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Decision making under risk and uncertainty

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Abstract

In this paper, the decision making in risk and decision making under uncertainty are studied, and some examples are presented. This paper is continuation of paper [29] in References.

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1. Decision making in risk

The problem of making a decision in risk arises in cases when with every decision x_i made, a *set* of possible results $\{y_1, \dots, y_n\}$ is connected, where the conditional probabilities $P(y_i/x_i)$ of y_i subject to x_i are known.

This problem can be presented in the form of the following table (matrix)

$x_i \backslash y_i$	y_1	$\dots$	y_j	$\dots$	y_n
x_1	u_{11}	$\dots$	u_{1j}	$\dots$	u_{1n}
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_i	u_{i1}	$\dots$	u_{ij}	$\dots$	u_{in}
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_m	u_{m1}	$\dots$	u_{mj}	$\dots$	u_{mn}

where u_{ij} is the utility of the result y_j when using the decision x_i.

Let the conditional probabilities $P(y_j/x_i)$, $i = 1, \dots, m$, $j = 1, \dots, n$, be given.

For each decision (strategy), an expected utility is introduced in terms of the mathematical expectation:

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$$E\{u(x_i)\} = \sum_{j=1}^{n} u_{ij} P(y_j/x_i), \quad i = 1, \dots, m. \tag{1}$$

The decision rule for determining the optimal strategy is determined as follows

$$E\{u(x_i)\} = \max_{x_k} E\{u(x_k)\}. \tag{2}$$

Problems and topics, considered in this paper, and related to them are studied in reference titles [1]-[32].

2. Decision making under uncertainty

One of the important factors in the problem of decision making under uncertainty is the external environment (situation), which can be in one of the states $S_1, \dots, S_k$, that are known to the decision maker.

The mathematical model of the decision problem under uncertainty can be stated as follows.

Given an $m \times n$ matrix U (as in the decision problem in risk), whose elements u_{ij} are the utilities of the result y_j, $j = 1, \dots, n$, when using the decision (strategy) x_i, $i = 1, \dots, m$, that is,

$$u_{ij} = U(y_j, x_i), \quad i = 1, \dots, m, j = 1, \dots, n.$$

Depending on the state of the environment, the result y_j is achieved with probability $P(y_j/x_i, S_k)$.

The observer does not know the probability distribution $P(S_k)$, and concerning the state of the environment he can express certain hypotheses. His assumptions about the possible state of the environment are called *subjective probabilities* $P_s(S_k)$, $k = 1, \dots, K$.

If the value $P(S_k)$ is known to the observer, then we would have a problem of making a decision in risk, considered in the previous section. In such a case, the decision rule x_i is determined as follows

$$\max \sum_{j=1}^{n} \sum_{k=1}^{K} u(y_j, x_i) P(y_j/x_i, S_k) P(S_k). \tag{3}$$

In practice, the state of the environment and the probability distributions $\{P(S_k)\}$, $k = 1, \dots, K$, are unknown.

There are several criteria for choosing an optimal strategy.

Wald criterion (*Wald test*, *criterion of the careful observer*, after Abraham Wald).

For this criterion, the utility is optimized assuming that the environment is in a state which is unfavorable to the observer. In this case, the decision rule is of the form

$$\max_{x_i} \min_{S_k} u(x_i, S_k),$$

where

$$u(x_i, S_k) = \sum_{j=1}^{n} u(y_j/x_i, S_k) P(y_j/x_i, S_k). \tag{4}$$

When applying this criterion, a strategy is chosen that gives a guaranteed profit in the worst-case scenario.

Hurwicz criterion (after Leonid Hurwicz).

This criterion is based on the following two assumptions: the environment (situation) may be in the most unfavorable state with probability $1-p$ and in the most favorable state with probability p, where p is a *confidence factor* (*coefficient of optimism*).

Then the decision rule is written as follows

$$\max_{x_i}\,[p \max_{S_k} u(x_i,S_k) + (1-p) \min_{S_k} u(x_i,S_k)], \quad 0 \le p \le 1. \tag{5}$$

If $p=0$, we obtain the Wald criterion. If $p=1$, we get a decision rule of the form

$$\max_{x_i}\ \max_{S_k}\ u(x_i,S_k),$$

which is called the *optimist's strategy* (*strategy of the optimist*), who believes in success.

Laplace criterion (*the equal likelihood criterion*, after Pierre-Simon Laplace).

If the environment states are unknown, then all states are considered equally likely:

$$P(S_1) = P(S_2) = \cdots = P(S_k).$$

The decision rule in this case is defined as in (3) with $P(S_k) = \frac{1}{K}$.

Savage criterion (*Savage minimax regret criterion*, after Leonard Savage).

