Journal of Statistics & Management Systems
ISSN 0972-0510 (Print), ISSN 2169-0014 (Online)
Vol. 28 (2025), No. 5, pp. 941–957
DOI : 10.47974/JSMS-1417

Extended two-piece double generalized gamma distribution and its applications

Tao Peng [†]
Xi-Yuan Qian *
School of Mathematics
East China University of Science and Technology
Shanghai 200237
China

Abstract

A novel bimodal modification of the gamma distribution is developed in this article by combining the double ganeralized gamma distribution with the extended two-piece transformation, named the extended two-piece double generalized gamma distribution (ETDGG). The epsilon-skew parameterization method is used to introduce asymmetric to this distribution. The statistical properties of the ETDGG distribution are derived, including cumulative distribution function (CDF), probability density function (PDF), moments, quantiles, and moment generation functions. Two parameter estimation methods (moment and MLE) are explored.In order to assess the performance and reliability of the estimation method, Monte Carlo simulation is carried out. Finally, to demonstrate the adaptability and effectiveness of the proposed distribution, two real world datasets are used. The findings present that the proposed ETDGG distribution provides a better fit than other considered distributions.

Subject Classification: *60E05, 62E15, 62F10, 91G15.*

Keywords: *Extended two-piece distribution, Asymmetric, Double generalized gamma distribution, Maximum likelihood estimation, Financial data analysis.*

1. Introduction

Bimodal distributions arise naturally in many different scenarios. Finding appropriate probabilistic models that can explain the occurrence of bimodality is an issue of vital importance. The bimodal distribution structure has been extensively investigated by scholars. Sulewski constructed a two-piece power Normal distribution ($TPPN$) [1] and Gómez et al. proposed a Bimodal Truncation Positive Normal distribution ($BTPN$) in [2]. Hassan and Hijazi introduced a novel

[†] *E-mail:* **1206022731@qq.com**
* *E-mail:* **xyqian@ecust.edu.cn (Corresponding Author)**

statistical model known as the bimodal exponential distribution [3]. Juric introduced the asymmetric double Weibull distribution (ADW) [4], Vila and Çankaya introduced the bimodal Weibull distribution ($BWeibull$) with the quadratic transformation based on the Weibull distribution [5]. Abdulah presented the ε-skew gamma distribution (ESG) and epsilon-skew inverted gamma distribution in [6] and [7], Among them, epsilon-skew is a method of adding skewness, whose construction makes the cumulants distributed on both sides of the scale parameter obey the binomial distribution, which may be helpful for the study of the difference in the proportion between two independent samples. For example,epidemiological research often investigates gender disparities in the occurrence of chronic health issues (like lung cancer, diabetes, hepatitis, severe pain, and depression) within both transnational and subnational contexts[8]. respectively. However, these distributions share a common limitation: regardless of parameter variations, the distribution shows two peaks at the same height level.Therefore, this distribution may not be suitable for datasets with two modes of unequal observation frequencies

Juric further introduced an asymmetric double Weibull type II (ADW^*) [4], which allows for controlling the relative sizes of left and right peaks through a skewness parameter. Nevertheless, this particular distribution also possesses evident drawbacks: when skewed to the left, the left peak becomes higher; when skewed to the right, the right peak becomes higher. Such correlation significantly limits its flexibility as a bimodal distribution. Therefore, Çankaya introduced the asymmetric bimodal exponential power distribution in [9], while Bulut et al. proposed the bimodal extended generalized gamma distribution in [10]. These two distributions address the limitations of the aforementioned bimodal distribution by controlling the peak value through manipulation of distinct shape parameters on either side of the position parameter, thereby ensuring independence between peak value and skewness. However, employing distinct shape parameters may lead to discontinuity in the distribution function.

In this paper, we proposed a novel bimodal distribution, the extended two-piece double generalized gamma distribution. Firstly, the two-piece distribution proposed by Rubio and Steel in [11] was extended and presented in Section 2. Then, in Section 3 we applied the extended two-piece transformation on the double gamma and double generalized gamma distributions to have extended two-piece double gamma ($ETDG$) distribution and extended two-piece double generalized gamma ($ETDGG$) distribution respectively. The term "double" refers to a construction method that extends the distribution from the original positive real domain to the whole real line by symmetrization of the density. A comprehensive moment estimation and maximum likelihood estimation process for the $ETDGG$ distribution is provided in Section 4, followed by an assessment of parameter estimation stability through simulation studies presented in Section 5. In section 6, we apply our novel model to real financial data. Finally, a discussion and summary are presented at the end of this paper.

2. An extended two-piece transformation for bimodal distributions

Let's revisit the definition of the two piece distribution. Let $f(x)$ denote a symmetric unimodal probability density function with mode at zero and support on the entire real line R. Its corresponding CDF is denoted as $F(x)$.

Definition 1: A random variable X is said to follow a two-piece distribution if its PDF is given by

$$f_{tp}(x;\mu,\sigma_1,\sigma_2,\theta)=\frac{2}{\sigma_1+\sigma_2}\left(f\left(\frac{x-\mu}{\sigma_1},\theta\right)I(x<\mu)+f\left(\frac{x-\mu}{\sigma_2},\theta\right)I(x\geq\mu)\right). \quad (1)$$

This distribution is formed by combining two half-densities that have distinct scale parameters,σ_1 and σ_2, centered around the location parameter μ. The shape parameter θ primarily influences the behavior of the distribution's tails.A commonly adopted approach is to define $\sigma_1=\sigma a(k)$ and $\sigma_2=\sigma b(k)$,where $a(k)$ and $b(k)$ are positive functions dependent on the skewness parameter k. Under this reparameterization, the corresponding PDF can be expressed as:

$$f_{tp}(x;\mu,\sigma,k,\theta)=\frac{2}{\sigma(a(k)+b(k))}\left(f\left(\frac{x-\mu}{\sigma a(k)},\theta\right)I(x<\mu)+f\left(\frac{x-\mu}{\sigma b(k)},\theta\right)I(x\geq\mu)\right). \quad (2)$$

However, when $f(x)$ deviates from a unimodal symmetric distribution to a bimodal distribution with zero values on the symmetric axis μ, the flexibility of the above two-piece transformation becomes constrained. Through calculations, it can be observed that the ratio of scale parameters on both sides in (1) and (2) is equal to the ratio of cumulative quantities on both sides, expressed as equation

$$\frac{\sigma_1}{\sigma_2}=\frac{\sigma a(k)}{\sigma b(k)}=\frac{a(k)}{b(k)}=\frac{F_{st}(\mu)}{1-F_{st}(\mu)},$$

where $F_{st}(x)$ represents the cumulative distribution function of $f_{st}(x)$. This strong correlation significantly impacts the flexibility of bimodal distributions. For instance, in (1) and (2), the peaks on both sides will always be identical. To address this limitation, we propose an extended two-piece transformation for bimodal distributions.

