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A three-species interaction model for crop-pest dynamics and environmental factors in agricultural ecosystems

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Abstract

This Study Examines a Three-Species Interaction Model Representing the Dynamics Between Crops, Pests, and a Third Environmental Factor in Agriculture. The Model, Driven by Nonlinear Differential Equations, Explores How Crop Growth is Influenced by Carrying Capacity, Pest Predation, and Agricultural Interventions like Fertilization and Pesticides. Pest Dynamics Depend on Competition with Crops and Interactions with the Third Species, Which Could Signify Predators or Ecological Stressors. Through Equilibrium Analysis and Stability Assessment via Jacobian Matrices and Eigenvalues, the Model Offers Insights into Sustainable Farming and Pest Management. The Findings Contribute to Strategies for Ecological Balance, Optimal Yield, and Long-Term Agricultural Sustainability.

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Keywords: *Three-species interaction, Agricultural ecosystem, Crop-pest dynamics, Nonlinear differential equations, Carrying capacity, Pest predation, Agricultural interventions, Equilibrium analysis, Stability, Jacobian matrix, Eigenvalue analysis, Sustainable farming, Pest management, Ecological balance.*

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Introduction

This System of Equilibrium describes a Three-Species Interaction model within an Ecological or Agricultural context, where the variables x, y, and z represent the Populations or Densities of Crops (x), Pests (y), and a Third Species (z), which could be a Predator or an Environmental Factor. The model captures their dynamic interactions over time, including Competition, Predation, and Environmental Influences. The primary focus is on how these populations evolve due to these factors, as originally discussed by pioneers in Ecological Modeling, including Lotka and Volterra [4], [9]. The Differential Equations define how each Population evolves based on several Ecological and Agricultural Factors: drawing on the foundational work by Holling [3] and May [5], the model integrates Predation Rates, Competition Dynamics, and Carrying Capacities to describe the behavior of Crops and Pests. It also incorporates Resource Competition and Community Structure Theory as elaborated by Tilman [8], reflecting the importance of Competition between Species and the Effects of Resource Limitation. Furthermore, the concept of Equilibrium in Ecological and Economic Models is crucial in understanding Population Dynamics, particularly under Resource Constraints and Spatial Heterogeneity. Perez and Pelegrin [10] discuss Equilibrium Quantities in Systems with Spatially Separated Markets under Production Constraints, which can provide additional insights into how Competition and Resource Limitations impact Species Distributions in Ecological Models.

The system consists of three differential equations: the first equation models the growth rate of the Crop Population (x), influenced by its Intrinsic Growth Rate (b_1) and limited by Carrying Capacity (K_x), in line with the Logistic Growth Models from early Population Dynamics studies [4], [9]. The crop's interaction with pests is governed by a Predation Term, moderated by a functional response that incorporates the predation rate (α) and Handling Time (h) as per Holling's Framework [3]. Additional factors, such as Agricultural Interventions like Fertilization (u_f) and Pesticide Application (u_p) modify the Crop's Growth Rate. The interaction with the Third Species (z) is captured through an Additional Pressure Term (εz). The second equation focuses on the Pest Population (y), which grows according to its Intrinsic Rate (b_2) but faces competition from the Crop Population, modelled by the Competition Coefficient (βx) and its own Carrying Capacity (K_y) [8]. The Pest Population is also impacted by interactions with the Third Species (z), influenced by the Interaction Coefficient (μ), and is subject to Agricultural Interventions such as Biological Control represented by the Control Parameter (u_c) [2]. The third equation describes the dynamics of the Third Species (z), which could be a Predator or an External Environmental Factor.

The Population of this species is subject to a Natural Death Rate (b_3), as per Murray's General Population Models [6], but can increase through interactions with both Crops and Pests, via the Interaction Terms $(d\varepsilon x)$ and $(d\mu y)$ [7], [1]. This model is particularly relevant for understanding complex agro-ecosystems where crops, pests, and external factors like biological control agents or environmental stressors interact. Building on the work of Pimm [17] and Hassell [2], the model helps explain how these systems balance competing demands, including pest control and crop productivity. Agricultural interventions like pesticide use, fertilization, and biological control are embedded in the model to reflect modern sustainable farming practices. These interventions aim to maintain long-term stability by preventing pest outbreaks or population collapses, as discussed by Beddington and May [1]. By applying stability analysis through the Jacobian matrix and eigenvalues, the model evaluates equilibrium points to determine whether specific agricultural practices lead to sustainable outcomes or unstable population dynamics, building on ecological stability theories [5], [7]. The study of equilibrium quantities and stability in population models also aligns with findings from economic equilibrium models, such as those discussed by Perez and Pelegrin [10], which explore how constraints in production and resource allocation influence stability in competitive systems. This framework offers valuable Insights for guiding sustainable agro-ecosystem management and improving pest control strategies in complex environments.

The Model of The System

The System of Equations of this Model is,

Prey- Population (x):

$$\frac{dx}{dt} = x\left(b_1 - \frac{x}{K_x} - \frac{\alpha y}{1+hx} - \varepsilon z + u_f - u_p\right) \tag{1}$$

Here, K_x Represents the Carrying Capacity of the Environment for the Prey Population.

Predator- Population (y):

$$\frac{dy}{dt} = y\left(b_2 - \beta x - \frac{y}{K_y} - \mu z - u_c\right) \tag{2}$$

Here, K_y Represents the Carrying Capacity of the Environment for the Predator Population.

Third Species or Environmental Factor (z):

$$\frac{dz}{dt} = z(-b_3 + d\varepsilon x + d\mu y) \tag{3}$$

Here z can be Another Species or an Environmental Factor Interacting with x (Prey) and y (Predator) where

- x − Represents the Population of Crops (Prey).
- y − Represents the Population of Pests (Predator).
- z − Represents a Third Species or Environmental factor Influencing the System by Interacting with Both Crops and Pests.
- $b_1, b_2, and\ b_3$ − Intrinsic Growth Rates (for Crop and Pest) or Decay Rates (for the Third Species).
- K_x and K_y − Carrying Capacities for Crops and Pests, respectively.

