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On a multivariate Lindley distribution of Kind-1

G. S. David Sam Jayakumar *
Jamal Institute of Management
Jamal Mohamed College (Affiliated to Bharathidasan University)
Trichy 620020
Tamil Nadu
India

W. Samuel †
Department of Business Administration
Jamal Mohamed College (Affiliated to Bharathidasan University)
Trichy 620020
Tamil Nadu
India

Abstract

This article introduces a multivariate (p-variate) extension of the univariate Lindley distribution, known as Kind-1 of Multivariate Lindley Distribution (MLD). Further, the Cumulative, Marginal, Multivariate conditional distributions are computed. Mandatory Constants such as product moments, Co-variance, Correlation Co-efficient and generating functions namely MGF, Cumulant, characteristic, survival, hazard, cumulative hazard functions are derived with some special cases of Multivariate Lindley distributions such as Inverse, Squared, Power, Extreme, Unit and logit Lindley distribution are also highlighted. Parameter estimation based on Maximum likelihood estimation and Non-linear Programming are also shown. The bivariate and conditional Lindley probability surfaces are visualized for different combination of scale parameters.

Subject Classification: *62H10, 62E15.*

Keywords: *Multivariate extension, Lindley distribution, Constants, Generating functions, Parameter estimation, Maximum likelihood estimation, Non-linear programming, Probability surfaces.*

* *ORCID-ID:* **0000-0002-6630-8783**
† *ORCID-ID:* **0000-0002-9112-9744**

* *E-mail:* **samjaya77@gmail.com (Corresponding Author)**
† *E-mail:* **wsamuel365@gmail.com**

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1. Introduction

Lindley distribution [6] [7] was introduced with a single scale parameter with the purpose of studying complex lifetime data. In recent times, different versions of Lindley distributions were explored and extensively studied. But only, few authors were attempted to extend the univariate Lindley distribution to its multivariate case. Gomez-Deniz, Sarabia, and Balakrishnan [5] explored the actuarial applications of the Lindley distribution by proposing the Multivariate Discrete Poisson-Lindley distribution. On the other hand, Mesfioui and Abouammoh [8] extended the generalized Lindley distribution to the Multivariate case and the distribution was constructed based on Several Lindley-distributed random variables are associated via a shared shock mechanism. Similarly, Calderín-Ojeda and Gómez-Déniz [2] proposed the multivariate negative Binomial-Lindley distribution and investigated its properties, advantages which the distribution helps to understand the Over dispersed data. Bashir and Mahmood [1] explored a multivariate area-biased Lindley distribution and elaborately discussed the properties and application of the distribution. A new one-parameter circular distribution, known as the Wrapped Modified Lindley Distribution, was introduced and derived through the wrapping technique by Chesneau, Tomy, and Gillariose [4]. They explored its characteristic function and fundamental properties. The parameter estimation was carried out using the maximum likelihood estimation method. The proposed model was applied to two empirical datasets and its performance was benchmarked against the Wrapped Lindley, Wrapped Exponential, and Transmuted Wrapped Exponential distributions. Moreover, Chesneau, Tomy, and Gillariose [3] introduced the Modified Lindley Distribution, a one-parameter formulation derived from the classical Lindley distribution. This model offers a balanced compromise between the Exponential and Lindley distributions, presenting a viable alternative to both. In the present study, the authors propose a new variant of the Multivariate Lindley Distribution, referred to as Kind-1. The structural properties of this distribution differ from those of existing models in the literature, as discussed in the following sections.

Multivariate Lindley Distribution of Kind-1

Definition 1.1: Let $X_1, X_2, X_3 \ldots, X_p$ are the random variables follows univariate Lindley distribution having scale parameter θ_j with mean and variance $(\theta_j + 2) \,/\, \theta_j(\theta_j + 1)$ and $(2(\theta_j + 3)\,(\theta_j + 1) - (\theta_j + 2)^2) \,/\, \theta_j^2(\theta_j + 1)^2$

respectively for all j (j =1to p),then the Multivariate Lindley distribution of Kind-1 and its PDF is defined as

$$f_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p) = \left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)\left(\prod_{j=1}^{p}\theta_j e^{-\theta_j x_j}\right) \quad (1.1)$$

where $0 < x_j < \infty, \theta_j > 0, p \geq 2$

Theorem 1.2: *The CDF of the **MLD of kind-1** can also be expressed as*

$$F_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p) = \left(\sum_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)(1-e^{-\theta_j})}\right)-\prod_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)}\right)-(p-2)\right)\left(\prod_{j=1}^{p}1-e^{-\theta_j}\right) \quad (1.2)$$

where, $0 < x_j < \infty, \theta_j > 0, p \geq 2$

Proof: Let the multivariate **CDF** be defined as

$$F_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p) = \int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_p}\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+u_j)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+u_j)}{1+\theta_j}\right)-(p-2)\right)\left(\prod_{j=1}^{p}\theta_j e^{-\theta_j u_j}\right)\prod_{j=1}^{p}du_j \quad (1.3)$$

From (1.3), it can be written by Splitting the Multiple Integral as

$$F_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p) = \begin{pmatrix} \int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_p}\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+u_j)}{1+\theta_j}\right)\right)\left(\prod_{j=1}^{p}\theta_j e^{-\theta_j u_j}\right)\prod_{j=1}^{p}du_j \\ -\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_p}\left(\prod_{j=1}^{p}\left(\frac{\theta_j(1+u_j)}{1+\theta_j}\right)\right)\left(\prod_{j=1}^{p}\theta_j e^{-\theta_j u_j}\right)\prod_{j=1}^{p}du_j \\ -\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_p}(p-2)\left(\prod_{j=1}^{p}\theta_j e^{-\theta_j u_j}\right)\prod_{j=1}^{p}du_j \end{pmatrix} \quad (1.4)$$

$$F_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=\left(\sum_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)(1-e^{-\theta_j})}\right)\right.$$
$$\left.-\prod_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)}\right)-(p-2)\right)\left(\prod_{j=1}^{p}1-e^{-\theta_j}\right)$$