Definition 2: A random variable X is considered to adhere to an extended two piece distribution when its PDF takes the following form:

$$f_{etp}(x;\mu,\sigma_1,\sigma_2,k,\theta)=\frac{2}{(a(k)+b(k))}\left(\frac{1}{\sigma_1}f\left(\frac{x-\mu}{\sigma_1 a(k)},\theta\right)I(x<\mu)+\frac{1}{\sigma_2}f\left(\frac{x-\mu}{\sigma_2 b(k)},\theta\right)I(x\geq\mu)\right). \quad (3)$$

Considering the writing habits of some distributions and the convenience of calculation, we can take the reciprocal of σ, that is, $\beta=\frac{1}{\sigma}$. Therefore, an extended two-piece model with a rate parameter version is provided, which can be written as

Letting $\beta=\frac{1}{\sigma}$ for convenient expression and computation, the extended two-piece distribution can be re-written as

$$f_{etp}(x;\mu,\beta_1,\beta_2,k,\theta) = \frac{2}{(a(k)+b(k))}(\beta_1 f(\beta_1 \frac{x-\mu}{a(k)},\theta)I(x<\mu) + \beta_2 f(\beta_2 \frac{x-\mu}{b(k)},\theta)I(x\geq\mu)). \quad (4)$$

Here β_1 and β_2 are the induced scale parameters.

It can be observed that redefining two different scale parameters allows the distribution on both sides of the symmetric axis to be unconstrained by the skewness parameter k, while the parameter k still has the ability to control the cumulative quantities on both sides of the symmetric axis, which makes it more flexible.

3. Bimodal with extended two-piece transformation

In this section, we apply the extended two-piece transformation to the double gamma distribution and the double generalized gamma distribution, choosing epsilon-skew parameterization particularly [12].

Extended two-piece double gamma distribution

A random variable X is considered to follow extended two-piece double gamma distribution ($ETDG$), denoted by X:ETDG$(\alpha,\beta_1,\beta_2,k)$, if its probability density function (PDF) is given by:

$$f(x) = \begin{cases} \frac{\beta_1^\alpha}{2\Gamma(\alpha)}\left(\frac{-x}{1+k}\right)^{\alpha-1} e^{\frac{\beta_1 x}{1+k}}, & x<0 \\ \frac{\beta_2^\alpha}{2\Gamma(\alpha)}\left(\frac{x}{1-k}\right)^{\alpha-1} e^{-\frac{\beta_2 x}{1-k}}, & x\geq 0 \end{cases}, \quad (5)$$

in which $\Gamma(\alpha)$ is gamma function, $\alpha \in (0,+\infty)$ is shape parameter, $\beta_1,\beta_2,\in (0,+\infty)$ are scale parameters, $k\in(-1,1)$ is skewness parameter.

Extended two-piece double generalized gamma distribution and basic properties

A random variable X is considered to follow extended two-piece double generalized gamma distribution (ETDGG), denoted by X: ETDGG $(\alpha,s,\beta_1,\beta_2,k)$, if its PDF has the form:

$$f(x) = \begin{cases} \frac{\beta_1 s}{2\Gamma(\alpha)}\left(\frac{-\beta_1 x}{1+k}\right)^{s\alpha-1} e^{(-(-\frac{\beta_1 x}{1+k})^s)}, & x<0 \\ \frac{\beta_2 s}{2\Gamma(\alpha)}\left(\frac{\beta_2 x}{1-k}\right)^{s\alpha-1} e^{-(\frac{\beta_2 x}{1-k})^s}, & x\geq 0 \end{cases}, \quad (6)$$

in which $\Gamma(\alpha)$ is gamma function, $\alpha, s \in (0,+\infty)$ is shape parameter, $\beta_1,\beta_2,\in (0,+\infty)$ is scale parameter, $k\in(-1,1)$ is skewness parameter.

Figure 1 shows the pdf for the ETDGG distribution under different parameter combinations.

If a random variable Z has a location parameter μ and follows the $ETDGG$ distribution, we can express it as Z:ETDGG$(\mu,s,\alpha,\beta_1,\beta_2,k)$ through a linear transformation $Z=\mu+X$.

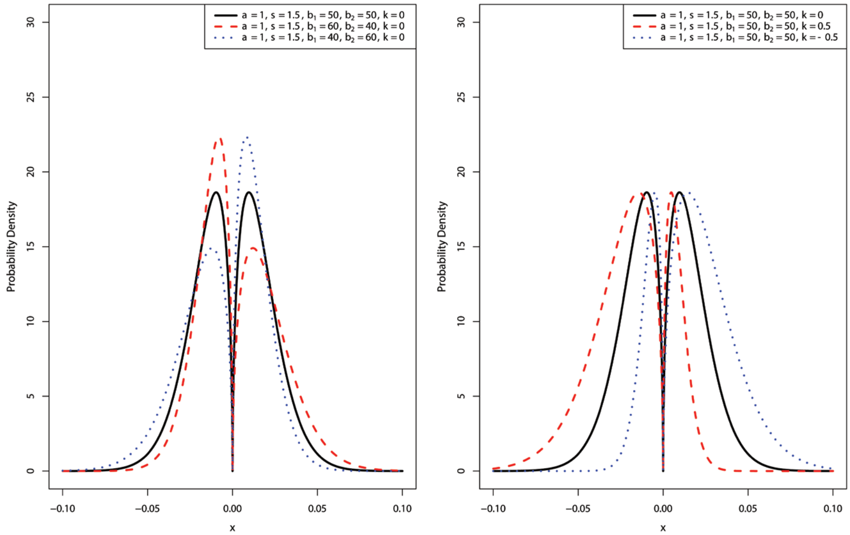

Figure 1
The behaviors of PDF of ETDGG distribution for different values of parameters.

