Rolling Hurst of the KLSE index volatility.

can be used to measure volatility, including the widely used GARCH model. In this study, we will use squared returns [17] as the proxy for volatility.

Figure 4 shows the volatility of the KLSE index, which varies between 0.68 and 0.83 and is considered quite high. From 2008 to 2010, the H experienced a significant increase from 0.69 to 0.83, indicating that the volatility was highly persistent during this period. Looking more generally at Figure 4, the volatility of the KLSE index is gradually decreasing, suggesting that the market is becoming more random over time.



Figure 5

Rolling Hurst of the NASDAQ index (IXIC) volatility.

Another result comes from the NASDAQ index as shown in Figure 5. Volatility rose to a peak of around 0.909, reflecting strong trending behaviour at that point in time. The market exhibited remarkable persistence at the beginning of 2010. Afterward, the H value rebounded to its normal level of around 0.75. During the period between 2012 and 2013, the *H* value dropped to its lowest point of 0.6618, indicating less trending

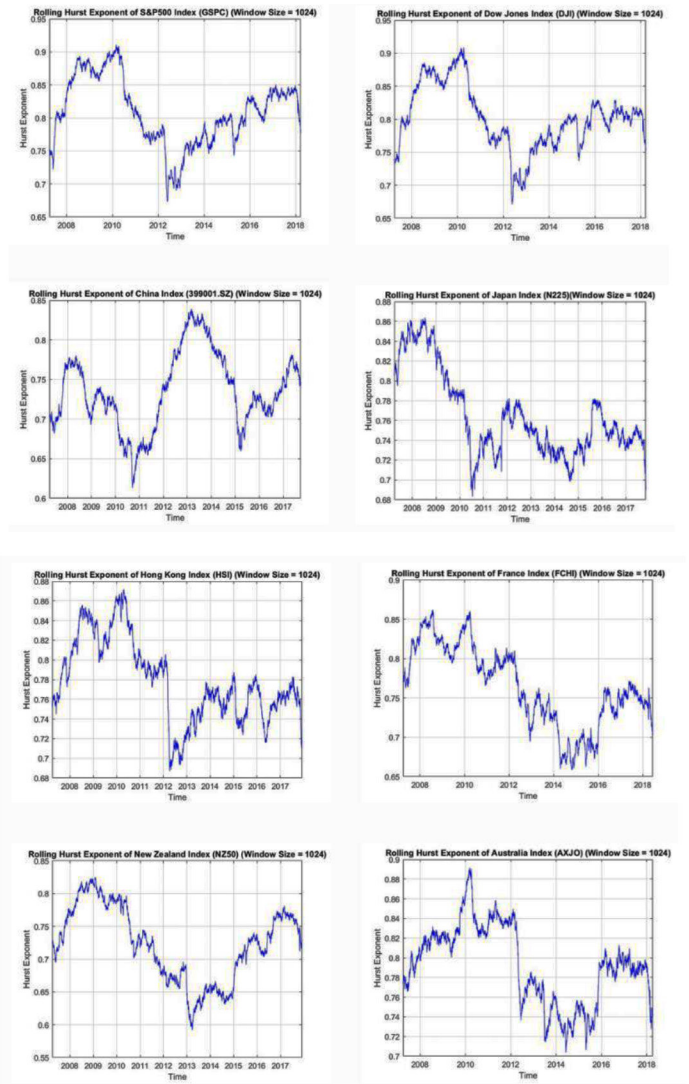


Figure 6

Rolling Hurst of the remaining world indices volatility

Table 3
Range of rolling Hurst exponents for the volatility of different world indices

Rolling Hurst Exponents for volatility		
Index	Minimum	Maximum
GSPC	0.6732	0.9107
DJI	0.6712	0.9085
IXIC	0.6618	0.9096
KLSE	0.6694	0.8321
N225	0.6832	0.8637
399001.SZ	0.6133	0.8389
HSI	0.6870	0.8719
FCHI	0.6584	0.8621
NZ50	0.5918	0.8251
AXJO	0.7039	0.8914

behaviour during this time. The time-varying H of volatility of the remaining world indices is shown in Figure 6. By visual look, most of the H values of the world stock indices fall in the interval [0.5, 1].

Table 3 summarizes the range of H for the volatility of various global indices. All the HHH values exceed 0.5, suggesting that these stock markets are neither anti-persistent nor consistent with a random walk model. Instead, the results indicate persistent volatility and clustering volatility.

4. Conclusion

This paper highlights the importance of analyzing dynamic H over time to assess financial market informationally efficiency, rather than relying solely on a single static measure of long memory. Our findings indicate that variations in the dynamics of the underlying return time series cause the H to fluctuate over time. This behavior cannot be fully explained by time-varying short memory or time-varying volatility, as the H remain time-varying even after adjusting for these factors. In examining the behavior of world stock market returns, our results show that the H tend to fluctuate around 0.5. According to the EMH, stock prices reflect all available information, implying that financial markets are largely efficient. While security prices often exhibit trends and non-stationary behavior, returns (price changes) are expected to follow random patterns, with no

predictable structure or memory. Therefore, it is not surprising that returns do not exhibit long memory.