- $\alpha, \beta, \varepsilon, and\ \mu$ – Interaction Coefficients Determining the Effects of Predators, Prey, and Environmental Factors on Population Dynamics.
- h – Represents the Handling Time in the Holling Type II Functional Response.
- $u_f, u_p, and\ u_c$ – Denote the Effects of Agricultural Practices such as Fertilization, Pesticide Application, and Biological Control, respectively.
- $d\varepsilon\ and\ d\mu$ – Represent the Effects of Crops and Pests on the Third Species or Environmental Factor.

This Model provides a Comprehensive Framework for studying the Complex Interactions between Agricultural Management Practices, Crop and Pest Dynamics, and Environmental Factors. By simulating Various Scenarios and Management Strategies, the Model contributes to Optimizing Agricultural Practices for Sustainable Food Production while considering Ecological Impacts. Insights gained from this Model can Guide Decision-Making in Agricultural Policy, Pest Management Strategies, and Ecosystem Conservation Efforts. **Prey (Crops):** The variable x represents the Crop Population (Prey). This is indicated by the Term $-\frac{\alpha y}{1+hx}$ in the Equation for $\frac{dx}{dt}$, which describes the Predation Rate following a Holling Type II Functional Response. The Crops are Consumed by Pests, reflecting their role as the Prey. **Predator (Pests):** The Variable y represents the Pest Population (Predator). The Term $-\beta x$ in the Equation for $\frac{dy}{dt}$ reflects the Dependence of Pest Growth on the Crop Population, indicating that Pests Feed on the Crops. **Third Species/Environmental Factor:** The Variable z represents a Third Species or an Environmental Factor that interacts with both the Crops and the Pests. The Dynamics of z are influenced by both x and y, as seen in the Equation for $\frac{dz}{dt}$.

The Parameters are,

- b_1 - Intrinsic Growth Rate of the Crop Population (x).
- K_x - Carrying Capacity of the Crop Population.
- α – Predation Rate of Pests on Crops, Representing the Strength of Pest-Crop Interaction.
- h - Handling Time of Pests, Which Moderates the Functional Response of Pest Predation on Crops.
- ε – Interaction Coefficient Representing the Effect of the Third Species or Environmental Factor (z) on Crops.
- u_f – Fertilization Rate or Positive Intervention that Enhances Crop Growth.
- u_p – Pesticide Application Rate or Negative Intervention that Reduces Crop Growth.
- b_2 – Intrinsic Growth Rate of the Pest Population (y).

- β – Competition Coefficient Representing the Strength of Competition Between Crops and Pests.
- K_y – Carrying Capacity of the Pest Population.
- μ – Interaction Coefficient Representing the Effect of the Third Species (z) on Pests.
- u_c – Control Parameter Related to Biological Control or Other Interventions that Reduce Pest Population.
- b_3 – Intrinsic Death Rate of the Third Species or Environmental Factor (z).
- d – Interaction Strength of the Third Species with Crops and Pests.
- $d\varepsilon$ – Interaction Coefficient Representing the Effect of the Crop Population on the Third Species or Environmental Factor.
- $d\mu$ – Interaction Coefficient Representing the Effect of the Pest Population on the Third Species.

These Parameters Define the Rates of Growth, Interaction, and Control Within the System, Shaping the Dynamics Between Crops, Pests, and External Factors.

Equilibrium Points of The System

The Equilibrium Points occur where all Rates of Change are Zero, Meaning the System is Stable and Not Changing Over Time.

The First Equilibrium Point $(0, 0, 0)$.

The Second Equilibrium Point $(K_x(b_1 + u_f - u_p), 0{,}0)$.

The Third Equilibrium Point$(\mathbf{0}, K_y(b_2 - u_c), 0)$.

The Fourth Equilibrium Point

$$\left(\frac{\boldsymbol{b_1(1+hx)-\alpha K_y b_2+\alpha K_y u_c+(u_f-u_p)(1+hx)}}{\boldsymbol{\frac{1}{K_x}(1+hx)-\alpha K_y\beta}}, \boldsymbol{K_y(b_2-\beta x-u_c), 0}\right).$$

The Fifth Equilibrium Point $\left(0, \frac{\boldsymbol{b_3}}{\boldsymbol{d\mu}}, \frac{\boldsymbol{1}}{\boldsymbol{\mu}}\left(\boldsymbol{b_2} - \frac{\boldsymbol{b_3}}{\boldsymbol{d\mu K_y}} - \boldsymbol{u_c}\right)\right)$.

The Sixth Equilibrium Point $(x*, y*, z*) = \left(\frac{\boldsymbol{\mu K_y}}{\boldsymbol{\beta\mu K_y-\varepsilon}}\left(\boldsymbol{b_2} - \frac{\boldsymbol{b_3}}{\boldsymbol{K_y d}} - \boldsymbol{\mu z^*} - \boldsymbol{u_c}\right), \frac{\boldsymbol{b_3}}{\boldsymbol{d\mu}} - \frac{\boldsymbol{\varepsilon x^*}}{\boldsymbol{\mu}}, \frac{\boldsymbol{(1+hx^*)(b_1-\frac{x^*}{K_x})-\alpha(\frac{b_3}{\mu d}-\frac{\varepsilon x^*}{\mu})+(u_f-u_p)(1+hx^*)}}{\boldsymbol{\varepsilon(1+hx^*)}}\right)$.

The Jacobian Matrix,

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{b_1} - \frac{\boldsymbol{2x}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y}}{\boldsymbol{1+hx}} + \frac{\boldsymbol{\alpha hxy}}{\boldsymbol{(1+hx)^2}} - \varepsilon z + u_f - u_p & \frac{-x\alpha}{1+hx} & -\varepsilon x \\ -\beta y & b_2 - \beta x - \frac{2y}{K_y} - \boldsymbol{\mu z} - u_c & -\mu y \\ d\varepsilon z & d\mu z & -b_3 + d\varepsilon x + d\mu y \end{bmatrix}.$$

Stability And Nature of The Equilibrium Points

To Determine the Stability of the System using the Routh-Hurwitz Criterion, we need to find the Eigenvalues of the Jacobian Matrix J at the given

Equilibrium Points. The Jacobian Matrix of the System about the Equilibrium Point $\boldsymbol{J_0}(\mathbf{0},\mathbf{0},\mathbf{0})$ is given by

$$J = \begin{bmatrix} b_1 + u_f - u_p & 0 & 0 \\ 0 & b_2 - u_c & 0 \\ 0 & 0 & -b_3 \end{bmatrix}.$$