Cumulative Distribution Function (CDF) of ETDGG

If X:ETDGG$(\alpha, s, \beta_1, \beta_2, k)$, then the cumulative distribution distribution function (CDF), $F(x)$, of X is

$$F(x)=\begin{cases}\frac{1+k}{2}\left(1-\frac{\gamma(\alpha,(\frac{-\beta_1 x}{1+k})^s)}{\Gamma(\alpha)}\right), & x<0\\ \frac{1+k}{2}+\frac{1-k}{2}\frac{\gamma(\alpha,(\frac{\beta_2 x}{1-k})^s)}{\Gamma(\alpha)}, & x\geq 0\end{cases}, \tag{7}$$

Analysis of the survival and hazard functions for ETDGG

The survival function $S(x)$ and hazard function $h(x)$, of X are

$$S(x)=\begin{cases}\frac{1-k}{2}+\frac{(1+k)\gamma(\alpha,(\frac{-\beta_1 x}{(1+k)})^s)}{2\Gamma(\alpha)}, & x<0\\ \frac{1-k}{2}-\frac{(1-k)\gamma(\alpha,(\frac{\beta_2 x}{(1-k)})^s)}{2\Gamma(\alpha)}, & x\geq 0\end{cases}, \tag{8}$$

$$h(x)=\begin{cases}\frac{\beta_1 s}{\Gamma(\alpha)}\frac{\left(\frac{-\beta_1 x}{1+k}\right)^{s\alpha-1}e^{(-(\frac{\beta_1 x}{1+k})^s)}}{1-k+\frac{(1+k)\gamma(\alpha,(\frac{-\beta_1 x}{(1+k)})^s)}{\Gamma(\alpha)}}, & x<0\\ \frac{\beta_2 s}{\Gamma(\alpha)}\frac{\left(\frac{\beta_2 x}{1-k}\right)^{s\alpha-1}e^{(-\frac{\beta_2 x}{1-k})^s}}{1-k-\frac{(1-k)\gamma(\alpha,(\frac{\beta_2 x}{(1-k)})^s)}{\Gamma(\alpha)}}, & x\geq 0\end{cases} \tag{9}$$

where $\gamma(\alpha, x)$ denotes the lower incomplete γ function.

Mode of $ETDGG$

By definition, the mode is the point at which the PDF reaches its maximum value or, equivalently, the logarithm of the probability density function. Taking the derivative of the logarithm of the PDF to have

$$lnf(x) = \begin{cases} \alpha\ln(\beta_1) + \ln s - \ln(2\Gamma(\alpha)) + (s\alpha - 1)\ln\left(\frac{-\beta_1 x}{1+k}\right) - (-\frac{\beta_1 x}{1+k})^s, & x < 0 \\ \alpha\ln(\beta_2) + \ln s - \ln(2\Gamma(\alpha)) + (s\alpha - 1)\ln\left(\frac{\beta_2 x}{1-k}\right) - (\frac{\beta_2 x}{1-k})^s, & x \geq 0 \end{cases} \quad (10)$$

Letting its derivative equals to zero, solving the equation yields at most two solutions:

$$x_1 = \frac{-(s\alpha-1)^{\frac{1}{s}}(1+k)}{\beta_1}, x_2 = \frac{(s\alpha-1)^{\frac{1}{s}}(1-k)}{\beta_2}$$

where $\alpha > 0$, $\beta_1, \beta_2 > 0, -1 < k < 1$. So that, we can obtain the following scenarios of mode.

(a) if $0 < s\alpha < 1$, the mode does not exist.

(b) if $s\alpha = 1$, the mode is equal to zero.

(c) if $s\alpha > 1, \beta_1 = \beta_2$, the mode is $x_1 = \frac{-(s\alpha-1)^{\frac{1}{s}}(1+k)}{\beta_1}, x_2 = \frac{(s\alpha-1)^{\frac{1}{s}}(1-k)}{\beta_2}$.

(d) if $s\alpha > 1, \beta_1 > \beta_2$, the mode is $x_1 = \frac{-(s\alpha-1)^{\frac{1}{s}}(1+k)}{\beta_1}$.

(e) if $s\alpha > 1, \beta_1 < \beta_2$, the mode is $x_2 = \frac{(s\alpha-1)^{\frac{1}{s}}(1-k)}{\beta_2}$.

The above results show that the mode of $ETDGG$ is independent of the skewness parameter k.

Moments of $ETDGG$

$$\begin{aligned} E(X^n) &= \int_{-\infty}^{+\infty} x^n f(x)\, dx \\ &= \int_{-\infty}^{0} x^n f(x) dx + \int_{0}^{+\infty} x^n f(x) dx. \end{aligned}$$

Let

$$L(n) = \int_{-\infty}^{0} x^n f(x) dx,$$
$$R(n) = \int_{0}^{+\infty} x^n f(x) dx,$$

based on the moments of the gamma distribution, we can obtain

$$L(n) = (-1)^n \frac{(1+k)^{n+1}\Gamma(\alpha+\frac{n}{s})}{2\Gamma(\alpha){\beta_1}^{\alpha}},$$
$$R(n) = \frac{(1-k)^{n+1}\Gamma(\alpha+\frac{n}{s})}{2\Gamma(\alpha){\beta_2}^{\alpha}}.$$

So the moments of $ETDGG$ are ——

$$E(X^n) = L(n) + R(n) = (-1)^n \frac{(1+k)^{n+1}\Gamma(\alpha+\frac{n}{s})}{2\Gamma(\alpha){\beta_1}^{\alpha}} + \frac{(1-k)^{n+1}\Gamma(\alpha+\frac{n}{s})}{2\Gamma(\alpha){\beta_2}^{\alpha}}. \quad (1)$$

So the expectation and variance of $ETDGG$ are

$$E(X) = -\frac{(1+k)^2\Gamma(\alpha+\frac{1}{s})}{2\Gamma(\alpha)\beta_1^{\alpha}} + \frac{(1-k)^2\Gamma(\alpha+\frac{1}{s})}{2\Gamma(\alpha)\beta_2^{\alpha}},$$

$$V(X) = -\frac{(1+k)^2\Gamma(\alpha+\frac{1}{s})}{2\Gamma(\alpha)\beta_1^{\alpha}} + \frac{(1-k)^2\Gamma(\alpha+\frac{1}{s})}{2\Gamma(\alpha)\beta_2^{\alpha}} - \frac{(1+k)^4\Gamma^2(\alpha+\frac{1}{s})}{4\Gamma^2(\alpha)\beta_1^{2\alpha}} - \frac{(1-k)^4\Gamma^2(\alpha+\frac{1}{s})}{2\Gamma^2(\alpha)\beta_2^{2\alpha}} + \frac{(1-k)^2(1+k)^2\Gamma^2(\alpha+\frac{1}{s})}{2\Gamma^2(\alpha)\beta_1^{\alpha}\beta_2^{\alpha}}.$$

Quantile function $Q(p)$ and random variable generation function of $ETDGG$

The $Q(p)$ of $ETDGG$ distribution is as follows:

$$Q(p) = \begin{cases} -\frac{(1+k)}{\beta_1}(Q_G\left(\frac{1-2p}{1+k},\alpha,1\right))^{\frac{1}{s}}, & 0 \le p \le \frac{1+k}{2} \\ \frac{(1-k)}{\beta_2}(Q_G\left(\frac{2p-1-k}{1-k},\alpha,1\right))^{\frac{1}{s}}, & \frac{1+k}{2} < p \le 1 \end{cases}, \quad (12)$$

where $Q_G(p,\alpha,\beta)$ is the p-th quantile function of gamma distribution. Based on this, we can naturally use the inverse method to obtain the generation function of the random variable Y following the $ETDGG$ distribution, which is given by:

$$Y = -\frac{(1+k)}{\beta_1}(Q_G\left(\frac{1-2p}{1+k},\alpha,1\right))^{\frac{1}{s}} \times I(0 \le U \le \frac{1+k}{2}) + \frac{(1-k)}{\beta_2}(Q_G\left(\frac{2p-1-k}{1-k},\alpha,1\right))^{\frac{1}{s}} \times I(\frac{1+k}{2} < U \le 1) \quad (13)$$

where $U \sim U(0,1)$.

