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Multi-scenario reliability stress-strength models based on Topp-Leone and generalized Rayleigh distributions

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Abstract

This paper examines the role of reliability theory in system analysis, focusing on system representation, quantification, and uncertainty modeling. It explores various multi-scenario reliability stress-strength models based on Topp-Leone and Generalized Rayleigh distributions, covering environments with multiple stresses, strength variability, and constrained stress conditions. The study addresses modeling complexity in stress-strength relationships for multi-component, n-standby, and cascade systems. Several parameter estimation methods are evaluated, including maximum likelihood estimation, the Jackknife, and Bayesian estimators, with a Monte Carlo simulation used to compare their performance. The results indicate that maximum likelihood estimation and Jackknife methods are superior due to their low bias and Mean Squared Error across various scenarios and sample sizes, demonstrating robustness and reliability. Bayesian methods offer flexibility but require careful management of priors and data volume, while non-parametric methods tend to have higher bias and MSE, particularly in complex scenarios or with smaller sample sizes.

Subject Classification: *90B25.*

Keywords: *Reliability theory, Stress-strength models, Topp-Leone distribution, Generalized rayleigh distribution, Multi-stress environments, Adaptability, Constrained stress, Multi-component systems, n-standby systems, Cascade systems.*

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1. Introduction

Reliability theory emerged as a distinct scientific field in the 1950s, underwent specialization in the 1960s, and became integrated into risk assessment in the 1970s. By the 1980s and 1990s, it was deeply incorporated into system analysis, marked by significant methodological advancements and practical applications. Since its inception, reliability theory has centered around three core tasks: system representation and modeling, quantification of system models, and modeling and quantification of uncertainty [1]. However, the increasing complexity of systems has made these tasks more challenging in the present context. When referring to models in this context, we primarily discuss mathematical models, particularly probabilistic models. These models in reliability theory broadly fall into three categories: time-dependent models, stress-strength (S-S) models (also known as interference models), and time-dependent S-S models [3, 4]. Time-dependent models do not consider stress, while S-S models do not consider time; however, this does not imply the absence of the neglected factor; rather, its effect is either disregarded. Interference theory, within reliability theory, explores the relationship between a system's strength and the stress it undergoes during operation, which is the primary cause of its failure [5-8].

Definition 1: ($\boldsymbol{S}$ - $\boldsymbol{S}$ models, the interference models) Let X and Y are continuous random variable's its probability density function are $f_X(x;\Theta_1)$ and $g_Y(y;\Theta_2)$ respectively, let X represents the strength of a system (or component) and Y the operating under stress, the reliability of system (R) is defined as:

$$R = P(X \geq Y) \tag{1}$$

such models are termed S - S models or interference models.

The intersection of probability density functions determines the failure probability (1-R) for strength (X) and stress (Y). Interference Theory, initially developed in a nonparametric context by Mann-Whitney [2], focuses on comparing two random variables. Key elements include assessing system reliability through strength-stress models and hypothesis testing, with a comprehensive review provided by Johnson [1]. S-S models emerged from inference problems related to estimation, with notable contributions by Birnbaum [2], Birnbaum and McCarty [3], and, more recently, Yousif et al. [4], who explored stress-strength reliability using the Exponentiated Inverse Rayleigh Distribution (EIRD). Their study compared three estimation methods, highlighting the superiority of maximum likelihood. Hussain and Karam [5] analyzed reliability using the Generalized Inverted Kumaraswamy Distribution (GIKD) and found Bayes estimation superior. Almarashi, Algarni, and Nassar [6], examined stress-strength reliability with the Weibull distribution, finding that the percentile and maximum product of spacing methods are efficient. Xavier and Jose [7] introduced a new model using a generalized power-transformed half-logistic distribution, suitable for heavy-tailed data. Yousif et al. [4] also applied the New Weibull-Pareto Distribution and found maximum likelihood estimation superior. Biswas, Chakraborty, and Mukherjee [8] proposed the log-Lindley distribution, noting advantages of Bayesian methods.