Then, perform the integration for (1.4), the result of CDF as

Theorem 1.3: *The* ***PDF*** *of the Conditional* ***MLD of Kind-1*** *of* X_1 *on* $X_2,X_3,\ldots X_p$ *is*

$$f_{X_1/X_2,\ldots,X_p}(x_1/x_2,\ldots,x_p)=\frac{\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)(\theta_1 e^{-\theta_1 x_1})}{\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-3)} \tag{1.5}$$

where $0<x_1<\infty,\theta_j>0,p\geq 2$

Proof: It is obtained from the following fact by substituting the Multivariate and Marginal PDF

$$f_{X_1/X_2,\ldots,X_p}(x_1/x_2,\ldots,x_p)=f_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p)/f_{X_2,\ldots,X_p}(x_2,\ldots,x_p)$$

Theorem 1.4: *If P random variables* $X_1,X_2,\ldots,X_p$ *has a conditional dependence on the* ***q*** *variables namely* $Y_1,Y_2,Y_3,\ldots,Y_q$, *then the density function of the Kind-1* ***MCLD*** *is given by*

$$f_{X_1,X_2,\ldots,X_p/Y_1,Y_2,\ldots,Y_q}(x_1,x_2,\ldots,x_p/y_1,y_2,\ldots,y_q)$$

$$=\frac{\left(\sum_{i=1}^{p}\left(\frac{\theta_i(1+x_i)}{1+\theta_i}\right)+\sum_{j=1}^{q}\left(\frac{\theta_i(1+y_i)}{1+\theta_i}\right)-\prod_{i=1}^{p}\left(\frac{\theta_i(1+x_i)}{1+\theta_i}\right)-\prod_{j=1}^{q}\left(\frac{\theta_i(1+y_i)}{1+\theta_i}\right)-(p+q-2)\right)\left(\prod_{i=1}^{p}\theta_i e^{-\theta_i x_i}\right)}{\sum_{j=1}^{q}\left(\frac{\theta_i(1+y_i)}{1+\theta_i}\right)-\prod_{j=1}^{q}\left(\frac{\theta_i(1+y_i)}{1+\theta_i}\right)-(q-2)} \tag{1.6}$$

where $0<x_j<\infty,\theta_j>0,p+q\geq 2$

Proof: It is obtained from the multivariate conditional law for p random variables $X_1, X_2, X_3, \ldots, X_p$ on the q variables $Y_1, Y_2, Y_3, \ldots, Y_q$ by substituting the appropriate PDFs, then it is given as

$$f_{X_1,X_2,\ldots,X_p/Y_1,Y_2,\ldots,Y_q}(x_1, x_2, \ldots, x_p / y_1, y_2, \ldots, y_q) = \frac{f_{X_1,X_2,\ldots,X_p}(x_1, x_2, \ldots, x_p)}{f_{Y_1,Y_2\ldots,Y_q}(y_1, y_2, \ldots, y_q)}$$

2. Constants

Theorem 2.1: *The conditional r^{th} moment of the Multivariate conditional Lindley distribution is*

$$E_{X_1/X_2,\ldots,X_p}(x_1^r / x_2, \ldots, x_p) =$$

$$\frac{\frac{\theta^2}{\theta+1}\left(\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}}\right)\left(1 - \prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)\right) + \frac{\Gamma(r+1)}{\theta^{r+2}}\left(\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right) - (p-2)\right)}{\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right) - \prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right) - (p-3)}$$

(2.1)

Proof: The Conditional r^{th} moment of a multivariate distribution is given as

$$E_{X_1/X_2,\ldots,X_p}(x_1^r / x_2, \ldots, x_p) = \int_0^\infty x_1^r f_{X_1/X_2,\ldots,X_p}(x_1 / x_2, \ldots, x_p) dx_1 \qquad (2.2)$$

Now Substituting the PDF of Conditional MLD in (2.2) and perform the Integration, the result derived as

$$E_{X_1/X_2,\ldots,X_p}(x_1^r / x_2, \ldots, x_p) =$$

$$\frac{\frac{\theta^2}{\theta+1}\left(\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}}\right)\left(1 - \prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)\right) + \frac{\Gamma(r+1)}{\theta^{r+2}}\left(\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right) - (p-2)\right)}{\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right) - \prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right) - (p-3)}$$

(2.3)

From the result (2.3) and if $r = 1, 2, 3, 4$ then the first four conditional moments of the Multivariate Conditional Lindley distribution are given as

$$E_{X_1/X_2,\ldots,X_p}(x_1 / x_2,\ldots,x_p) = \frac{\frac{\theta^2}{\theta+1}\left(\frac{1}{\theta^2}+\frac{2}{\theta^3}\right)\left(1-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)\right)+\frac{1}{\theta^3}\left(\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)}{\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-3)}$$

$$E_{X_1/X_2,\ldots,X_p}(x_1^2 / x_2,\ldots,x_p) = \frac{\frac{\theta^2}{\theta+1}\left(\frac{2}{\theta^3}+\frac{6}{\theta^4}\right)\left(1-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)\right)+\frac{2}{\theta^4}\left(\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)}{\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-3)}$$

$$E_{X_1/X_2,\ldots,X_p}(x_1^3 / x_2,\ldots,x_p) = \frac{\frac{\theta^2}{\theta+1}\left(\frac{6}{\theta^4}+\frac{24}{\theta^5}\right)\left(1-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)\right)+\frac{6}{\theta^5}\left(\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)}{\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-3)}$$

$$E_{X_1/X_2,\ldots,X_p}(x_1^4 / x_2,\ldots,x_p) = \frac{\frac{\theta^2}{\theta+1}\left(\frac{6}{\theta^5}+\frac{120}{\theta^6}\right)\left(1-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)\right)+\frac{6}{\theta^6}\left(\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)}{\sum_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=2}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-3)}$$