Special cases of $ETDGG$

1. If $s = 1, \beta_1 = \beta_2, k = 0$, the $ETDGG$ distribution degenerates into the double gamma distribution.
2. If $s = 1, \beta_1 = \beta_2$, the $ETDGG$ distribution degenerates into the epsilon skew gamma distribution [6].
3. If $\alpha = 1, \beta_1 = \beta_2, k = 0$, the $ETDGG$ distribution degenerates into the double Weibull distribution.
4. If $\alpha = 1, \beta_1 = \beta_2$, the $ETDGG$ distribution degenerates into the epsilon skew Weibull distribution.
5. If $\alpha = n/2, \beta_1 = \beta_2 = \frac{\sqrt{2}}{2}, k = 0$, the $ETDGG$ distribution degenerates into the double Chi-squared distribution.
6. If $\alpha = 1/2, s = 2, \beta_1 = \beta_2 = \frac{\sqrt{2}}{2}, k = 0$, the $ETDGG$ distribution degenerates into the standard Normal distribution.
7. If $\alpha = 1/2, s = 2, \beta_1 = \beta_2 = \frac{\sqrt{2}}{2}$,the $ETDGG$ distribution degenerates into the epsilon skew Normal distribution [12].

4. Estimation Methods

In this part, we study the parameter estimation process of $ETDGG$ distribution. To estimate the parameters, we utilize two methods: the moments estimation and the maximum likelihood estimation. Let us consider a random sample of n observations $X = (x_1, x_2, x_3, \dots, x_n)^T$ drawn from the $ETDGG$ distribution and let $(\alpha, s, \beta_1, \beta_2, k)$ be the parameters to be estimated.

4.1 *Method of Moments Estimation (MME)*

Let X_+ be the section of X for $x_i > 0$ and X_- be the section of X for $x_i < 0$, then

$$E(X_+) = \frac{\sum_{x_i>0} x_i}{n}, \quad E(X_-) = \frac{\sum_{x_i<0} x_i}{n},$$

and

$$E({X_+}^2) = \frac{\sum_{x_i>0} x_i^2}{n}, \quad E({X_-}^2) = \frac{\sum_{x_i<0} x_i^2}{n},$$

So, we can obtain from (1) and (2)

$$E(X_-) = -\frac{\Gamma(\frac{1}{s}+\alpha)(1+k)^2}{2\Gamma(\alpha){\beta_1}^1}, \tag{14}$$

$$E({X_-}^2) = \frac{\Gamma(\frac{2}{s}+\alpha)(1+k)^3}{2\Gamma(\alpha){\beta_1}^2}, \tag{15}$$

$$E(X_+) = \frac{\Gamma(\frac{1}{s}+\alpha)(1-k)^2}{2\Gamma(\alpha){\beta_2}^2}, \tag{16}$$

$$E({X_+}^2) = \frac{\Gamma(\frac{2}{s}+\alpha)(1-k)^3}{2\Gamma(\alpha){\beta_2}^2}. \tag{17}$$

Solving the above equations, we can have the MME of the skewness parameter k, which is given by

$$\hat{k} = 1 - \frac{2\mathrm{E}^2(\mathrm{X}_+)E({X_-}^2)}{\mathrm{E}^2(\mathrm{X}_-)E({X_+}^2)}.$$

Assuming s is known, taking the following transformation on the original sample data X

$$Y_+ = \left(\frac{X_+}{1-k}\right)^s, Y_- = \left(-\frac{X_-}{1-k}\right)^s,$$

we can obtain four equations

$$E(Y_-) = \frac{(1+k)\alpha}{2{\beta_1}^s},$$

$$E(Y_+) = \frac{(1-k)\alpha}{2{\beta_2}^s},$$

$$E({Y_-}^2) = \frac{\alpha(\alpha+1)(1+k)}{2{\beta_1}^{2s}},$$

$$E({Y_+}^2) = \frac{\alpha(\alpha+1)(1-k)}{2{\beta_2}^{2s}}.$$

Solving the above equations, we can have the MME of the parameters α, β_1 and β_2

$$\hat{\alpha} = \frac{\frac{E^2(Y_-)}{E(Y_-{}^2)} + \frac{E^2(Y_+)}{E(Y_+{}^2)}}{1 - \frac{E^2(Y_-)}{E(Y_-{}^2)} - \frac{E^2(Y_+)}{E(Y_+{}^2)}},$$

$$\hat{\beta}_1 = \left(\frac{E(Y_-)}{E(Y_-{}^2)\left(1 - \frac{E^2(Y_-)}{E(Y_-{}^2)} - \frac{E^2(Y_+)}{E(Y_+{}^2)}\right)}\right)^{\frac{1}{s}},$$

$$\hat{\beta}_2 = \left(\frac{E(Y_+)}{E(Y_+{}^2)\left(1 - \frac{E^2(Y_-)}{E(Y_-{}^2)} - \frac{E^2(Y_+)}{E(Y_+{}^2)}\right)}\right)^{\frac{1}{s}}.$$

We employ the numerical method to find the estimation of the parameter s. Using grid search method to find the minimum of the Kolmogorov-Smirnov distance between the pdf and the sample data.

4.2 *Maximum Likelihood Estimation (MLE)*

The log-likelihood function for $(\alpha, s, \beta_1, \beta_2, k)$ is proportional to

$$l(X;\alpha,s,\beta_1,\beta_2,k) = n_l\left[\log\left(\frac{\beta_1 s}{2\Gamma(\alpha)}\right)\right] + (s\alpha-1)\sum_{x_i<0}\log\left(\frac{-\beta_1 x_i}{1+k}\right) - \sum_{x_i<0}\left(-\frac{\beta_1 x_i}{1+k}\right)^s + n_r[\log(\frac{\beta_2 s}{2\Gamma(\alpha)})] + (s\alpha-1)\sum_{x_i>0}\log(\frac{\beta_2 x_i}{1-k}) - \sum_{x_i<0}(\frac{\beta_2 x_i}{1-k})^s$$

Table 1
The results of simulation study.