Hamad and Salman [9] used the power Lomax distribution, evaluating several estimation methods. Rao, Mbwambo, and Pak [10] compared classical and Bayesian estimations of multicomponent stress-strength reliability using the exponentiated inverse Rayleigh distribution. Jana and Bera [11] focused on interval estimation for k-out-of-n systems with inverse Weibull distributions, comparing various confidence intervals. Cruz, Salinas, and Meza, [12] proposed a novel estimation procedure using the Unit-half-normal distribution and discussed multi-scenario reliability models with Topplion-Generalized Rayleigh Distributions. Overall, the paper contributes by developing a multi-scenario reliability model using distinct distributions for stress and strength variables, demonstrating various estimators' performance, and applying the model to edge-computing architecture systems. The paper's structure is similar to Patowary, Hazarika, and Sriwastav [13].

1.1 *Topp - Leone Distribution (TPL)*

The Topp-Leone (TPL) distribution, introduced in 1955, is a J-shaped continuous distribution used as an alternative to the Beta Distribution. Initially applied to failure data, its applications in reliability were expanded by Nadarajah and Kotz [14], who provided expressions for its moments and characteristic function. Ghitany, Kotz, and Xie [15] highlighted its useful properties, including the bathtub-shaped hazard rate and reversed hazard rate. Zghoul [16] explored its order statistics. The TPL distribution is particularly effective for modeling finite support datasets in reliability and life testing, making it valuable for stress-strength analysis.

Definition 2: The random variable X has the Topp-Leone Distribution if its probability density function $f(x)$ are given by:

$$f_{TPL}(x; b, v) = \begin{cases} \frac{2v}{b}\left(\frac{x}{b}\right)^{v-1}\left(1-\frac{x}{b}\right)\left(2-\frac{x}{b}\right)^{v-1} & \text{for } 0 < x < b \\ 0 & \text{else} \end{cases} \tag{2}$$

and its Cumulative Density Function (CDF) $F(x)$ is given by:

$$F_{TPL}(x; b, v) = \begin{cases} \left(\frac{x}{b}\right)^{v}\left(2-\frac{x}{b}\right)^{v} & \text{if } 0 \le x \le b \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge b \end{cases} \tag{3}$$

where $0 < v < 1$ represents the scale parameter and $b > 0$ represents the shape parameter.

One of the famous shapes of (2) are discussed in many authors when $(b = 1)$.

1.2 *Generalized Rayleigh Distribution (GRL)*

(Yousif, et al., [4] introduced cumulative distribution functions for modeling lifetime data, with Burr-Type X and XII distributions receiving notable attention. Additionally, Raqab and Madi [17], developed the

Generalized Rayleigh (GRL) distribution, a two-parameter model that has proven effective in analyzing lifetime data and stress-strength scenarios.

Definition 3: The random variable Y distributed as Generalized Rayleigh Distribution, then its Probability Density Function is given by:

$$g_{GRL}(y;\alpha,\lambda)=\begin{cases}2\alpha\lambda^2 y e^{-(\lambda y)^2}\left(1-e^{-(\lambda y)^2}\right)^{\alpha-1} & \text{if } y>0,\alpha>0,\lambda>0,\\ 0 & \text{otherwise}\end{cases} \tag{5}$$

and its Cumulative Density Function is given by:

$$G_{GRL}(y;\alpha,\lambda)=\left(1-e^{-(\lambda y)^2}\right)^{\alpha};\ y>0,\alpha>0,\lambda>0. \tag{6}$$

Where, $\alpha>0$ is the shape parameter and $\beta>0$ is scale, and denoting $\sim GR\,L(\alpha,\beta)$.

2. Reliability Structures

A system can be either single-component or multi-component. In multi-component systems, each component's reliability is assessed individually, with each treated as a system. The overall reliability of the system is determined by its structure and probability principles. Typically, the stress-strength (S-S) relationship for components is assumed independent unless otherwise specified. While most studies focus on single-component systems, early works on $P(X>Y)$ under parametric assumptions were incorrectly attributed to Owen, Craswell, and Hanson [18], as Lloyd and Lipow [19] previously studied normal scenarios for X and Y. Later studies (Disney, Sheth, and Lipson [20]) examined failure probabilities for systems where both stress and strength followed normal, gamma, or Weibull distributions.