Theorem 2.2: *The Marginal Co-variance between two Lindley random variables* X_1 *and* X_2 *is given as*

$$E_{X_i^m X_j^n}(x_i^m x_j^n) =$$

$$\left(\begin{array}{l} \dfrac{\theta_i^2}{\theta_i^2+1}\left(\dfrac{\Gamma(m+1)}{\theta_i^{m+1}}+\dfrac{\Gamma(m+2)}{\theta_i^{m+2}}\right)\left(\dfrac{\Gamma(n+1)}{\theta_j^{n+2}}\right) \\ +\dfrac{\theta_j^2}{\theta_j^2+1}\left(\dfrac{\Gamma(n+1)}{\theta_j^{n+1}}+\dfrac{\Gamma(n+2)}{\theta_j^{n+2}}\right)\left(\dfrac{\Gamma(m+1)}{\theta_i^{m+2}}\right) \\ -\dfrac{\theta_i^2\theta_j^2}{(\theta_i^2+1)(\theta_j^2+1)}\left(\dfrac{\Gamma(m+1)}{\theta_i^{m+1}}+\dfrac{\Gamma(m+2)}{\theta_i^{m+2}}\right)\left(\dfrac{\Gamma(n+1)}{\theta_j^{n+1}}+\dfrac{\Gamma(n+2)}{\theta_j^{n+2}}\right) \end{array} \right) \tag{2.4}$$

Proof: Let the product moment of the **MLD of Kind-1** is given as

$$E_{X_i^m X_j^n}(x_i^m x_j^n) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty x_i^m x_j^n f_{X_1, X_2, \ldots, X_p}(x_1, x_2, \ldots, x_p) \prod_{j=1}^{p} dx_j \tag{2.5}$$

After the substitution of **MLD of Kind-1** in **(**2.5), perform the integration and the result is written as

$$E_{X_i^m X_j^n}(x_i^m x_j^n) = \left(\begin{array}{l} \dfrac{\theta_i^2}{\theta_i^2+1}\left(\dfrac{\Gamma(m+1)}{\theta_i^{m+1}}+\dfrac{\Gamma(m+2)}{\theta_i^{m+2}}\right)\left(\dfrac{\Gamma(n+1)}{\theta_j^{n+2}}\right) \\ +\dfrac{\theta_j^2}{\theta_j^2+1}\left(\dfrac{\Gamma(n+1)}{\theta_j^{n+1}}+\dfrac{\Gamma(n+2)}{\theta_j^{n+2}}\right)\left(\dfrac{\Gamma(m+1)}{\theta_i^{m+2}}\right) \\ -\dfrac{\theta_i^2\theta_j^2}{(\theta_i^2+1)(\theta_j^2+1)}\left(\dfrac{\Gamma(m+1)}{\theta_i^{m+1}}+\dfrac{\Gamma(m+2)}{\theta_i^{m+2}}\right)\left(\dfrac{\Gamma(n+1)}{\theta_j^{n+1}}+\dfrac{\Gamma(n+2)}{\theta_j^{n+2}}\right) \end{array} \right)$$

Remark: If $m = n = 1$, then the result can be extended to the Computation of Co-variance and Correlation between the i^{th} and j^{th} Lindley random variables are given as

$$COV_{X_i, X_j}(x_i, x_j) = \frac{\theta_i^2}{\theta_j^3(\theta_i^2+1)}\left(\frac{1}{\theta_i^2}+\frac{2}{\theta_i^3}\right)+\frac{\theta_j^2}{\theta_i^3(\theta_j^2+1)}\left(\frac{1}{\theta_j^2}+\frac{2}{\theta_j^3}\right)$$
$$-\left(\frac{\theta_i^2\theta_j^2}{(\theta_i^2+1)(\theta_j^2+1)}\left(\frac{1}{\theta_i^2}+\frac{2}{\theta_i^3}\right)\left(\frac{1}{\theta_j^2}+\frac{2}{\theta_j^3}\right)+\frac{(\theta_i+2)(\theta_j+2)}{\theta_i\theta_j(\theta_i+1)(\theta_j+1)}\right) \tag{2.6}$$

$$\rho_{X_i, X_j}(x_i, x_j) =$$

$$\frac{\theta_i^3(\theta_i+1)}{\theta_j^2(\theta_i^2+1)(\theta_j+1)}\left(\frac{1}{\theta_i^2}+\frac{2}{\theta_i^3}\right)+\frac{\theta_j^3(\theta_j+1)}{\theta_i^2(\theta_i+1)(\theta_j^2+1)}\left(\frac{1}{\theta_j^2}+\frac{2}{\theta_j^3}\right)$$
$$-\left(\frac{\theta_i^3\theta_j^3}{(\theta_i^2+1)(\theta_i+1)(\theta_j^2+1)(\theta_j+1)}\left(\frac{1}{\theta_i^2}+\frac{2}{\theta_i^3}\right)\left(\frac{1}{\theta_j^2}+\frac{2}{\theta_j^3}\right)+\frac{(\theta_i+2)(\theta_j+2)}{(\theta_i+1)^2(\theta_j+1)^2}\right)$$
$$\frac{}{\sqrt{2(\theta_i+3)(\theta_i+1)-(\theta_i+2)^2}\sqrt{2(\theta_j+3)(\theta_j+1)-(\theta_j+2)^2}} \quad (2.7)$$