Parameters		α	s	β_1	β_2	k
	n	$\hat{\alpha}$	$\hat{s}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{k}$
Bias	500	0.3754	-0.0663	1.0606	0.0402	-0.0038
	2000	0.1350	-0.0839	0.6245	-0.1086	-0.0051
	5000	0.0081	-0.0722	-0.1226	0.0732	0.0027
SE	500	0.0277	0.0048	0.1680	0.9040	0.0007
	2000	0.0081	0.0035	0.0799	0.0443	0.0003
	5000	0.0076	0.0040	0.0594	0.0471	0.0002
RMSE	500	0.4227	0.0743	1.5839	0.6341	0.0059
	2000	0.1465	0.0874	0.8386	0.3285	0.0055
	5000	0.0535	0.0775	0.4336	0.3379	0.0031

where n_l and n_r are the sample size of X greater than 0 and less than 0 respectively, from which the normal equations are obtained. These equations are given by,

$$\frac{\partial \ell}{\partial \alpha} = -n\psi(\alpha) + s[\sum_{x_i<0}\log(-x_i) + \sum_{x_i>0}\log(x_i)] + s[n_l\log(\frac{\beta_1}{1+k}) + n_r\log(\frac{\beta_2}{1-k})] = 0,$$

$$\frac{\partial \ell}{\partial s} = \frac{n}{s} + \alpha[\sum_{x_i<0}\log(-x_i) + \sum_{x_i>0}\log(x_i)] + \alpha[n_l\log(\frac{\beta_1}{1+k}) + n_r\log(\frac{\beta_2}{1-k})] - \sum_{x_i<0}(-\frac{\beta_1 x_i}{1+k})^s\log(-\frac{\beta_1 x_i}{1+k}) - \sum_{x_i<0}(\frac{\beta_2 x_i}{1-k})^s\log(\frac{\beta_2 x_i}{1-k}) = 0,$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{n_l\alpha s}{\beta_1} - s\sum_{x_i<0}(-\frac{\beta_1 x_i}{1+k})^{s-1}(-\frac{x_i}{1+k}) = 0,$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{n_r \alpha s}{\beta_2} - s\sum_{x_i>0} (\frac{\beta_2 x_i}{1-k})^{s-1}(\frac{x_i}{1-k}) = 0,$$

$$\frac{\partial \ell}{\partial k} = -\frac{(\alpha s-1)n_l}{1+k} + \frac{(\alpha s-1)n_r}{1-k} + \frac{s}{1+k}\sum_{x_i<0} (-\frac{\beta_1 x_i}{1+k})^s - \frac{s}{1-k}\sum_{x_i>0} (\frac{\beta_2 x_i}{1-k})^s = 0,$$

where $\psi(\alpha) \equiv \frac{\Gamma\prime(\alpha)}{\Gamma(\alpha)}$, is digamma function [13]. Obviously, these equations are nonlinear and rather complex, having no closed-form solutions. Therefore, we solve these equations using the numerical methods such as the $Newton-Raphson$ method [14] with an iterative process

$$\theta_{n+1} = \theta_n - H^{-1}\nabla l(\theta_n),$$

where $\theta = (\alpha, s, \beta_1, \beta_2, k)^T$ and H is the Hessian matrix of the log-likelihood function (see Appendix for details of Hessian matrix).

5. Simulation study

In this part, we provide three sets of simulation studies to evaluate the behavior of maximum likelihood estimation under finite samples.We consider 50 Monte Carlo simulations for three samples. size=500, 2000 and 5000. We set $\alpha = 1.2, s = 1.1, \beta_1 = 6, \beta_2 = 4, k = 0.1$. The results are presented in Table 1 and Figures 2, 3, 4 respectively.

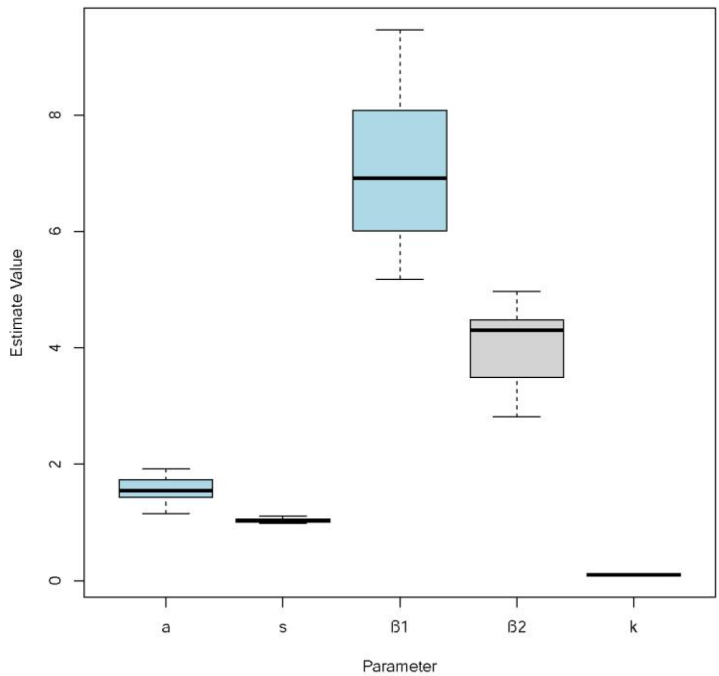

Figure 2

Boxplot of the parameters estimated results for sample size=500.

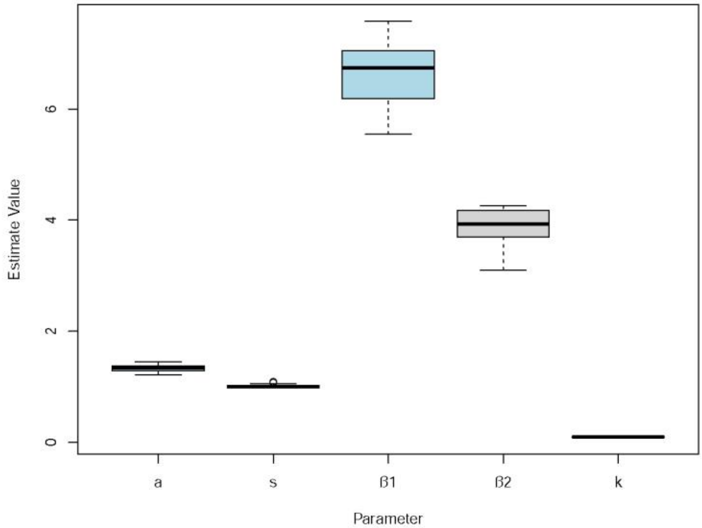

Figure 3
Boxplot of the parameters estimated results for sample size=2000.