Hence, the corresponding Eigenvalues of this System are

$$\lambda_1 = b_1 + u_f - u_p\ ,\ \lambda_2 = b_2 - u_c,\ \lambda_3 = -b_3.$$

Here, $b_1 + u_f - u_p > 0,\ b_2 - u_c > 0,\ -b_3 < 0.$

Thus, $b_1 + u_f - u_p > 0,\ b_2 - u_c > 0,\ b_3 > 0.$

Now considering this Equilibrium Point (0,0,0) and using the Parameter Values $b_1 = 0.1,\ k_x = 3.0,\ \alpha = 0.05,\ h = 1,\ \varepsilon = 0.05,\ u_f = 0.02,\ u_p = 0.01,$ $b_2 = 0.2,\ \beta = 0.05,\ k_y = 3.0,\ \mu = 1,\ u_c = 0.1,\ b_3 = 0.3,\ d = 3.0,$ we obtain the Eigenvalues (using MATLAB Code) $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 > 0$.

Since, we have one Negative Eigenvalue $\lambda_1 < 0$ and two Positive Eigenvalues $\lambda_2 > 0, \lambda_3 > 0$, the System is Unstable at the Equilibrium Point (0,0,0). Therefore, the System is **Unstable.**

Next, we consider the Jacobian Matrix of the System about the Equilibrium Point $J_1\left(K_x\left(b_1 + u_f - u_p\right), 0,0\right)$ is given by,

$$J = \begin{bmatrix} -b_1 - u_f + u_p & -\frac{\alpha K_x(b_1+u_f-u_p)}{1+hK_x(b_1+u_f-u_p)} & -\varepsilon K_x\left(b_1 + u_f - u_p\right) \\ 0 & b_2 - \beta K_x\left(b_1 + u_f - u_p\right) - u_c & 0 \\ 0 & 0 & -b_3 + d\varepsilon K_x\left(b_1 + u_f - u_p\right) \end{bmatrix}.$$

Proof: We know that $|J - \lambda I| = 0$.

$$|J - \lambda I| = \begin{vmatrix} -b_1 - u_f + u_p - \lambda & -\dfrac{\alpha K_x\left(b_1 + u_f - u_p\right)}{1 + hK_x\left(b_1 + u_f - u_p\right)} & -\varepsilon K_x\left(b_1 + u_f - u_p\right) \\ 0 & b_2 - \beta K_x\left(b_1 + u_f - u_p\right) - u_c - \lambda & 0 \\ 0 & 0 & -b_3 + d\varepsilon K_x\left(b_1 + u_f - u_p\right) - \lambda \end{vmatrix} = 0$$

Since the Matrix is Upper Triangular Matrix (all elements below the diagonal elements are zero), so simply the product of the diagonal elements: $|J - \lambda I| = \left(-b_1 - u_f + u_p - \lambda\right)\left(b_2 - \beta K_x\left(b_1 + u_f - u_p\right) - u_c - \lambda \right)\left(-b_3 + d\varepsilon K_x\left(b_1 + u_f - u_p\right)\lambda\right) = 0$. Hence, the corresponding Eigenvalues of this System are $\lambda_1 = -b_1 - u_f + u_p$, $\lambda_2 = b_2 - \beta K_x\left(b_1 + u_f - u_p\right) - u_c$, $\lambda_3 = -b_3 + d\varepsilon K_x\left(b_1 + u_f - u_p\right)$. Here, $-b_1 - u_f + u_p < 0$, $b_2 - \beta K_x\left(b_1 + u_f - u_p\right) - u_c < 0$, $-b_3 + d\varepsilon K_x\left(b_1 + u_f - u_p\right) < 0$. But, consider this Equilibrium Point $\left(K_x\left(b_1 + u_f - u_p\right), 0,0\right)$ and used this Parameter Values $b_1 = 0.1, k_x = 3.0, \alpha = 0.05, h = 1, \varepsilon = 0.05, u_f = 0.02, u_p = 0.01, b_2 = 0.1, \beta = 0.05, k_y = 5.0, \mu = 1, u_c = 0.1, b_3 = 0.4, d = 3.0,$ we get the Eigenvalues (using MATLAB CODE) $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$. Therefore, the System is **Stable**.

Next, we consider the Jacobian Matrix of the System about the Equilibrium Point $J_2(0, K_y(b_2 - u_c), 0)$, is given by,

$$J = \begin{bmatrix} b_1 - \alpha K_y(b_2 - u_c) + u_f - u_p & 0 & 0 \\ -\beta K_y(b_2 - u_c) & -b_2 + u_c & -\mu K_y(b_2 - u_c) \\ 0 & 0 & -b_3 + d\mu K_y(b_2 - u_c) \end{bmatrix}.$$

Proof: We know that $|J - \lambda I| = 0$.

$$|J - \lambda I| = \begin{vmatrix} b_1 - \alpha K_y(b_2 - u_c) + u_f - u_p - \lambda & 0 & 0 \\ -\beta K_y(b_2 - u_c) & -b_2 + u_c - \lambda & -\mu K_y(b_2 - u_c) \\ 0 & 0 & -b_3 + d\mu K_y(b_2 - u_c) - \lambda \end{vmatrix} = 0.$$

Since, $|J - \lambda I| = (b_1 - \alpha K_y(b_2 - u_c) + u_f - u_p - \lambda)(-b_2 + u_c - \lambda)(-b_3 + d\mu K_y(b_2 - u_c) - \lambda) = 0$.