When applying this criterion, we minimize a value, equal to the change of the utility of the result at a given state of the environment with respect to its best possible state.

The value to be minimized is determined as follows. Construct a matrix $U = (u_{ik})$, where $u_{ik} = u(x_i,S_k)$, $i = 1,\dots,m$; $k = 1,\dots,K$.

The maximal element u_k of each column of this matrix is found:

$$u_k = \max_i\ u_{ik}, \quad k = 1,\dots,K.$$

The value of the maximal element u_k is subtracted from each element of this column.

Then we construct the matrix

$$U_S = (u_{ik}^S), \quad u_{ik}^S = u_{ik} - u_k.$$

The decision rule for choosing an optimal strategy according to the Savage criterion can be presented as follows

$$\max_{x_i} \min_{S_k} u_{ik}^S(x_i,S_k).$$

Consider the application of the criteria for decision making under uncertainty in the following real-world situation.

Example 1: Transport carrier plans to organize passenger transports during the summer months. The number x of vehicles, and the number of crews to be provided and prepared for the incoming summer, is a variable and it is determined by the actual needs for the carriage of passengers in the upcoming season. Let x

can take the values 10, 20, 30, 40, and 50 vehicles. The actual need for passenger transport is a random variable, which depends on many unknown factors. Let a plan be made about the operating costs and let the value of the expected income be determined as a result of the realization of the transportation, depending on the number of vehicles x_i and the real need of them to fully satisfy the needs S_k for carriage of passengers. The expected income values for all possible values of x_i and S_k are given in the following table with elements u_{ik}:

$x_i \backslash S_k$	10	20	30	40	50
10	60	60	60	60	60
20	10	110	110	110	110
30	–48	30	160	160	160
40	–100	–50	200	240	240
50	–150	–100	50	200	340

Determine the optimal number x_{opt} of the necessary vehicles, so that the expected income be *maximal*.

Using the criteria described above, we will find the optimal number of vehicles.

Wald criterion. Using this criterion, we have to find

$$\max_{x_i} \min_{S_k} u_{ik} = \max_{x_i} \{60; 10; -48; -100; -150\} = 60.$$

Since the maximum value 60 is attained for the first row of the table, then $x_{opt} = 10$.

Hurwicz criterion. For this criterion, the optimal solution x_{opt} is determined by the condition

$$\max_{x_i} [p \max_k u_{ik} + (1-p) \min_k u_{ik}], \quad 0 \le p \le 1.$$

The table (matrix) $H = (h_{ip})$ of expected income according to this criterion is

$x_i \backslash p$	0.1	0.2	0.5	0.8	0.9
10	60	60	60	60	60
20	20	30	60	90	110
30	–27.2	–6.4	56	118.4	139.2
40	–66	–32	70	172	206
50	–101	–52	95	242	291

where

$$h_{ip} = p \max_{S_k} u_{ik} + (1-p) \min_{S_k} u_{ik}.$$

Therefore, according to the Hurwicz criterion, the optimal number of vehicles, depending on p, will be

p	0.1	0.2	0.5	0.8	0.9
x_{opt}	**20**	20	20	**50**	50

Laplace criterion. Calculate

$$\max_{x_i} \frac{1}{5}\sum_{k=1}^{5} u_{ik} = \max\frac{1}{5}\{60+60+60+60+60; 10+110+110+110 +110; -48+30+160+160+160; -100+(-50) +200+240+240; -150+(-100)+50+200+340\}$$
$$= \max\{60; 90; 92.4; 106; 72\} = 106.$$

Therefore $x_{opt} = 40$ because the maximal value 106 is attained for the 4th row of the table for x_i and S_k.

Savage criterion. Construct the matrix $U_S = (u_{ik}^S)$ and enter the results in the following table

$x_i \backslash S_k$	10	20	30	40	50
10	0	–50	–140	–180	–280
20	–50	0	–90	–130	–230
30	**–108**	–80	–40	–80	–80
40	–160	–160	0	0	–100
50	–210	–210	–150	–40	0

Calculate

$$\max_{x_i} \min_{S_k} u_{ik}^S = \max\{-280; -230; -108; -160; -210\} = -108.$$

Therefore

$$x_{opt} = 30.$$

The obtained results can be interpreted economically as follows:

- using Wald criterion, 10 vehicles should be provided,
- using Hurwicz criterion, 20 vehicles should be provided, if carrier managers are pessimists, and 50 vehicles if they are optimists,
- using Laplace criterion, 40 vehicles should be provided,
- using Savage criterion, 30 vehicles should be provided.

3. Conclusions

The decision theory is considered in this paper. Decision making in risk and decision making under uncertainty are studied, and some examples are presented. This paper is continuation of paper [29] in References.

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