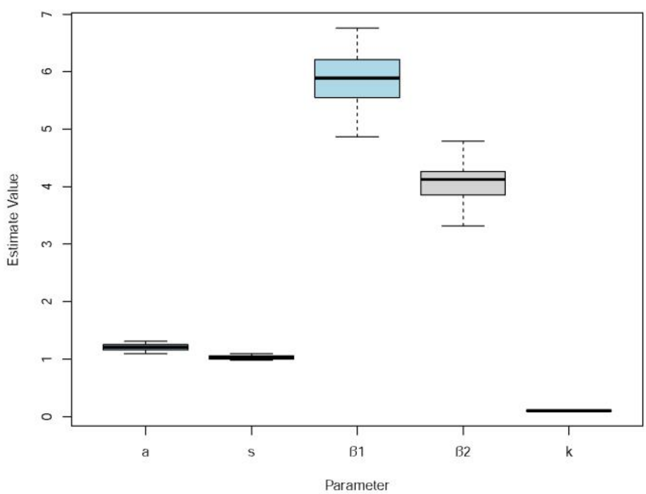

Figure 4
Boxplot of the parameters estimated results for sample size=5000.

Table 1 summarizes the empirical bias, SE and RMSE). In general terms, when the sample size increases, the bias, SE and RMSE items decrease, suggesting the consistency of the MLE. As also shown in the figures, Typically, larger sample sizes result in lower bias and reduced variability, implying the estimators are well estimated and the consistency of the MLE.

6. Applications

6.1 *Bimodality in S&P's distortion*

Table 2
Descriptive statistics for the S&P's distortion data.

Max	Min	Mean	Median	Variance	Sd	Skewness	Kurtosis
0.97400	-0.63441	0.20436	0.20487	0.09736905	0.3120401	0.03988599	2.320289

Table 3
The criteria results of parameter estimation.

	p-value	K-S distance	AIC	BIC
$BTPN$	0.007251	0.040185	825.1484	846.995
ESG	0.008275	0.03971	926.0849	947.9315
$TPPN$	2.2e-16	0.155789	919.3812	941.2278
ADW^*	0.1204	0.035232	830.2833	852.1299
$ETDG$	0.04349	0.033168	882.2451	904.0916
$ETDGG$	**0.2351**	**0.024793**	**803.9791**	**836.749**

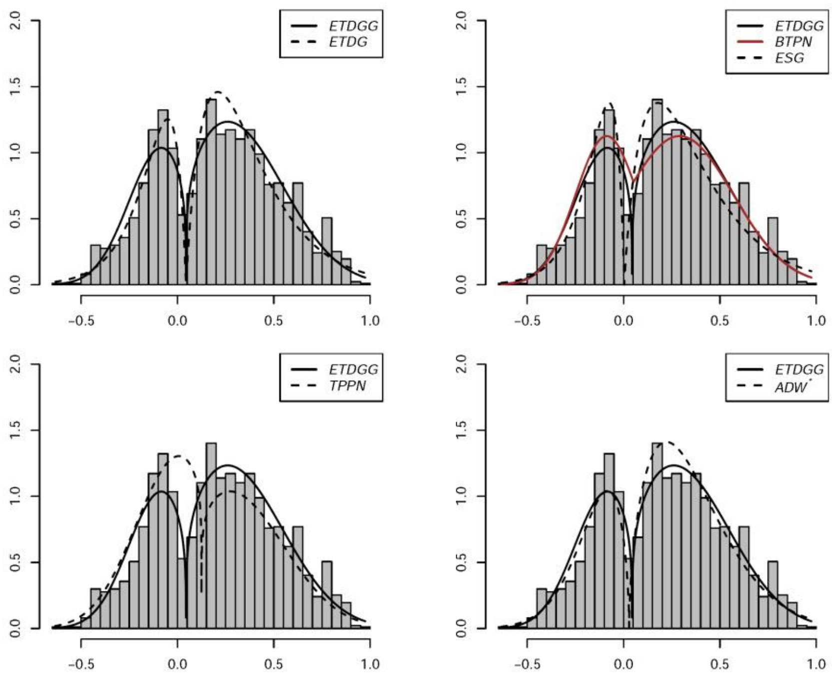

Figure 5
Histogram plot for the S&P's distortion data and fitted results with the distributions.

Empirical studies shows that The S&P 500 distortion distribution, that is, The log-transformed discrepancy between the market index and its true fundamental value, has a non-unimodal distribution. Although S&P 500 exhibits fluctuations around its basic value, it tends to persist longer in both bull and bear market conditions, and fluctuates less near its basic value. Therefore, the distorted distribution of S&P 500 is opposite to the expectation, instead of conforming to a unimodal shape, the distribution is distinctly bimodal.**[15]**.

We have obtained the data from [15] for the period January 1871 to December 2015. Table 2 presents the descriptive statistics of this dataset. The histgram plot of this data set presented in Figure 5 shows obviously asymmetry and bimodality. We compare our proposed $ETDG$ and $ETDGG$ distribtions with ESG, $BTPN$, $TPPN$, ADW^* distributions. Table 3 presents Kolmogorov-Smirnov (K-S) test, AIC and BIC criteria for the employed distributions. All these criteria show that our proposed $ETDGG$ distribution performs the best. Figure 5 shows the histogram plot for this dataset and the fitted pdfs for the considered distributions. The top left is the fitted results with $ETDGG$ and $ETDG$ distributions, constructed based on the extended two-piece transformation, which can capture the cumulative quantities as well as the peak value on both sides of the data. The top right shows the fitted results with ESG distribution and $BTPN$ distribution. They can only capture the cumulantive quantities while cannot capture the peak values on both sides. The bottom left panel presents the fitted result with $TPPN$ distribution, which can only capture the peak value while cannot capture the cumulative quantities on both sides. The bottom right shows the fitted result with ADW^* distribution, although it can capture the peak value and cumulative quantities on both sides, but it cannot flexibly capture either one of them alone.

6.2 *Other applications*

Table 4
Summary statistics for the currency exchange rates data.

Max	Min	Mean	Median	Variance	Sd	Skewness	Kurtosis
0.03682	-0.04121	-0.0002879	-0.0000859	0.00004334	0.0065865	-0.3211597	5.454019

Table 5
The K-S test results for distributions.

	p-value	Kolmogorov-Smirnov (K-S) distance
$Normal$	3.119e-13	0.059671
ESG	0.2983	0.015144
AL	0.03539	0.022071
ADW	0.1189	0.018468
ADW^*	0.1689	0.017279
$ETDG$	**0.7322**	**0.010685**
$ETDGG$	**0.5283**	**0.012589**

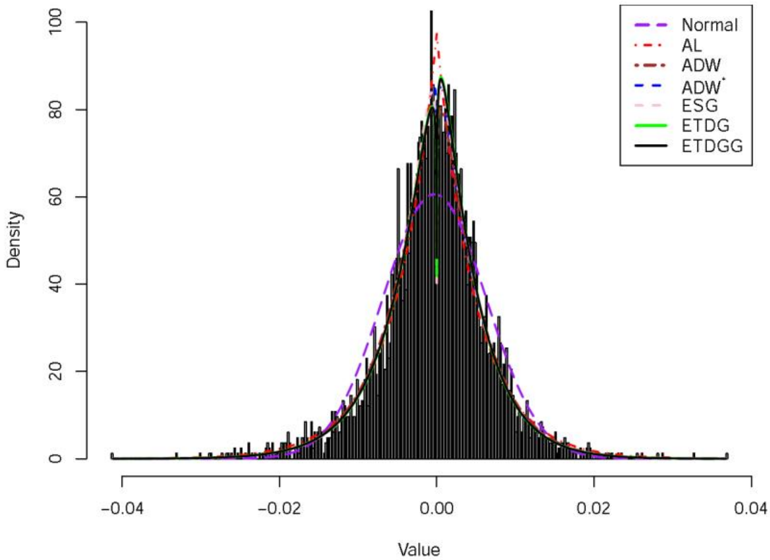

Figure 6
Histogram plot for the currency exchange rates and fitted results with the considered distributions.