Hence, the corresponding Eigenvalues of this System are $\lambda_1 = b_1 - \alpha K_y(b_2 - u_c) + u_f - u_p$, $\lambda_2 = -b_2 + u_c$, $\lambda_3 = -b_3 + d\mu K_y(b_2 - u_c)$.
Here, $\lambda_1 = b_1 - \alpha K_y(b_2 - u_c) + u_f - u_p < 0$, $\lambda_2 = -b_2 + u_c < 0$, $\lambda_3 = -b_3 + d\mu K_y(b_2 - u_c) < 0$. But, consider this Equilibrium Point $(\mathbf{0}, \boldsymbol{K_y(b_2 - u_c)}, \mathbf{0})$ and used this Parameter Values $b_1 = 0.5, K_x = 1, \alpha = 1, h = 1, \varepsilon = 0.8, u_f = 0.1, u_p = 0.05, b_2 = 1.5, \beta = 1, K_y = 1, \mu = 1, u_c = 0.5, b_3 = 1.2, d = 0.7$. we get the Eigenvalues (using MATLAB CODE) $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$. Therefore, the System is **Stable. []**

Next, we consider the Jacobian Matrix of the System about the Equilibrium Point

$$\boldsymbol{J_3}\left(\frac{\boldsymbol{b_1(1+hx) - \alpha K_y b_2 + \alpha K_y u_c + (u_f - u_p)(1+hx)}}{\boldsymbol{\frac{1}{K_x}(1+hx) - \alpha K_y \beta}}, \boldsymbol{K_y(b_2 - \beta x - u_c)}, \mathbf{0}\right).$$

First, let's denote the Equilibrium Point as (x_e, y_e, z_e).

$$x_e = \frac{\boldsymbol{b_1(1+hx) - \alpha K_y b_2 + \alpha K_y u_c + (u_f - u_p)(1+hx)}}{\boldsymbol{\frac{1}{K_x}(1+hx) - \alpha K_y \beta}}, y_e = \boldsymbol{K_y(b_2 - \beta x - u_c)}, z_e = 0.$$

The Jacobian Matrix is given by,

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{b_1} - \frac{\boldsymbol{2x_e}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y_e}}{\boldsymbol{1 + hx_e}} + \frac{\boldsymbol{\alpha h x_e y_e}}{\boldsymbol{(1 + hx_e)^2}} & \boldsymbol{u_f} - u_p & \frac{-\alpha x_e}{1 + hx_e} \\ -\varepsilon x_e - \beta y_e & b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c & -\mu \boldsymbol{y_e} \\ 0 & 0 & -b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e} \end{bmatrix}$$

Proof: We know that $|J - \lambda I| = 0$.

$$|J - \lambda I| = \begin{vmatrix} \boldsymbol{b_1} - \frac{2x_e}{K_x} - \frac{\alpha y_e}{1 + hx_e} + \frac{\alpha h x_e y_e}{(1 + hx_e)^2} - \lambda & \boldsymbol{u_f} - u_p & \frac{-\alpha x_e}{1 + hx_e} \\ -\varepsilon x_e - \beta y_e & b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c - \lambda & -\mu \boldsymbol{y_e} \\ 0 & 0 & -b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e} - \lambda \end{vmatrix} = 0.$$

Since, $|J-\lambda I| = ((\boldsymbol{b_1} - \frac{\boldsymbol{2x_e}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y_e}}{\boldsymbol{1+hx_e}} + \frac{\boldsymbol{\alpha h x_e y_e}}{(\boldsymbol{1+hx_e})^2} - \lambda)(b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c - \lambda)(-b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e} - \lambda)) - ((\boldsymbol{u_f} - u_p)((-\varepsilon x_e - \beta y_e)(-b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e} - \lambda))) = 0.$

Simply the above equation, we get the Characteristic Equation can be rewritten as, $(\lambda - (\boldsymbol{b_1} - \frac{\boldsymbol{2x_e}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y_e}}{\boldsymbol{1+hx_e}} + \frac{\boldsymbol{\alpha h x_e y_e}}{(\boldsymbol{1+hx_e})^2}))(\lambda - (b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c))$ $\left(\lambda - (-b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e})\right) - (\boldsymbol{u_f} - u_p)(-\varepsilon x_e - \beta y_e)(\lambda - (-b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e})) = 0.$

There are two main terms:

1. **Product of the Diagonal Elements**: $(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ where $\lambda_1 = \boldsymbol{b_1} - \frac{\boldsymbol{2x_e}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y_e}}{\boldsymbol{1+hx_e}} + \frac{\boldsymbol{\alpha h x_e y_e}}{(\boldsymbol{1+hx_e})^2}$, $\lambda_2 = b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c\, b_2 - \lambda_3 = -b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e}.$
2. **Coupling of Term:** $-(\boldsymbol{u_f} - u_p)(-\varepsilon x_e - \beta y_e)(\lambda - \lambda_3)$

   Hence, the corresponding Eigenvalues of this System are $\lambda_1 = \boldsymbol{b_1} - \frac{\boldsymbol{2x_e}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y_e}}{\boldsymbol{1+hx_e}} + \frac{\boldsymbol{\alpha h x_e y_e}}{(\boldsymbol{1+hx_e})^2}$, $\lambda_2 = b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c$, $\lambda_3 = -b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e}.$

   Since, $\lambda_1 = \boldsymbol{b_1} - \frac{\boldsymbol{2x_e}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y_e}}{\boldsymbol{1+hx_e}} + \frac{\boldsymbol{\alpha h x_e y_e}}{(\boldsymbol{1+hx_e})^2} < \boldsymbol{0}, \lambda_2 = b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c <$

$0,\ \lambda_3 = -b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e}. > 0.$

But, consider this Equilibrium Point

$$\left(\frac{\boldsymbol{b_1(1+hx) - \alpha K_y b_2 + \alpha K_y u_c + (u_f - u_p)(1+hx)}}{\boldsymbol{\frac{1}{K_x}(1+hx) - \alpha K_y \beta}}, \boldsymbol{K_y(b_2 - \beta x - u_c), 0}\right),$$

and used this Parameter Values $b_1 = 0.1, k_x = 4.0, \alpha = 0.05, h = 1, \varepsilon = 0.05, u_f = 0.02, u_p = 0.01, b_2 = 0.2, \beta = 0.05, k_y = 3.0, \mu = 1, u_c = 0.1, b_3 = 0.2, d = 4.0$, we get the Eigenvalues (using MATLAB Code) $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 > 0$. Therefore, the System is **Unstable.**