2.1 *Stress-strength models adhering to TPL and GRL distributions*

Similar to (Pandit and Sriwastav [21]) and (Kapur and Lamberson [22]), we introduce stress and strength as follows:

Definition 4: Let $R^{(\Theta_1,\Theta_2)}=P(X^{\Theta_1}>Y^{\Theta_2})$ are the reliability model based on X and Y, where $\Theta=(\Theta_1,\Theta_2);\Theta_1=(b,v)$ and $\Theta_2=(\alpha,\beta)$. Then:

$$R^{(\Theta_1,\Theta_2)}=\int_0^{\infty}[1-F_X(y;\Theta_1)]g_Y(y;\Theta_2)dy \tag{7}$$

If we take into account $X\sim \mathrm{TPL}(b,v)$ and $Y\sim \mathrm{GR\,L}(\alpha,\beta)$ then:

$$R^{(\Theta_1,\Theta_2)}=G_Y(b)-m(\alpha,\beta,v,b) \tag{8}$$

where: $m(\alpha,\beta,v,b)=2\alpha\beta^2\left(\frac{2}{b}\right)^{v}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\left(\frac{1}{2b}\right)^{j}\frac{\Gamma(\alpha)}{i!\Gamma(\alpha-i)}\Gamma\left(\frac{1}{2}v+\frac{1}{2}j+1,\beta^2(i+1)b^2\right)$

2.2 *Scenarios of Reliability*

2.2.1 Reliability in One-strength Multi-Stress Model

A system or component can experience failure in multiple ways. When a system is exposed to various stresses simultaneously, it will continue to function if each stress is below the system's strength. For instance, an electrical device might malfunction due to voltage fluctuations, mechanical stresses (such as vibrations), moisture, and so forth.

Definition 5: Suppose a system with strength X can fail in k distinct ways, each associated with a different stress $Y_1, Y_2, \dots, Y_k$. In such cases, the reliability of the system is determined by:

$$R = \prod_{i=1}^{k} P[X \geq Y_i] \tag{9}$$

Let Y_i are independent stress distributed as $g\left(y, \Theta_2^{(i)}\right)$, then One-strength multi-Stress model define as:

$$R = \prod_{i=1}^{k} P\left(X^{\Theta_1} > Y_2^{\Theta_2^{(i)}}\right) = \prod_{i=1}^{k} R^{\left(\Theta_1, \Theta_2^{(i)}\right)} \tag{10}$$

2.2.2 Multi-component systems

In multi-component systems, each component's strength-stress (S-S) relationship is considered, necessitating an understanding of how different components are arranged within the system and their respective S-S distributions. Additionally, it's important to determine whether components operate independently, incorporate redundancy, or exhibit other characteristics. Depending on these factors, various S-S models can be applied, resulting in diverse forms of reliability expressions. Multi-component systems encompass various types, including series, parallel, standby, or combinations thereof. A variant of the series system is a chain, wherein the overall strength equals that of its weakest link. The reliability, denoted as R_n, of a chain comprising n identical links is expressed as:

$$R_n = \int_0^{\infty} [1 - F(x)]^n g(x) dx \tag{12}$$

where $g(y)$ represents the density function of stress Y applied to the chain, and $F(x)$ signifies the distribution function of the strength of chain links (citation). if $X \sim TPL(b, v)$ then $F(x)^k = F(x; b, vk)$, and:

$$R_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k m(\alpha, \beta, vk, b) \tag{13}$$

where $m(\alpha, \beta, vk, b)$ are defined as ()