We also consider the daily average currency exchange rates for British Pounds to Japanese Yen from the Bank of England spanning from January 1, 1980, to May 21, 1996. Table 4 summarizes the descriptive statistics for this data set. Kozubowski and Podgórski[16], Juric [4] had previously fitted this data set using the asymmetric Laplace distribution (AL) and type 2 of asymmetric double Weibull distribution (ADW^*) respectively. Here we employ the Normal, ESG, AL, ADW, ADW^*, $ETDG$ and $ETDGG$ distributions to fit this dataset respectively. Table 5 presents Kolmogorov-Smirnov (K-S) test results of the fitness with these employed models.The histogram of dataset and the fitted PDFs of the selected distributions are illustrated in Figure 6.

Note that, based on the K-S test results, both extended two-piece transformed distributions as $ETDG$ and $ETDGG$ distributions provide a superior fit compared to the others. Figure 6 also presents outperformance for the $ETDG$ and $ETDGG$ distributions.

7. Discussion and conclusion

When the data is bimodal distribution, it is difficult because there is no traditional distribution with this property.This study contributes to the analysis of bimodal distributions, the proposed extended two-piece double generalized gamma distribution have the following advantages are observed: a well-defined support over the real line, exact mathematical formulations for the CDF, and analytically tractable moments. flexible to handle skewness and kurtosis and the ablility to generalize a lot of Normal distributions such as standard Normal, Weibull, Gamma and Chi-square distributions. The proposed distribution was the best choice of fitting comparing with the other distributions for the real applications. In the future work, We believe that this distribution is useful for practitioners in many different fields.

Appendix

The Hessian matrix of the log-likelihood function for the $ETDGG$ distribution is

$$H=\begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\,\partial s} & \frac{\partial^2\ell}{\partial\alpha\,\partial\beta_1} & \frac{\partial^2\ell}{\partial\alpha\,\partial\beta_2} & \frac{\partial^2\ell}{\partial\alpha\,\partial k} \\ \frac{\partial^2\ell}{\partial s\,\partial\alpha} & \frac{\partial^2\ell}{\partial s^2} & \frac{\partial^2\ell}{\partial s\,\partial\beta_1} & \frac{\partial^2\ell}{\partial s\,\partial\beta_2} & \frac{\partial^2\ell}{\partial s\,\partial k} \\ \frac{\partial^2\ell}{\partial\beta_1\,\partial\alpha} & \frac{\partial^2\ell}{\partial\beta_1\,\partial s} & \frac{\partial^2\ell}{\partial{\beta_1}^2} & \frac{\partial^2\ell}{\partial\beta_1\,\partial\beta_2} & \frac{\partial^2\ell}{\partial\beta_1\,\partial k} \\ \frac{\partial^2\ell}{\partial\beta_2\,\partial\alpha} & \frac{\partial^2\ell}{\partial\beta_2\,\partial s} & \frac{\partial^2\ell}{\partial\beta_2\,\partial\beta_1} & \frac{\partial^2\ell}{\partial{\beta_2}^2} & \frac{\partial^2\ell}{\partial\beta_2\,\partial k} \\ \frac{\partial^2\ell}{\partial k\,\partial\alpha} & \frac{\partial^2\ell}{\partial k\,\partial s} & \frac{\partial^2\ell}{\partial k\,\partial\beta_1} & \frac{\partial^2\ell}{\partial k\,\partial\beta_2} & \frac{\partial^2\ell}{\partial k^2} \end{pmatrix}$$

The elements in matrix H are as follows,

$$\frac{\partial^2\ell}{\partial\alpha^2}=-n\psi'(\alpha),$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial s}=\frac{\partial^2\ell}{\partial s\,\partial\alpha}=\sum_{x_i<0}\log(-x_i)+\sum_{x_i>0}\log(x_i)+n_l\log(\tfrac{\beta_1}{1+k})+n_r\log(\tfrac{\beta_2}{1-k}),$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial\beta_1}=\frac{\partial^2\ell}{\partial\beta_1\,\partial\alpha}=\frac{n_l s}{\beta_1},$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial\beta_2}=\frac{\partial^2\ell}{\partial\beta_2\,\partial\alpha}=\frac{n_r s}{\beta_2},$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial k}=\frac{\partial^2\ell}{\partial k\,\partial\alpha}=\frac{n_r s}{1-k}-\frac{n_l s}{1+k},$$

$$\frac{\partial^2\ell}{\partial s^2}=-\frac{n}{s^2}-\sum_{x_i<0}(-\tfrac{\beta_1x_i}{1+k})^s\log^2(-\tfrac{\beta_1x_i}{1+k})-\sum_{x_i<0}(\tfrac{\beta_2x_i}{1-k})^s\log^2(\tfrac{\beta_2x_i}{1-k}),$$

$$\frac{\partial^2\ell}{\partial s^2}=-\frac{n}{s^2}-\sum_{x_i<0}(-\tfrac{\beta_1x_i}{1+k})^s\log^2(-\tfrac{\beta_1x_i}{1+k})-\sum_{x_i>0}(\tfrac{\beta_2x_i}{1-k})^s\log^2(\tfrac{\beta_2x_i}{1-k}),$$

$$\frac{\partial^2\ell}{\partial s\,\partial\beta_1}=\frac{\partial^2\ell}{\partial\beta_1\,\partial s}=\frac{n_l\alpha}{\beta_1}-\sum_{x_i<0}(-\tfrac{\beta_1x_i}{1+k})^{s-1}(-\tfrac{x_i}{1+k})-$$
$$s\sum_{x_i<0}(-\tfrac{\beta_1x_i}{1+k})^{s-1}\log(-\tfrac{\beta_1x_i}{1+k})(-\tfrac{x_i}{1+k}),$$