where $i \neq j$, $-1 \leq \rho_{ij} \leq +1$

3. Generating Functions

Theorem 3.1: *The **MGF** of the Kind-1 Multivariate Lindley Distribution (MLD) is expressed as*

$$M_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=$$
$$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-t_j)}{(\theta_j+1)(\theta_j-t_j)}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-t_j)}{(\theta_j+1)(\theta_j-t_j)}\right)-(p-2)\right)\prod_{j=1}^{p}\left(\frac{\theta_j}{\theta_j-t_j}\right) \quad (3.1)$$

Proof: Let the **MGF** of a multivariate distribution be defined as

$$M_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=\int_0^\infty\int_0^\infty\ldots\int_0^\infty e^{\sum_{j=1}^{p}t_jx_j}f_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)\prod_{j=1}^{p}dx_j$$

$$M_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=\int_0^\infty\int_0^\infty\ldots\int_0^\infty e^{\sum_{j=1}^{p}t_jx_j}$$
$$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)\left(\prod_{j=1}^{p}\theta_je^{-\theta_jx_j}\right)\prod_{j=1}^{p}dx_j \quad (3.2)$$

Thus, by Integration

$$M_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=$$
$$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-t_j)}{(\theta_j+1)(\theta_j-t_j)}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-t_j)}{(\theta_j+1)(\theta_j-t_j)}\right)-(p-2)\right)\prod_{j=1}^{p}\left(\frac{\theta_j}{\theta_j-t_j}\right)$$

Theorem 3.2: *The Cumulant of the **MLD of kind-1** distribution is*

$$C_{X_1,X_2\ldots X_p}(t_1,t_2,\ldots,t_p)=$$
$$\sum_{j=1}^{p}\log\left(\frac{\theta_j}{\theta_j-t_j}\right)+\log\left(\sum_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-t_j)}{(\theta_j+1)(\theta_j-t_j)}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-t_j)}{(\theta_j+1)(\theta_j-t_j)}\right)-(p-2)\right) \quad (3.3)$$

Proof: It is found from

$$C_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=\log(M_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p))$$

Theorem 3.3: *The* ***CF*** *of* ***MLD of kind-1*** *distribution is*

$$\phi_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=$$
$$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-it_j)}{(\theta_j+1)(\theta_j-it_j)}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-it_j)}{(\theta_j+1)(\theta_j-it_j)}\right)-(p-2)\right)\prod_{j=1}^{p}\left(\frac{\theta_j}{\theta_j-it_j}\right) \quad (3.4)$$

Proof: Let the **CF** of a multivariate distribution be defined as

$$\phi_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=\int_0^\infty\int_0^\infty\ldots\int_0^\infty e^{i\sum_{j=1}^{p}t_jx_j}f_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)\prod_{j=1}^{p}dx_j$$

$$\phi_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=\int_0^\infty\int_0^\infty\ldots\int_0^\infty e^{i\sum_{j=1}^{p}t_jx_j}$$
$$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)\left(\prod_{j=1}^{p}\theta_je^{-\theta_jx_j}\right)\prod_{j=1}^{p}dx_j$$

Obviously, by integration

$$\phi_{X_1,X_2,\ldots,X_p}(t_1,t_2,\ldots,t_p)=$$
$$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-it_j)}{(\theta_j+1)(\theta_j-it_j)}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(\theta_j+1-it_j)}{(\theta_j+1)(\theta_j-it_j)}\right)-(p-2)\right)\prod_{j=1}^{p}\left(\frac{\theta_j}{\theta_j-it_j}\right)$$

Theorem 3.4: *The survival function of the* ***MLD of kind-1*** *is*

$$S_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=1-\left(\sum_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_jx_j}-2\theta_je^{-\theta_jx_j}}{(1+\theta_j)(1-e^{-\theta_j})}\right)\right.$$

$$-\prod_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)}\right)-(p-2)\Bigg)\left(\prod_{j=1}^{p}1-e^{-\theta_j}\right) \tag{3.5}$$

Proof: It is obtained from

$$S_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=1-F_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)$$

Theorem 3.5: *The hazard function of the* ***MLD of kind-1*** *distribution is*

$$h_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=$$

$$\frac{\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_j)}{1+\theta_j}\right)-(p-2)\right)\left(\prod_{j=1}^{p}\theta_j e^{-\theta_j x_j}\right)}{1-\left(\sum_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)(1-e^{-\theta_j})}\right)-\prod_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)}\right)-(p-2)\right)\left(\prod_{j=1}^{p}1-e^{-\theta_j}\right)} \tag{3.6}$$

Proof: It is obtained from

$$h_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=\frac{f_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)}{S_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)}$$

and $S_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=1-F_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)$

Theorem 3.6: *The* ***CHF*** *of the* ***MLD of kind-1*** *is*

$$H_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=-\log\Bigg(1-\Bigg(\sum_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)(1-e^{-\theta_j})}\right)$$
$$-\prod_{j=1}^{p}\left(\frac{1+\theta_j-e^{-\theta_j x_j}-2\theta_j e^{-\theta_j x_j}}{(1+\theta_j)}\right)-(p-2)\Bigg)\left(\prod_{j=1}^{p}(1-e^{-\theta_j})\right)\Bigg) \tag{3.7}$$

Proof: It is Obtained from

$$H_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p)=-\log(S_{X_1,X_2,\ldots,X_p}(x_1,x_2,\ldots,x_p;\theta_1,\theta_2,\ldots,\theta_p))$$

4. Special cases

Result 4.1: Table 1 below presents selected special cases of the Kind-1 Multivariate Lindley Distribution, derived from equation (1.1) using the multidimensional Jacobian transformation.