2.2.3 n-standby system

Definition 6: The an *n-standby system* reliability, denoted as R_n, expressed as following: (Sriwastav & Kakati 1981a)

$$R_n = R(1) + R(2) + \cdots + R(n) \tag{14}$$

where $R(i)$ represents the marginal reliability attributable to the i^{th} component, defined as:

$$R(i) = P[X_1 < Y_1, X_2 < Y_2, \dots, X_{i-1} < Y_{i-1}, X_i \geq Y_i] \tag{15}$$

with (X_i, Y_i) representing the strength-stress pair of the i th component, for $i = 1,2, \dots, n$. If we consider (X_i, Y_i) to be independent random variables distributed as TP $\mathrm{L}\left(\Theta_1^{(i)}\right)$ and GR $\mathrm{L}\left(\Theta_2^{(i)}\right)$ respectively for all $i = 1,2, \dots, n$, then:

$$R(i) = P\left(X^{\Theta_1^{(i)}} > Y_2^{\Theta_2^{(i)}}\right) \prod_{k=1}^{i-1} P\left(X^{\Theta_1^{(k)}} < Y_2^{\Theta_2^{(k)}}\right) \tag{16}$$

2.2.4 Cascade system

An n-cascade system, as described by Pandit and Srivastav (Pandit and Srivastav [21], functions as an n-standby system with a distinctive feature: the stress faced by a new component replacing a failed one is attenuated by a factor denoted as k. This k factor could be a constant, a component-specific parameter, or even a random variable, referred to as an attenuation factor. Consequently, after the failure of the i th component $(i = 1,2,\dots,n)$, the $(i+1)^{th}$ component encounters a stress $Y_{i+1} = k_i Y_i$.

Definition 7: The an n-standby system reliability, computed similarly to (14), where:

$$R(i) = P[X_1 < Y_1, X_2 < k_1, Y_1, \dots, X_{i-1} < k_{i-2}Y_1, X_i \geq k_{i-1}Y_1]. \tag{17}$$

In contrast to Equation (15), Equation (17) adds complexity by making $Y_1,, \dots, Y_n$ dependent on Y_1, eliminating the use of the product rule for independent events as in Equation (). As a result, deriving general expressions for $R(i)$ in cascade systems becomes more difficult. However, if we assume $k_i = c > 0$ as constant for all iii, simplifications can be made. Let $\Theta_2^{*(i)} = (\alpha_i, \beta_i/c)\forall i$ and $c > 1$. Then:

$$\begin{aligned} R(i) &= P[X_1 < Y_1, X_2 < cY_1, \dots, X_{i-1} < c^{i-2}Y_1, X_i \geq c^{i-1}Y_1] \\ &= R^{\left(\Theta_1^{(i)}, \Theta_2^{*(1)}\right)} \prod_{k=1}^{i-1} \left[1 - R^{\left(\Theta_1^{(k)}, \Theta_2^{*(k)}\right)}\right]. \end{aligned} \tag{18}$$

3. Model Inference

After choosing a reliability model for a system, the next step is to estimate various reliability characteristics and test related hypotheses through system testing. This inference problem can be addressed using two main approaches:

3.1 *Maximum Likelihood Estimation*

Without loss of generality, if we take into account Nadarajah's model (where $\mathrm{b} = 1$), the log-likelihood function $\mathcal{L}(v, \alpha, \beta)$ is expressed as:

$$\mathcal{L}(v,\alpha,\beta) = \sum_{i=1}^{n} \ln\left(4v\alpha\beta^2 y_i \cdot x_i^{v-1} \cdot (1-x_i) \cdot (2-x_i)^{v-1} \cdot e^{-(\beta y_i)^2 + (\alpha-1)\ln\left(1-e^{-(\beta y_i)^2}\right)}\right). \tag{19}$$

So $\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}, \hat{v}_{MLE}$ are given by solving the following nonlinear equations:

$$\frac{\partial \mathcal{L}}{\partial v} = n \cdot \frac{1}{v} + \sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} \ln(2 - x_i) = 0, \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = 2n \cdot \frac{1}{\beta} - 2\beta \sum_{i=1}^{n} (y_i)^2\left(1 - e^{-(\beta y_i)^2}\right) = 0, \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln\left(1 - e^{-(\beta y_i)^2}\right) = 0. \tag{22}$$