$$\frac{\partial^2\ell}{\partial s\,\partial\beta_2}=\frac{\partial^2\ell}{\partial\beta_2\,\partial s}=\frac{n_r\alpha}{\beta_2}-\sum_{x_i>0}(\tfrac{\beta_2x_i}{1-k})^{s-1}(\tfrac{x_i}{1-k})-s\sum_{x_i>0}(\tfrac{\beta_2x_i}{1-k})^{s-1}\log(\tfrac{\beta_2x_i}{1-k})(\tfrac{x_i}{1-k}),$$

$$\frac{\partial^2\ell}{\partial k\,\partial s}=\frac{\partial^2\ell}{\partial s\,\partial k}=-\frac{\alpha n_l}{1+k}+\frac{\alpha n_r}{1-k}+\frac{1}{1+k}\sum_{x_i<0}(-\tfrac{\beta_1x_i}{1+k})^s+$$
$$\frac{s}{1+k}\sum_{x_i<0}(-\tfrac{\beta_1x_i}{1+k})^s\log(-\tfrac{\beta_1x_i}{1+k})$$
$$-\frac{1}{1-k}\sum_{x_i>0}(-\tfrac{\beta_2x_i}{1-k})^s-\frac{s}{1-k}\sum_{x_i>0}(\tfrac{\beta_2x_i}{1-k})^s\log(\tfrac{\beta_2x_i}{1-k}),$$

$$\frac{\partial^2\ell}{\partial{\beta_1}^2}=-\frac{n_l\alpha s}{{\beta_1}^2}-s(s-1)\sum_{x_i<0}(-\tfrac{\beta_1x_i}{1+k})^{s-2}(-\tfrac{x_i}{1+k})^2,$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \ell}{\partial \beta_2 \partial \beta_1} = 0,$$
$$\frac{\partial^2 \ell}{\partial \beta_1 \partial k} = \frac{\partial^2 \ell}{\partial k \partial \beta_1} = \frac{s^2}{1+k} \sum_{x_i<0} \left(-\frac{\beta_1 x_i}{1+k}\right)^{s-1} \left(-\frac{x_i}{1+k}\right),$$
$$\frac{\partial^2 \ell}{\partial {\beta_2}^2} = -\frac{n_l \alpha s}{{\beta_1}^2} - s(s-1) \sum_{x_i>0} \left(\frac{\beta_2 x_i}{1-k}\right)^{s-2} \left(\frac{x_i}{1-k}\right)^2,$$
$$\frac{\partial^2 \ell}{\partial \beta_2 \partial k} = \frac{\partial^2 \ell}{\partial k \partial \beta_2} = -\frac{s^2}{1-k} \sum_{x_i<0} \left(\frac{\beta_2 x_i}{1-k}\right)^{s-1} \left(\frac{x_i}{1-k}\right),$$
$$\frac{\partial^2 \ell}{\partial k^2} = \frac{(\alpha s-1) n_l}{(1+k)^2} + \frac{(\alpha s-1) n_r}{(1-k)^2} + \frac{s(s+1)}{(1+k)^2} \sum_{x_i<0} \left(-\frac{\beta_1 x_i}{1+k}\right)^s - \frac{s(s+1)}{(1-k)^2} \sum_{x_i>0} \left(\frac{\beta_2 x_i}{1-k}\right)^s,$$
where $\psi'(\alpha)$ is the derivative of $\psi(\alpha)$, called trigamma function[13].

References

[1] Piotr Sulewski, Two-piece power normal distribution, *Communications in Statistics – Theory and Methods*, vol. 50, no. 11, pp. 2619–2639 (2021).

[2] Héctor J. Gómez, Wilson E. Caimanque, Yenny M. Gómez, Tiago M. Magalhães, Mauricio Concha, and Daniel I. Gallardo, A bimodal model based on truncation positive normal with application to height data, *Symmetry*, vol. 14, no. 4, article 665 (2022).

[3] Mohammed Y. Hassan and Rafiq H. Hijazi, A bimodal exponential power distribution, *Pakistan Journal of Statistics*, vol. 26, no. 2, pp. 379–396 (2010).

[4] Vedran Juric, Asymmetric Double Weibull Distributions, M.S. thesis, University of Nevada, Reno (2003).

[5] Ramón Vila and Mehmet Niyazi Çankaya, A bimodal Weibull distribution: Properties and inference, *Journal of Applied Statistics*, vol. 49, no. 12, pp. 3044–3062 (2022). [Online]. Available: https://doi.org/10.1080/02664763.2021.1931822

[6] Ehab Abdulah and Hossam Elsalloukh, Analyzing skewed data with the epsilon skew gamma distribution, *Journal of Statistics Applications & Probability*, vol. 2, no. 3, pp. 195–204 (2013).

[7] Ehab K. Abdulah and Hossam Elsalloukh, Bimodal class based on the inverted symmetrized gamma distribution with applications, *Journal of Statistics Applications & Probability*, vol. 3, no. 1, pp. 1–10 (2014).

[8] Deo Rahardja, Hsinchu Wu, Zhen Zhang, and Anthony D. Tiedt, Maximum likelihood estimation for the proportion difference of two-sample binomial data subject to one type of misclassification, *Journal of Statistics and Management Systems*, vol. 22, no. 8, pp. 1365–1379 (2019).

[9] Mehmet Niyazi Çankaya, Asymmetric bimodal exponential power distribution on the real line, *Entropy,* vol. 20, no. 1, article 23 (2018).

[10] Yusuf Mehmet Bulut, Osman Arslan, et al., A bimodal extension of the generalized gamma distribution, Revista Colombiana de Estadística, vol. 38, no. 2, pp. 371–384 (2015).

[11] Francisco J. Rubio and Mark F. J. Steel, Inference in two-piece location-scale models with Jeffreys priors (2014). [Unpublished manuscript or preprint – clarify if it's a journal article]

[12] Govind S. Mudholkar and Alan D. Hutson, The epsilon–skew–normal distribution for analyzing near-normal data, *Journal of Statistical Planning and Inference*, vol. 83, no. 2, pp. 291–309 (2000).

[13] Masaaki Sibuya, Digamma and trigamma distributions, in *Encyclopedia of Statistical Sciences*, vol. 3 (2004).

[14] Thomas J. Ypma, Historical development of the Newton–Raphson method, *SIAM Review*, vol. 37, no. 4, pp. 531–551 (1995).

[15] Norbert Schmitt and Frank Westerhoff, On the bimodality of the distribution of the S&P 500's distortion: Empirical evidence and theoretical explanations, *Journal of Economic Dynamics and Control*, vol. 80, pp. 34–53 (2017).

[16] Tomasz J. Kozubowski and Krzysztof Podgórski, Asymmetric Laplace laws and modeling financial data, *Mathematical and Computer Modelling,* vol. 34, no. 9–11, pp. 1003–1021 (2001).

Received July, 2024