Table-1

Case	Multivariate Lindley Distribution	Transformation (y_j)	Multivariate Density function $f_{Y_1,Y_2,\ldots,Y_p}(y_1,y_2,\ldots,y_p;\theta_1,\theta_2,\ldots,\theta_p)$
1	Inverse	$1/x_j$	$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+y_j)}{(1+\theta_j)y_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+y_j)}{(1+\theta_j)y_j}\right)-(p-2)\right)$ $\prod_{j=1}^{p}((\theta_j/y_j^2)e^{-\theta_j/y_j})$ where, $0\le y_j<\infty$, $\theta_j>0$
2	Squared	$\sqrt{x_j}$	$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+y_j^2)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+y_j^2)}{1+\theta_j}\right)-(p-2)\right)$ $2^p\prod_{j=1}^{p}(\theta_j y_j e^{-\theta_j y_j^2})$ where, $0\le y_j<\infty$, $\theta_j>0$
3	Power	$\sqrt[q]{x_j}$	$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+y_j^q)}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+y_j^q)}{1+\theta_j}\right)-(p-2)\right)q^p\prod_{j=1}^{p}(\theta_j y_j^{q-1}e^{-\theta_j y_j^q})$ where, $0\le y_j<\infty, \theta_j, q>0$
4	Extreme	$\log x_j$	$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+e^{y_j})}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+e^{y_j})}{1+\theta_j}\right)-(p-2)\right)$ $\prod_{j=1}^{p}\left(\theta_j e^{-(\theta_j e^{y_j}-y_j)}\right)$ where, $-\infty<y_j<+\infty$, $-\infty<\theta_j<+\infty$

Contd...

5	Unit	$x_j/(1+x_j)$	$\left(\sum_{j=1}^{p}\left(\frac{\theta_j}{(1+\theta_j)(1-y_j)}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j}{(1+\theta_j)(1-y_j)}\right)-(p-2)\right)$ $\prod_{j=1}^{p}\left(\frac{\theta_j}{(1-y_j)^2}e^{-\theta_j\left(\frac{y_j}{1-y_j}\right)}\right)$ where, $0\le y_j\le 1, \theta_j>0$
6	logit	$1/(1+e^{x_j})$	$\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+\log(y_j/(1-y_j)))}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+\log(y_j/(1-y_j)))}{1+\theta_j}\right)-(p-2)\right)$ $\prod_{j=1}^{p}(\theta_j/y_j(1-y_j)e^{-\theta_j\log(y_j/(1-y_j))})$ where, $0\le y_j\le 1$, $\theta_j>0$

5. Parameter Estimation

Result 5.1: As an initial approach, the authors employ the maximum likelihood method (MLE) for estimation. The log-likelihood function for a multivariate random sample is given by x_{1i}, x_{2i}, x_{3i}, …, $x_{(p-1)(n-1)}$, x_{pn} from (1.1) is

$$\log L(\theta_1,\theta_2,\ldots,\theta_p)=\sum_{i=1}^{n}\left(\log\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-\left(p-2\right)\right)\right)$$
$$+n\sum_{j=1}^{p}\log\theta_j-\theta_j\sum_{i=1}^{n}x_{ji} \quad (5.1)$$

The first order derivatives of (5.1) with respect to 'p' parameters are

$$\frac{\partial\log L}{\partial\theta_j}=\sum_{i=1}^{n}\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-(p-2)\right)^{-1}$$
$$\left(\sum_{j=1}^{p}\left(\frac{1+x_{ji}}{(1+\theta_j)^2}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\sum_{j=1}^{n}\left(\frac{1}{\theta_j(1+\theta_j)}\right)\right)+n\sum_{j=1}^{p}\left(\frac{1}{\theta_j}\right)-\sum_{i=1}^{n}x_{ji} \quad (5.2)$$

By setting these expressions to zero and solving them simultaneously, the authors obtain the ML estimates of the 'p' number of parameters. To construct the Fisher Information matrix, the second-order and cross partial derivatives of equation (5.1) are required.

$$\frac{\partial^2 \log L}{\partial \theta_j^2} = -\sum_{i=1}^{n}\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-(p-2)\right)^{-2}$$
$$\left(\sum_{j=1}^{p}\left(\frac{1+x_{ji}}{(1+\theta_j)^2}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\sum_{j=1}^{n}\left(\frac{1}{\theta_j(1+\theta_j)}\right)\right)-n\sum_{j=1}^{p}\left(\frac{1}{\theta_j^2}\right) \tag{5.3}$$

$$\frac{\partial^2 \log L}{\partial \theta_j \partial \theta_k} =$$

$$\left(\left(-\sum_{i=1}^{n}\left(\sum_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)+\sum_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-\prod_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\prod_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-(p-2)\right)^{-2}\right)\right.$$
$$\left.\left(\sum_{k=s+1}^{p}\left(\frac{1+x_{ki}}{(1+\theta_k)^2}\right)-\prod_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\prod_{k=s+1}^{k}\left(\frac{1}{\theta_k(1+\theta_k)}\right)\right)-n\sum_{k=s+1}^{p}\left(\frac{1}{\theta_k^2}\right)\right) \tag{5.4}$$

$$\frac{\partial^2 \log L}{\partial \theta_k \partial \theta_j} =$$

$$\left(\left(-\sum_{i=1}^{n}\left(\sum_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)+\sum_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-\prod_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\prod_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-(p-2)\right)^{-2}\right)\right.$$
$$\left.\left(\sum_{k=s+1}^{p}\left(\frac{1+x_{ki}}{(1+\theta_k)^2}\right)-\prod_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\prod_{k=s+1}^{p}\left(\frac{1}{\theta_k(1+\theta_k)}\right)\right)-n\sum_{k=s+1}^{p}\left(\frac{1}{\theta_k^2}\right)\right) \tag{5.5}$$