3.2 *Bayesian Estimator*

Given the joint density function $f_{X,Y}(x, y; v, \alpha, \beta)$, the squared error loss function, and the gamma priors for v, α, and β, we can compute the joint posterior distributions as:

$$f(v, \alpha, \beta \mid x, y) \propto f_{X,Y}(x, y; v, \alpha, \beta) \cdot f_{\Gamma(\delta_1)}(v) \cdot f_{\Gamma(\delta_2)}(\alpha) \cdot f_{\Gamma(\delta_3)}(\beta) \qquad (13)$$

where $f_{\Gamma(\delta_1)}(v), f_{\Gamma(\delta_2)}(\alpha)$, and $f_{\Gamma(\delta_3)}(\beta)$ are the gamma priors with parameters $\delta_i = (a_i, b_i)\ \forall i = 1,2,3$. According to (28), obtaining the closed form of the posterior distribution is analytically challenging and **practically impossible** using classical estimation methods. Therefore, we can utilize sampling methods such as Markov Chain Monte Carlo (MCMC).

3.3 *Non-parametric Estimator*

(Birnbaum [2]) pioneered the non-parametric estimation of $= P(X < Y) = 1 - R$, under the assumption of X and Y being independent continuous random variables. He expressed p as $p = U_{nm}$, where n and m are the sample sizes of X and Y, respectively, and U represents the Mann-Whitney U statistic. This statistic is defined as the count of pairs (X_i, Y_j) where $X_i < Y_j$, with $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$. Birnbaum derived one-sided and two-sided confidence intervals for large sample sizes. Other researchers who delved into this problem include Birnbaum and McCarty [22-25].

3.4 *Bootstrap estimation and Bootstrap confidence interval*

To find Bootstrap estimation and Bootstrap confidence interval we use the following algorithms like (Kumar, Dey, and Saha [26]

Algorithm : Bootstrap method

Step 1: Based on $(X, Y) = (x_i, y_i)_{\mathrm{i=1}}^{\mathrm{n}}$ SS sample, Find $\hat{\theta}_1^{MLE}$ and $\hat{\theta}_2^{MLE}$ MLE Estimators.
Step 2: Generate $B = 10000$ parametric Bootstrap Samples i.e., generate parametric
Bootstrap samples (X_{ij}, Y_{ij}) Based on $\hat{\theta}_1^{MLE}, \hat{\theta}_2^{MLE}\ \forall(i = 1 \dots B\ \&\ j = 1, \dots, n)$
Step 3: Compute $\hat{\theta}_1^{(i)}$ and $\hat{\theta}_2^{(i)}$ for each bootstrap sample, then compute $\hat{R}^{(i)}$ $\forall i = 1 \dots B$
Step 4: Compute Bootstrap Estimator $\bar{R} = \frac{\sum_i \hat{R}^{(i)}}{B}$.
Step 5: Compute the differences $d_i = \hat{R}^{(i)} - \bar{R}, \forall i = 1 \dots B$. and Sort d_i in increasing order.
Step 6: Compute Percentiles.
(i) Compute lower percentile = 100(1 - α / 2).
(ii) Compute upper percentile = 100(α / 2).
Step 7: Compute Confidence Interval.
(i) Compute lower limit of CLI = $\bar{R}$ - lower percentile of d.
(ii) Compute upper limit of CUI = $\bar{R}$ - upper percentile of d.

4. Monte Carlo Simulation

4.1 *R-Monte Carlo Simulation*

In Monte Carlo simulation, multiple replications are executed (1000 times) to accommodate the intrinsic randomness and variability in the dataset. Various sample sizes are investigated. $(25,50)$ to elucidate how the performance of estimators evolves with different data quantities. Two sets of parameters are scrutinized: Θ_1, representing the parameters of the Toppleone distribution with values of $\nu = 0.3, 0.5, 0.7$, and 0.9, while $b = 1$; and Θ_2, representing parameters of the Generalized Rayleigh distribution. Θ_2 is delineated as pairs of values, including $(0.5,0.5), (1,1), (0.5,1), (2,2), (3,\ 3.5)$, and $(4.5,1)$. The simulation evaluates various estimators, encompassing *MLE*, *Jackknife estimators*, and *Bayesian* estimation using a standard gamma prior for all distributions. The R-VGAM package is employed to generate random samples and furnish tailored functions for the proposed model.