Next, we consider the Jacobian Matrix of the System about the Equilibrium Point

$$\boldsymbol{J_4}\left(0, \frac{\boldsymbol{b_3}}{\boldsymbol{d\mu}}, \frac{\boldsymbol{1}}{\boldsymbol{\mu}}\left(\boldsymbol{b_2} - \frac{\boldsymbol{b_3}}{\boldsymbol{d\mu K_y}} - \boldsymbol{u_c}\right)\right) \text{ is given by,}$$

$$J = \begin{bmatrix} b_1 - \frac{\alpha b_3}{d\mu} - \varepsilon\left(\frac{b_2}{\mu} - \frac{b_3}{d\mu^2 K_y} - \frac{u_c}{\mu}\right) + u_f - u_p & 0 & 0 \\ -\frac{\beta b_3}{d\mu} & b_3 - \frac{b_3}{d\mu K_y} - u_c & -b_3 \\ \varepsilon\left(\frac{b_2}{\mu} - \frac{b_3}{d\mu^2 K_y} - \frac{u_c}{\mu}\right) & b_2 - \frac{b_3}{d\mu K_y} - u_c & 0 \end{bmatrix}.$$

Proof: We know that $|J - \lambda I| = 0$.

$$|J-\lambda I| = \begin{vmatrix} b_1-\frac{\alpha b_3}{d\mu}-\varepsilon\left(\frac{b_2}{\mu}-\frac{b_3}{d\mu^2K_y}-\frac{u_c}{\mu}\right)+u_f-u_p-\lambda & 0 & 0 \\ -\frac{\beta b_3}{d\mu} & b_3-\frac{b_3}{d\mu K_y}-u_c-\lambda & -b_3 \\ \varepsilon\left(\frac{b_2}{\mu}-\frac{b_3}{d\mu^2K_y}-\frac{u_c}{\mu}\right) & b_2-\frac{b_3}{d\mu K_y}-u_c & -\lambda \end{vmatrix} = 0.$$

$$|J-\lambda I| = \left(b_1-\frac{\alpha b_3}{d\mu}-\varepsilon\left(\frac{b_2}{\mu}-\frac{b_3}{d\mu^2K_y}-\frac{u_c}{\mu}\right)+u_f-u_p-\lambda\right)$$

$$\left[\left(b_3-\frac{b_3}{d\mu K_y}-u_c-\lambda\right)(-\lambda)-(-b_3)\left(b_2-\frac{b_3}{d\mu K_y}-u_c\right)\right]=0.$$

$$\left(b_1-\frac{\alpha b_3}{d\mu}-\varepsilon\left(\frac{b_2}{\mu}-\frac{b_3}{d\mu^2K_y}-\frac{u_c}{\mu}\right)+u_f-u_p-\lambda\right)$$

$$\left(\lambda^2-\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)\lambda+b_3\left(b_2-\frac{b_3}{d\mu K_y}-u_c\right)\right)=0.$$

This equation gives the Eigenvalues,

$$\lambda_1 = b_1-\frac{\alpha b_3}{d\mu}-\varepsilon\left(\frac{b_2}{\mu}-\frac{b_3}{d\mu^2K_y}-\frac{u_c}{\mu}\right)+u_f-u_p.$$

The other two Eigenvalues λ_2 *and* λ_3 are the roots of the Quadratic Equation:

$$\lambda^2-\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)\lambda+b_3\left(b_2-\frac{b_3}{d\mu K_y}-u_c\right)=0.$$

Here, $\lambda_1 = b_1-\frac{\alpha b_3}{d\mu}-\varepsilon\left(\frac{b_2}{\mu}-\frac{b_3}{d\mu^2K_y}-\frac{u_c}{\mu}\right)+u_f-u_p.$

$$\boldsymbol{\lambda_2} = \frac{1}{2}\ [\ (b_3-\frac{b_3}{d\mu K_y}-u_c\ )\ +\sqrt{\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)^2-4(\ b_3)(b_2-\frac{b_3}{d\mu K_y}-u_c}\ \ ].$$

$$\boldsymbol{\lambda_3} = \frac{1}{2}\ [\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)-\sqrt{\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)^2-4(\ b_3)(b_2-\frac{b_3}{d\mu K_y}-u_c}\ \ ].$$

Since, $\lambda_1 = b_1-\frac{\alpha b_3}{d\mu}-\varepsilon\left(\frac{b_2}{\mu}-\frac{b_3}{d\mu^2K_y}-\frac{u_c}{\mu}\right)+u_f-u_p>0.$

$$\boldsymbol{\lambda_2} = \frac{1}{2}\left[\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)+\sqrt{\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)^2-4(\ b_3)(b_2-\frac{b_3}{d\mu K_y}-u_c}\right]>0.$$

$$\boldsymbol{\lambda_3} = \frac{1}{2}\ [\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)-\sqrt{\left(b_3-\frac{b_3}{d\mu K_y}-u_c\right)^2-4(\ b_3)(b_2-\frac{b_3}{d\mu K_y}-u_c}\ \ ]>\mathbf{0}.$$

But, consider this Equilibrium Point $\left(0,\boldsymbol{\frac{b_3}{d\mu}},\boldsymbol{\frac{1}{\mu}}\left(\boldsymbol{b_2}-\boldsymbol{\frac{b_3}{d\mu K_y}}-\boldsymbol{u_c}\right)\right)$**,** and used this Parameter Values $b_1=0.05, k_x=4.0, \alpha=0.05, h=1, \varepsilon=0.05, u_f=0.02, u_p=0.01, b_2=0.15, \beta=0.05, k_y=5.0, \mu=1, u_c=$

$0.1, b_3 = 0.2, d = 4.0$ we get the Eigenvalues(using MATLAB code) $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$. Therefore, the System is **Unstable.**

Interior Equilibrium Point: Stability Analysis

Theorem: *Stability of the System at the Equilibrium Point*

$$(x^*, y^*, z^*) = [\frac{\mu K_y}{\beta\mu K_y - \varepsilon}\left(\boldsymbol{b_2} - \frac{b_3}{K_y d} - \boldsymbol{\mu z^*} - \boldsymbol{u_c}\right), \frac{b_3}{d\mu} - \frac{\varepsilon x^*}{\mu}, \frac{(1+hx^*)(b_1 - \frac{x^*}{K_x}) - \alpha(\frac{b_3}{\mu d} - \frac{\varepsilon x^*}{\mu}) + (u_f - u_p)(1+hx^*)}{\varepsilon(1+hx^*)}].$$