In Ben Jacobsen's [18] study, he found no evidence suggesting that indices from countries in Europe, the United States, and Japan exhibit long memory. However, Jacobsen also demonstrated that the H for stock return series does not necessarily imply the presence of long memory. Despite this, we further examine the volatility of each stock index. The H for volatility consistently exceeded 0.5, indicating persistent behavior and long memory in the volatility series. This suggests a high autocorrelation: if the market is volatile for a period, it is likely to remain volatile for some time. This phenomenon can be described as "calm markets stay calm, and turbulent markets stay turbulent." Moreover, a high value of H in the volatility series implies that risk persists over time. Investors and traders should consider longer risk horizons and investigate the factors contributing to the high H value. These factors may include economic trends, geopolitical uncertainties, and the overall market structure.

References

[1] E. F. Fama, "The behaviour of stock-market prices," *The Journal of Business*, vol. 38, no. 1, pp. 34–105 (1965).

[2] B. Guo-ping and Y. Yi-rong, "Hurst exponent estimation based on modified aggregated variance method," pp. 51–56 (2006).

[3] A. Plastun, R. Mynhardt, and I. Makarenko, "Behavior of financial markets efficiency during the financial market crisis: 2007-2009," *Corporate Ownership and Control*, vol. 11, pp. 531–546 (Sep. 2014).

[4] A. Tiţan, "The efficient market hypothesis: Review of specialized literature and empirical research," *Procedia Economics and Finance*, vol. 32, pp. 442–449 (Dec. 2015).

[5] B. B. Mandelbrot, *Fractals: Form, Chance and Dimension*. New York: Free Press, 1977.

[6] J. T. Barkoulas and C. F. Baum, "Long-term dependence in stock returns," *Economics Letters*, vol. 53, no. 3, pp. 253–259 (1996).

[7] D. O. Cajueiro and B. M. Tabak, "The Hurst exponent over time: Testing the assertion that emerging markets are becoming more efficient," *Physica A: Statistical Mechanics and its Applications*, vol. 336, no. 8, pp. 521–537 (2004).

[8] T. Graves, R. Gramacy, N. Watkins, and C. Franzke, "A brief history of long memory: Hurst, Mandelbrot and the road to ARFIMA, 1951–1980," vol. 19, no. 9, p. 437 (2017).

[9] H. E. Hurst, "Long-term storage capacity of reservoirs," *Transactions of the American Society of Civil Engineers*, vol. 116, no. 1, pp. 770–799 (1951).

[10] J. Flemming and C. Kirby, "Long memory in volatility and trading volume," *Journal of Banking & Finance*, vol. 35, no. 7, pp. 1714–1726 (2011).

[11] T. Bollerslev and D. Jubinski, "Equity trading volume and volatility: Latent information arrivals and common long-run dependencies," *Journal of Business & Economic Statistics*, vol. 17, no. 1, pp. 9–21 (1999).

[12] C. Park, F. Hernández-Campos, L. Le, J. Marron, J. Park, V. Pipiras, F. Smith, R. Smith, M. Trovero, and Z. Zhu, "Long-range dependence analysis of internet traffic," *Journal of Applied Statistics*, vol. 38, pp. 1407–1433 (Jul. 2011).

[13] S. V. Muniandy, S. C. Lim, and R. Murugan, "Inhomogeneous scaling behaviors in Malaysian foreign currency exchange rates," *Physica A: Statistical Mechanics and its Applications*, vol. 301, no. 1, pp. 407–428 (2001).

[14] C.W. Cheong, L.M. Cherng, L.C. Zhi, Z.Y. Huai, "Do general elections affect fractal structure of stock market?," *Journal of Statistics and Management Systems*, vol. 24, no. 5, 951–964 (2021).

[15] R. L. Costa and G. L. Vasconcelos, "Long-range correlations and nonstationarity in the Brazilian stock market," *Physica A: Statistical Mechanics and its Applications*, vol. 329, no. 1, pp. 231–248 (2003).

[16] G. Zournatzidou and C. Floros, "Hurst exponent analysis: Evidence from volatility indices and the volatility of volatility indices," *Journal of Risk and Financial Management*, vol. 16, no. 5 (2023).

[17] D. O. Cajueiro and B. M. Tabak, "Ranking efficiency for emerging equity markets II," *Chaos, Solitons & Fractals*, vol. 23, no. 2, pp. 671–675 (2005).

[18] B. Jacobsen, "Long term dependence in stock returns," *Journal of Empirical Finance*, vol. 3, no. 4, pp. 393–417 (1996).

Received December, 2024