4.2 *n-standby -Monte Carlo Simulation*

For n=25, *MLE* shows moderate accuracy with *biases* ranging from -0.025 to 0.025 and *MSE* from 0.0014 to 0.0037, indicating reliable performance across various ν and Θ_2 settings. Jackknife exhibits similar bias levels to *MLE*, reflecting stable performance. Bayesian methods perform well with lower ν values but show increased bias with higher v. non-parametric methods generally have higher bias and *MSE*, indicating challenges with smaller sample sizes or higher variability. For $n = 50$, *MLE* maintains stable performance with biases from -0.019 to 0.022 and *MSE* from 0.001 to 0.0015. *Jackknife* continues to demonstrate consistent results. Bayesian methods show similar trends to *n*=25, while *NP* methods display higher bias and *MSE*, especially in scenarios with larger *v* values and smaller sample sizes. Increasing *n* improves bias and *MSE* performance across all methods. *MLE* and Jackknife are robust for parameter estimation in *R n-standby* scenarios. Bayesian methods are flexible but require careful management, while non-parametric methods need larger sample sizes or more robust techniques for effective reliability analysis.

4.3 *n-casecade -Monte Carlo Simulation*

The results demonstrate the performance of the ML, JAC, and BAYESIAN estimators across different values of cc. For the ML estimator, both bias and MSE decrease as cc increases, indicating improved accuracy and reduced error with larger cc. In contrast, the JAC estimator shows minimal bias and consistently lower MSE compared to ML, regardless of cc, demonstrating its robustness. The BAYESIAN estimator exhibits variable behavior; it tends to underestimate with a negative bias that intensifies with higher cc, and its MSE lacks a clear trend due to its stochastic nature.

Overall, the ML estimator performs better with higher cc, the JAC estimator remains stable and reliable, and the BAYESIAN estimator shows inconsistent performance. Future work should focus on optimizing these estimators, particularly improving BAYESIAN performance and understanding its variability.

6. Conclusion

*This paper introduced a Streng*th-Stress model derived from topplone and generalized relight models, which was subsequently expanded into several variants, including the One-strength multi-Stress model, Strength Variability in Stress-Reliability Modeling, Constrained Stress Environments, Mixing Modeling in Stress-Strength, Multi-component Systems, n-standby System, and Cascade System. Notably, the model described in Definition 4 shows increasing behavior in one parameter and decreasing behavior in another. The models in Definitions 5 exhibit decreasing reliability with an increasing number of elements k, indicating that higher element counts lead to greater system failure likelihood. The simulation analysis reveals distinct performance characteristics among various estimators. *MLE* and *Jackknife* estimators generally exhibit low bias and *MSE*, with Jackknife showing more consistent performance across scenarios. Bayesian estimators display variable *bias* and *MSE*, performing well with lower complexity but increasing bias at higher complexity. *Non-Parametric* estimators consistently show higher *bias* and *MSE*, especially in complex scenarios, suggesting the need for larger samples or enhanced robustness. Increasing sample size from 25 to 50 improves *bias* and *MSE* performance across all methods. *MLE* and *Jackknife* methods are particularly robust, while Bayesian methods require careful management of priors and data volume, and *Non-Parametric* methods exhibit limitations in accuracy. In standby simulations, for $n = 25$, *MLE* shows moderate accuracy with biases from *-0.025* to *0.025* and *MSE* from *0.0014 to 0.0037*. *Jackknife* performs similarly to MLE. Bayesian methods vary, with increased bias at higher v , while NP methods show higher *bias* and *MSE*. For n = 50, *MLE* maintains stability with *biases* from -0.019 to 0.022 and *MSE* from 0.001 to 0.0015. *Jackknife* remains consistent, *Bayesian* methods show similar patterns to $n = 25$, and NP methods continue to have higher *bias* and *MSE*. Overall, increasing *n* from *25 to 50* improves performance across methods, with MLE and Jackknife methods proving robust for parameter estimation. Performance across various values of c shows that the *MLE* benefits from higher c values, improving accuracy and reliability. The *JAC* estimator remains stable with minimal *bias* and lower MSE compared to ML. *BAYESIAN* exhibits variability with more pronounced negative *bias* and inconsistent *MSE*, highlighting the need for future optimization of *BAYESIAN* performance.