Proof: Consider the Jacobian Matrix $J_5(x^*, y^*, z^*)$ of the System about the Equilibrium Point

$$(x^*, y^*, z^*) = [\frac{\mu K_y}{\beta\mu K_y - \varepsilon}\left(\boldsymbol{b_2} - \frac{b_3}{K_y d} - \boldsymbol{\mu z^*} - \boldsymbol{u_c}\right), \frac{b_3}{d\mu} - \frac{\varepsilon x^*}{\mu}, \frac{(1+hx^*)(b_1 - \frac{x^*}{K_x}) - \alpha(\frac{b_3}{\mu d} - \frac{\varepsilon x^*}{\mu}) + (u_f - u_p)(1+hx^*)}{\varepsilon(1+hx^*)}].$$

The Jacobian Matrix is given by,

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{b_1} - \frac{2x}{K_x} - \frac{\alpha y}{1+hx} + \frac{\alpha hxy}{(1+hx)^2} - \varepsilon z + u_f - u_p & \frac{-x\alpha}{1+hx} & -\varepsilon x \\ -\beta y & b_2 - \beta x - \frac{2y}{K_y} - \boldsymbol{\mu z} - u_c & -\mu y \\ d\varepsilon z & d\mu z & -b_3 + d\varepsilon x + d\mu y \end{bmatrix}.$$

Replace $x = x^*,\ y = y^*,\ z = z^*$ in the Jacobian Matrix.

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{b_1} - \frac{2x^*}{K_x} - \frac{\alpha y^*}{1+hx^*} + \frac{\alpha hx^*y^*}{(1+hx^*)^2} - \varepsilon z^* + u_f - u_p & \frac{-x^*\alpha}{1+hx^*} & -\varepsilon x^* \\ -\beta y^* & b_2 - \beta x^* - \frac{2y^*}{K_y} - \boldsymbol{\mu z^*} - u_c & -\mu y^* \\ d\varepsilon z^* & d\mu z^* & -b_3 + d\varepsilon x^* + d\mu y^* \end{bmatrix}.$$

We know that $|J - \lambda I| = 0$.

$$|J - \lambda I| = \begin{vmatrix} \boldsymbol{b_1} - \frac{2x^*}{K_x} - \frac{\alpha y^*}{1+hx^*} + \frac{\alpha hx^*y^*}{(1+hx^*)^2} - \varepsilon z^* + u_f - u_p - \lambda & \frac{-x^*\alpha}{1+hx^*} & -\varepsilon x^* \\ -\beta y^* & b_2 - \beta x^* - \frac{2y^*}{K_y} - \boldsymbol{\mu z^*} - u_c - \lambda & -\mu y^* \\ d\varepsilon z^* & d\mu z^* & -b_3 + d\varepsilon x^* + d\mu y^* - \lambda \end{vmatrix}.$$

$$|J - \lambda I| = \left(\boldsymbol{b_1} - \frac{\boldsymbol{2x^*}}{\boldsymbol{K_x}} - \frac{\boldsymbol{\alpha y^*}}{\boldsymbol{1+hx^*}} + \frac{\boldsymbol{\alpha hx^*y^*}}{\boldsymbol{(1+hx^*)^2}} - \varepsilon z^* + u_f - u_p - \lambda\right)$$
$$\left[\left(b_2 - \beta x^* - \frac{2y^*}{K_y} - \mu z^* - u_c - \lambda\right)(-b_3 + d\varepsilon x^* + d\mu y^* - \lambda) - (-\mu y^*)(d\mu z^*)\right]$$
$$-\left(\frac{-x^*\alpha}{1+hx^*}\right)[(-\beta y^*)(-b_3 + d\varepsilon x^* + d\mu y^* - \lambda) - (d\varepsilon z^*)(-\mu y^*)] + (-\varepsilon x^*)$$
$$\left[(-\beta y^*)(d\mu z^*) - (d\varepsilon z^*)\left(b_2 - \beta x^* - \frac{2y^*}{K_y} - \mu z^* - u_c - \lambda\right)\right] = 0.$$

But, consider this equilibrium point $\left(\frac{\boldsymbol{\mu K_y}}{\boldsymbol{\beta\mu K_y-\varepsilon}}\left(\boldsymbol{b_2}-\frac{\boldsymbol{b_3}}{\boldsymbol{K_y d}}-\boldsymbol{\mu z^*}-\boldsymbol{u_c}\right), \frac{\boldsymbol{b_3}}{\boldsymbol{d\mu}}-\frac{\boldsymbol{\varepsilon x^*}}{\boldsymbol{\mu}}, \frac{(\boldsymbol{1+hx^*})(\boldsymbol{b_1}-\frac{\boldsymbol{x^*}}{\boldsymbol{K_x}})-\boldsymbol{\alpha}(\frac{\boldsymbol{b_3}}{\boldsymbol{\mu d}}-\frac{\boldsymbol{\varepsilon x^*}}{\boldsymbol{\mu}})+(\boldsymbol{u_f-u_p})(\boldsymbol{1+hx^*})}{\boldsymbol{\varepsilon(1+hx^*)}}\right)$, and used this Parameter Values $b_1 = 0.1,\ k_x = 3.0,\ \alpha = 0.05,\ h = 1,\ \varepsilon = 0.05,\ u_f = 0.02,\ u_p = 0.01,\ b_2 = 0.2,\ \beta = 0.05,\ k_y = 3.0,\ \mu = 1,\ u_c = 0.1,\ b_3 = 0.3,\ d = 3.0$, we get the complex Eigenvalues, (using MATLAB code) but the Real Part of the Eigenvalues are Real and Negative $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$.Therefore, the System is Asymptotically Stable. Hence the System is Stable.