One can exhibit the elements of the Fisher Information matrix for (5.3), (5.4) and (5.5) are

$$E\left(-\frac{\partial^2 \log L}{\partial \theta_j^2}\right) = \sum_{i=1}^{n} E\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-(p-2)\right)^{-2}$$
$$\left(\sum_{j=1}^{p}\left(\frac{1+E(x_{ji})}{(1+\theta_j)^2}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+E(x_{ji}))}{1+\theta_j}\right)\sum_{j=1}^{n}\left(\frac{1}{\theta_j(1+\theta_j)}\right)\right)-n\sum_{j=1}^{p}\left(\frac{1}{\theta_j^2}\right) \tag{5.6}$$

$$E\left(-\frac{\partial^2 \log L}{\partial\theta_j \partial\theta_k}\right) =$$

$$\left(\begin{array}{l}\left(\sum_{i=1}^{n} E\left(\sum_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)+\sum_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-\prod_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\prod_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-(p-2)\right)^{-2}\right) \\ \left(\sum_{k=s+1}^{p}\left(\frac{1+E(x_{ki})}{(1+\theta_k)^2}\right)-\prod_{j=1}^{q}\left(\frac{\theta_j(1+E(x_{ji}))}{1+\theta_j}\right)\prod_{k=s+1}^{k}\left(\frac{1}{\theta_k(1+\theta_k)}\right)\right)-n\sum_{k=s+1}^{p}\left(\frac{1}{\theta_k^2}\right)\end{array}\right)$$

(5.7)

$$E\left(-\frac{\partial^2 \log L}{\partial\theta_k \partial\theta_j}\right) =$$

$$\left(\begin{array}{l}\left(\sum_{i=1}^{n} E\left(\sum_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)+\sum_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-\prod_{j=1}^{s}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)\prod_{k=s+1}^{p}\left(\frac{\theta_k(1+x_{ki})}{1+\theta_k}\right)-(p-2)\right)^{-2}\right) \\ \left(\sum_{k=s+1}^{p}\left(\frac{1+E(x_{ki})}{(1+\theta_k)^2}\right)-\prod_{j=1}^{s}\left(\frac{\theta_j(1+E(x_{ji}))}{1+\theta_j}\right)\prod_{k=s+1}^{k}\left(\frac{1}{\theta_k(1+\theta_k)}\right)\right)-n\sum_{k=s+1}^{p}\left(\frac{1}{\theta_k^2}\right)\end{array}\right)$$

(5.8)

Result 5.2: Alternatively, the authors realized the computational complexity of the MLE arises from the non-linearity of the parameters. Therefore, a nonlinear programming technique using the Constrained ML method is employed to estimate the parameters of the Kind-1 Multivariate Lindley Distribution and the idea is to Maximizing the log-likelihood function from (5.1) under some restriction and parameter constraints as follows:

$$Max(\log L(\theta_1,\theta_2,\ldots,\theta_p)) = \sum_{i=1}^{n}\left(\log\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-(p-2)\right)\right) + n\sum_{j=1}^{p}\log\theta_j - \theta_j\sum_{i=1}^{n}x_{ji} \quad (5.9)$$

Subject to the constraints

$$\sum_{i=1}^{n}\left(\log\left(\sum_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-\prod_{j=1}^{p}\left(\frac{\theta_j(1+x_{ji})}{1+\theta_j}\right)-(p-2)\right)\right)+n\sum_{j=1}^{p}\log\theta_j-\theta_j\sum_{i=1}^{n}x_{ji}\le 0$$

$$\theta_1,\theta_2,\theta_3,\ldots,\theta_{p-1},\theta_p>0$$

6. Bivariate and Conditional Lindley distribution of Kind-1

Result 6.1: From (1.1), if p =2, then the PDF of **MLD of kind-1** reduces to **BLD of kind-1** as

$$f_{X_1,X_2}(x_1,x_2;\theta_1,\theta_2)=\left(\frac{\theta_1(1+x_1)}{1+\theta_1}+\frac{\theta_2(1+x_2)}{1+\theta_2}-\frac{\theta_1\theta_2(1+x_1)(1+x_2)}{(1+\theta_1)(1+\theta_2)}\right)(\theta_1\theta_2e^{-(\theta_1x_1+\theta_2x_2)}) \tag{6.1}$$

where $0<x_1,x_2<\infty$, $\theta_1,\theta_2>0$

Result 6.2: Figures 1-21 illustrates the shapes of the bivariate probability surfaces of equation (6.1) under various scale parameter settings.