7. Appendix

7.1 *Notification 1*

According to Prudnikov,(1986), we have:

$$(1 + x)^{\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)} x^{j} \tag{14}$$

7.2 *Notification 2*

$$\int_0^b y^{v+j+1} e^{-\beta^2(i+1)y^2} dy = \frac{1}{2\beta^{v+j+2}(i+1)^{\frac{1}{2}v+\frac{1}{2}j+1}} \Gamma\left(\frac{1}{2}v + \frac{1}{2}j + 1, \beta^2(i+1)b^2\right) \tag{15}$$

where, $\Gamma(a,x)$ is the lower incomplete gamma function defined as: $\Gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$

7.3 *Definition (4) Proof:*

$$R = \int_0^b \bar{F}_X(y) g_Y(y) dy = G_Y(b) - 2\alpha\beta^2 \int_0^b \left(\frac{y}{b}\right)^v \left(2-\frac{y}{b}\right)^v \cdot y \cdot e^{-(\beta y)^2} \left(1-e^{-(\beta y)^2}\right)^{\alpha-1} dy$$

According to Notification 1, we have:

$$\left(2-\frac{y}{b}\right)^v = 2^v\left(1-\frac{y}{2b}\right)^v = 2^v \sum_{j=0}^{\infty} \frac{\Gamma(v+1)}{j!\Gamma(v+1-j)}\left(-\frac{1}{2b}\right)^j y^j, \quad (16)$$

$$\left(\frac{y}{b}\right)^v\left(2-\frac{y}{b}\right)^v = \left(\frac{2}{b}\right)^v y^v \left(1-\frac{y}{2b}\right)^v = \left(\frac{2}{b}\right)^v \sum_{j=0}^{\infty} \frac{\Gamma(v+1)}{j!\Gamma(v+1-j)}\left(-\frac{1}{2b}\right)^j y^{j+v}, \quad (17)$$

$$\left(1-e^{-(\beta y)^2}\right)^{\alpha-1} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha)}{i!\Gamma(\alpha-i)}(-1)^i\left(e^{-\beta^2y^2}\right)^i = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha)}{i!\Gamma(\alpha-i)}(-1)^i e^{-i\beta^2y^2}, \quad (18)$$

$$e^{-\beta^2y^2}\left(1-e^{-\beta^2y^2}\right)^{\alpha-1} = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha)}{j!\Gamma(\alpha-i)}(-1)^i e^{-(i+1)\beta^2y^2}, \quad (19)$$

Then:

$$R = G_Y(b) - 2\alpha\beta^2\left(\frac{2}{b}\right)^v \sum_{j=0}^{\infty} \frac{\Gamma(v+1)}{j!\Gamma(v+1-j)}\left(-\frac{1}{2b}\right)^j \left[\sum_{i=0}^{b} \frac{\Gamma(\alpha)}{j!\Gamma(\alpha-i)}(-1)^i \int_0^b y^{j+v+1}e^{-(i+1)\beta^2y^2}\right]$$

$$= G_Y(b) - 2\alpha\beta^2\left(\frac{2}{b}\right)^v \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (-1)^{i+j}\left(\frac{1}{2b}\right)^j \frac{\Gamma(\alpha)}{i!\Gamma(\alpha-i)}\frac{\Gamma(v+1)}{j!\Gamma(v+1-j)}\left(\int_0^b y^{j+v+1}e^{-(i+1)\beta^2y^2}\right), \quad (20)$$

so using notification 2 we have:

$$R = G_Y(b) - 2\alpha\beta^2\left(\frac{2}{b}\right)^v \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (-1)^{i+j}\left(\frac{1}{2b}\right)^j \frac{\Gamma(\alpha)}{i!\Gamma(\alpha-i)}$$
$$\times \frac{\Gamma(v+1)}{j!\,\Gamma(v+1-j)} \frac{1}{2\beta^{v+j+2}(i+1)^{\frac{1}{2}v+\frac{1}{2}j+1}} \Gamma\left(\frac{1}{2}v+\frac{1}{2}j+1, \beta^2(i+1)b^2\right),$$

where $\Gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ are is lower incomplate gamma function.

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