Numerical Analysis

In Numerical Experiments we use MATLAB Software and, in this System (1,2,3), we illustrate some findings of the System numerically around the Stability for a range of Parameter Values and also plot the corresponding Graph. First, we consider the Parameter Values for $b_1 = 0.1, k_x = 3.0, \alpha = 0.05, h = 1, \varepsilon = 0.05, u_f = 0.02, u_p = 0.01, b_2 = 0.2, \beta = 0.05, k_y = 3.0, \mu = 1, u_c = 0.1, b_3 = 0.3, d = 3.0$. Analyzing the Stability of a System at the Equilibrium Point $\boldsymbol{J_0(0,0,0)}$ using the Routh-Hurwitz Criterion. The corresponding Eigenvalues of the Jacobian Matrix is $\lambda_1 = b_1 + u_f - u_p$, $\lambda_2 = b_2 - u_c$, $\lambda_3 = -b_3$. Therefore, $\lambda_1 = 0.11 > 0$, $\lambda_2 = 0.1 > 0$, $\lambda_3 = -0.3 < 0$. Since the System has one Negative Eigenvalue $\lambda_3 < 0$ and two Positive Eigenvalues $\lambda_1 > 0$, $\lambda_2 > 0$, the Equilibrium Point $\boldsymbol{J_0(0,0,0)}$ is Unstable. For an Equilibrium Point to be Stable, all the Eigenvalues must have Negative Real Parts. The presence of any Positive Eigenvalue indicates Instability, as the System would have at least one direction in which Perturbations grow over time. Given the Parameter Values and the resulting Eigenvalues, the System is Unstable at the Equilibrium Point $\boldsymbol{J_0(0,0,0)}$. The Negative Eigenvalue λ_3 suggests that Perturbations along the z-axis would decay, while the Positive Eigenvalues λ_1 *and* λ_2 indicate that Perturbations in the x and y Directions would grow, leading to overall Instability (see Fig.1 and Fig.2). Therefore, the System is Unstable at this point. **Secondly,** we consider the Parameter Values for $b_1 = 0.1, k_x = 3.0, \alpha = 0.05, h = 1, \varepsilon = 0.05, u_f = 0.02, u_p = 0.01, b_2 = 0.1, \beta = 0.05, k_y = 5.0, \mu = 1, u_c = 0.1, b_3 = 0.4, d = 3.0$. Analysing the Stability of a System at the Equilibrium Point $\boldsymbol{J_1\left(K_x\left(b_1+u_f-u_p\right),0,0\right)}$, using the Routh-Hurwitz Criterion. The corresponding Eigenvalues of the Jacobian Matrix are $\lambda_1 = -b_1 - u_f + u_p$, $\lambda_2 = b_2 - \beta K_x\left(b_1 + u_f - u_p\right) - u_c$, $\lambda_3 = -b_3 + d\varepsilon K_x\left(b_1 + u_f - u_p\right)$. Therefore, $\lambda_1 = -0.11 < 0$, $\lambda_2 = -0.025 < 0$, $\lambda_3 = -0.355 < 0$. Since all Eigenvalues are Negative, the Equilibrium Point $\boldsymbol{J_1}$ is **Stable** according to the Routh-Hurwitz Criterion. This indicates that small Perturbations around this Equilibrium Point will decay over time, and the System will return to Equilibrium, confirming Stability (see Fig. 4 and Fig.5). Thirdly, we consider the Parameter Values for $b_1 = 0.5, K_x = 1, \alpha = 1, h = 1, \varepsilon = 0.8, u_f = 0.1, u_p = 0.05, b_2 = 1.5, \beta = 1, K_y = 1, \mu = 1, u_c =$

$0.5, b_3 = 1.2, d = 0.7$. Analysing the Stability of a System at the Equilibrium Point $\boldsymbol{J_2(0, K_y(b_2 - u_c), 0)}$ using the Routh-Hurwitz Criterion.

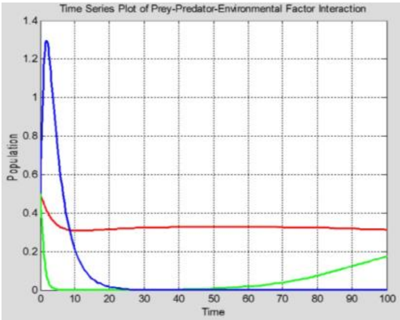

Figure 1
Time Series Plot of Prey-Predator-Environmental Factor Interaction

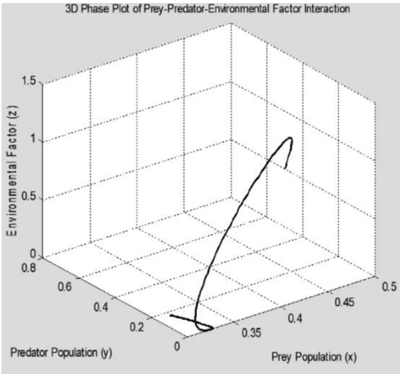

Figure 2
3D Phase Plot of Prey-Predator Environmental Factor Interaction

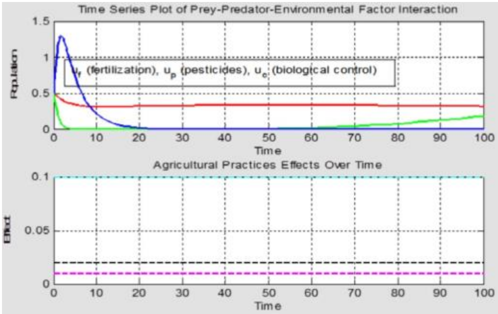

Figure 3
Subplot of Time Series Plot of Prey-Predator Environmental Factor Interaction with Time and Population and Agricultural Practices Effect Over Time with Time and Effect.

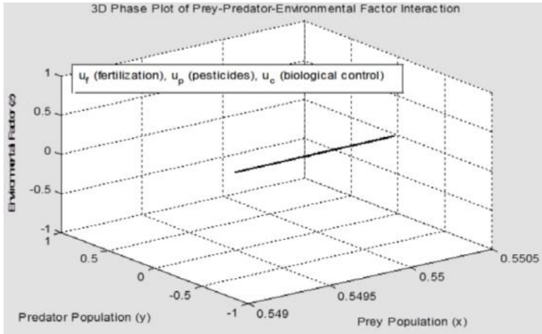

Figure 4
3D Phase Plot of Prey-Predator Environmental Factor Interaction

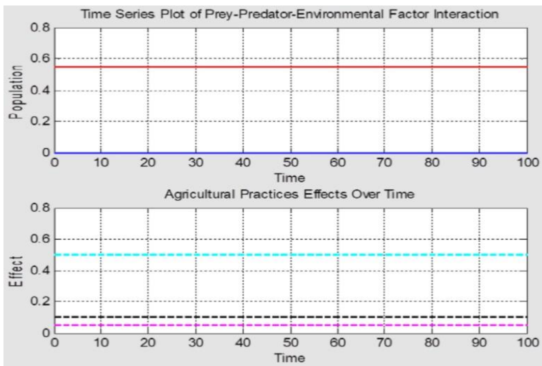

Figure 5
Subplot of Time Series Plot of Prey-Predator Environmental Factor Interaction with Time and Population and Agricultural Practices Effect Over Time with Time and Effect.