(θ_1,θ_2) = (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3) and (4,4) are given as follows:

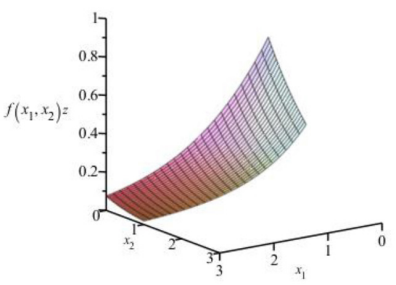


Figure 1

(θ_1, θ_2) = (1, 1)

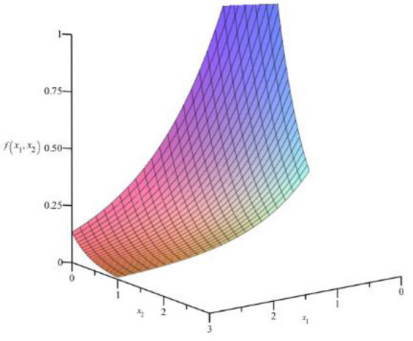


Figure 2

(θ_1, θ_2) = (1, 2)

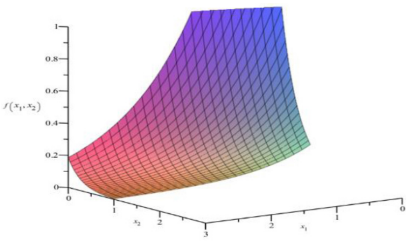


Figure 3
$(\theta_1, \theta_2) = (1, 3)$

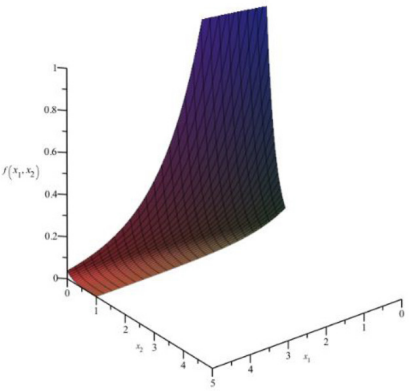


Figure 4
$(\theta_1, \theta_2) = (1, 4)$

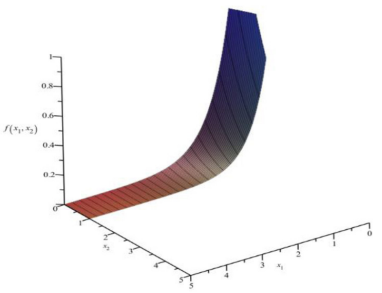


Figure 5
$(\theta_1, \theta_2) = (2, 1)$

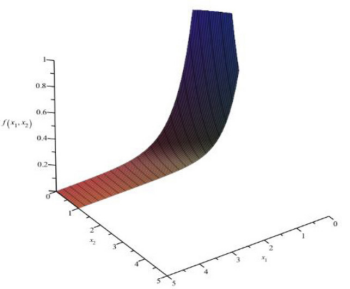


Figure 6
$(\theta_1, \theta_2) = (2, 2)$

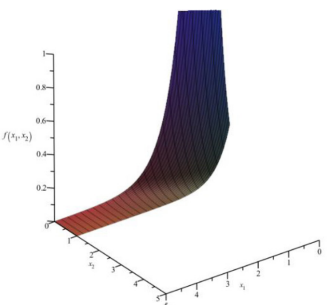


Figure 7
$(\theta_1, \theta_2) = (2, 3)$

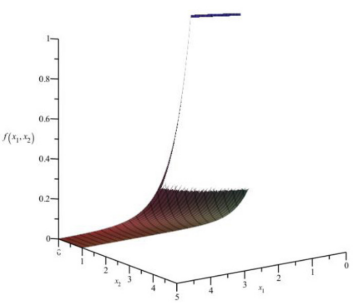


Figure 8
$(\theta_1, \theta_2) = (2, 4)$

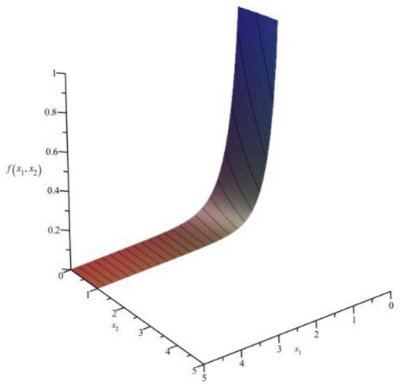


Figure 9

$(\theta_1, \theta_2) = (3, 1)$

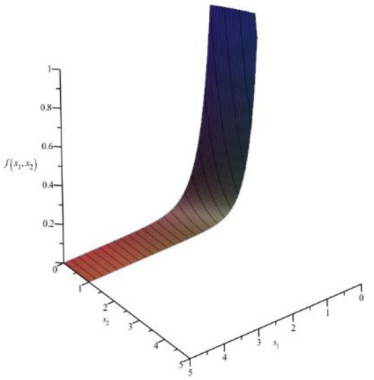


Figure 10

$(\theta_1, \theta_2) = (3, 2)$

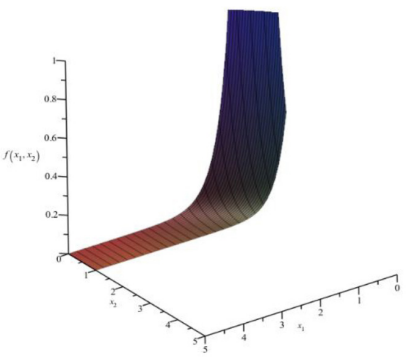


Figure 11

$(\theta_1, \theta_2) = (3, 3)$

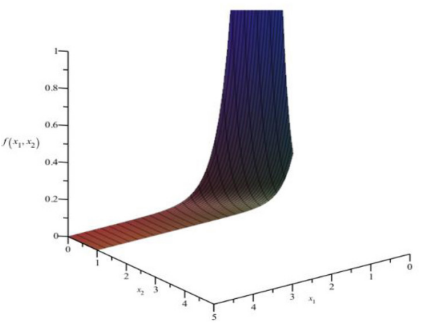


Figure 12

$(\theta_1, \theta_2) = (3, 4)$

Result 6.3: From (1.4), if p =2, then the density of Conditional MLD of Kind-1 is reduces to **Conditional BLD of Kind-1**

$$f_{X_1/X_2}(x_1 / x_2; \theta_1, \theta_2) =$$

$$\frac{1+\theta_2}{\theta_2(1+x_2)}\left(\frac{\theta_1(1+x_1)}{1+\theta_1}+\frac{\theta_2(1+x_2)}{1+\theta_2}-\frac{\theta_1\theta_2(1+x_1)(1+x_2)}{(1+\theta_1)(1+\theta_2)}\right)(\theta_1 e^{-\theta_1 x_1}) \qquad (6.2)$$

where $0 < x_1, x_2 < \infty$, $\theta_1, \theta_2 > 0$

Result-6.4 - The graph below depicts the shape of the conditional bivariate probability surfaces of equation (6.2) for different scale parameter settings.

(θ_1, θ_2) = (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2) and (3,3) are given as:

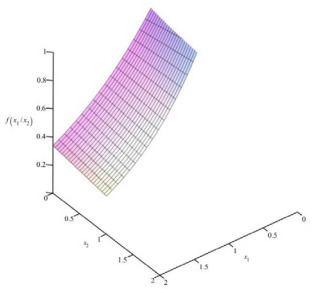


Figure 13

(θ_1, θ_2) = (1, 1)

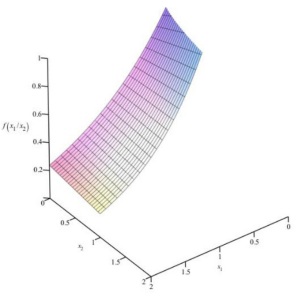


Figure 14

(θ_1, θ_2) = (1, 2)

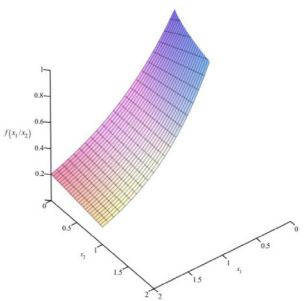


Figure 15

(θ_1, θ_2) = (1, 3)

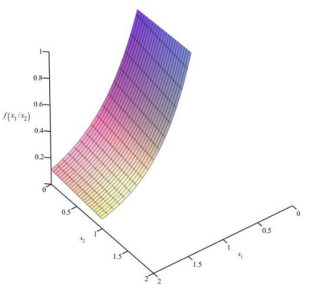


Figure 16

(θ_1, θ_2) = (2, 1)

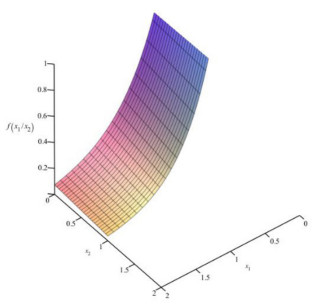


Figure 17

(θ_1, θ_2) = (2, 2)

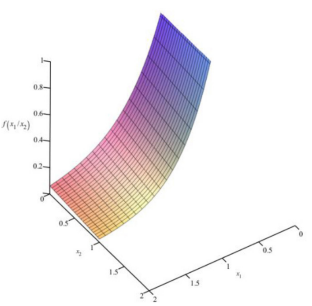


Figure 18

(θ_1, θ_2) = (2, 3)

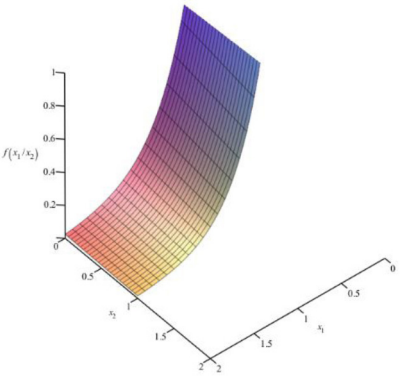


Figure 19

$(\theta_1, \theta_2) = (3, 1)$

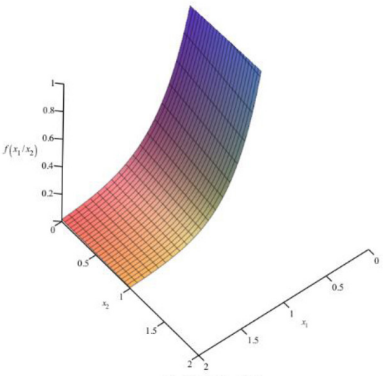


Figure 20

$(\theta_1, \theta_2) = (3, 2)$

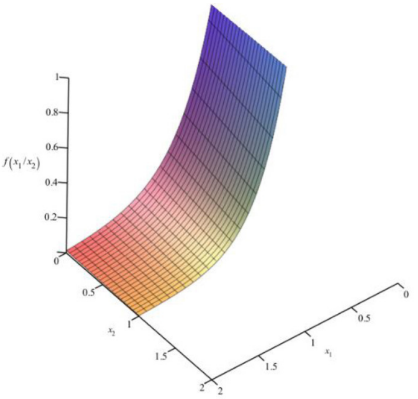


Figure 21

$(\theta_1, \theta_2) = (3, 3)$

7. Conclusion

To date, no natural multivariate generalization of the univariate Lindley distribution has been developed. In this work, the authors propose a new form called the Kind-1 Multivariate Lindley distribution. Notably, the marginal univariate distributions of the Kind-1 MLD are all univariate Lindley distributions. The Multivariate lifetime data can be easily modelled by the proposed distribution in the real time applications. Finally, the parameter estimation of the MLD of Kind-1 based on **MLE** is complex and

the advent of sophisticated mathematical softwares leads to solve the computational complexity of the parameter estimation.

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