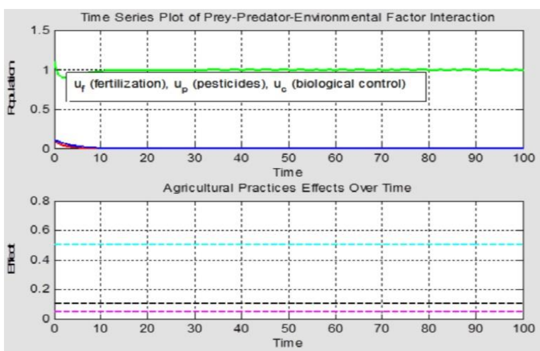

Figure 6
Subplot of Time Series Plot of Prey-Predator Environmental Factor Interaction with Time and Population and Agricultural Practices Effect Over Time with Time and Effect.

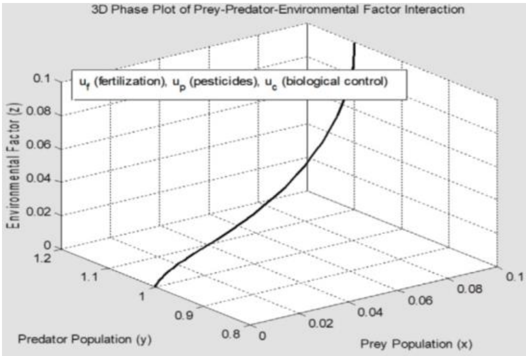

Figure 7
3D Phase Plot of Prey-Predator Environmental Factor Interaction

The Eigenvalues are $\lambda_1 = b_1 - \alpha K_y(b_2 - u_c) + u_f - u_p, \lambda_2 = -b_2 + u_c, \lambda_3 = -b_3 + d\mu K_y(b_2 - u_c)$. Therefore, $\lambda_1 = -0.95 < 0$, $\lambda_2 = -1 < 0$, $\lambda_3 = -0.5 < 0$. Since all Eigenvalues are Negative, the Equilibrium Point $\boldsymbol{J_2}$ is Stable (see Fig.6 and Fig.7) according to the Routh-Hurwitz Criterion. Fourthly, we consider the Parameter Values for$b_1 = 0.1, k_x = 4.0, \alpha = 0.05, h = 1, \varepsilon = 0.05, u_f = 0.02, u_p = 0.01, b_2 = 0.2, \beta = 0.05, k_y = 3.0, \mu = 1, u_c = 0.1, b_3 = 0.2, d = 4.0$. Analysing the Stability of a System at the Equilibrium Point

$$\boldsymbol{J_3}\left(\frac{\boldsymbol{b_1(1+hx) - \alpha K_y b_2 + \alpha K_y u_c + (u_f - u_p)(1+hx)}}{\boldsymbol{\frac{1}{K_x}(1+hx) - \alpha K_y \beta}}, \boldsymbol{K_y(b_2 - \beta x - u_c), 0} \right)$$ using

the Routh-Hurwitz Criterion. The corresponding Eigenvalues of the Jacobian Matrix are, $\lambda_1 = b_1 - \frac{2x_e}{K_x} - \frac{\alpha y_e}{1+hx_e} + \frac{\alpha h x_e y_e}{(1+hx_e)^2} < 0$,

$$\lambda_2 = b_2 - \beta x_e - \frac{2y_e}{K_y} - u_c < 0, \lambda_3 = -b_3 + d\varepsilon x_e + d\mu \boldsymbol{y_e} > 0.$$

Therefore, $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 < 0$.

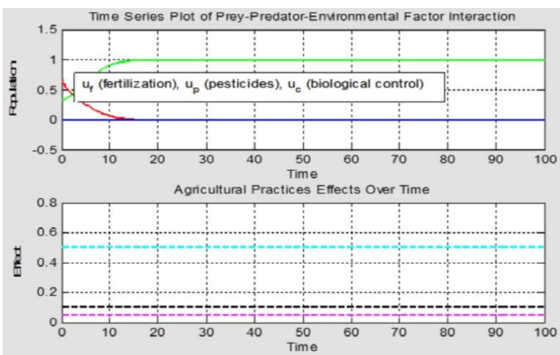

Figure 8
Subplot of Time Series Plot of Prey-Predator Environmental Factor Interaction with Time and Population and Agricultural Practices Effect Over Time with Time and Effect.

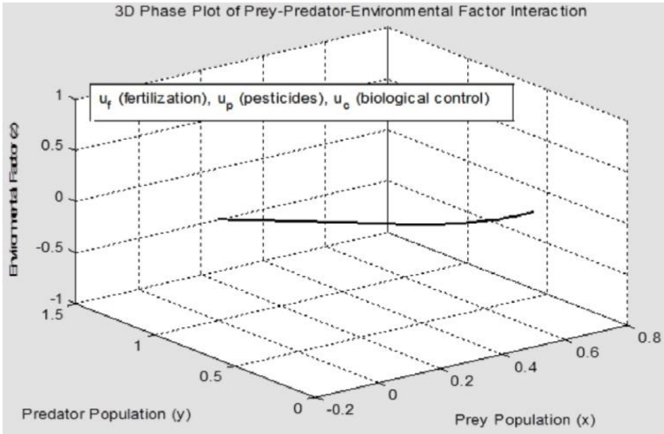

Figure 9
3D Phase Plot of Prey-Predator Environmental Factor Interaction

The Eigenvalues indicate that while two Eigenvalues are Negative, one is Positive. This implies that the System is **Unstable** (see Fig.8 and Fig.9) at the Equilibrium Point $\boldsymbol{J_3}$. The Positive Eigenvalue $\boldsymbol{\lambda_3}$ causes the System to diverge from the Equilibrium, leading to Instability.

Fifthly, we consider the Parameter Values for

$b_1 = 0.05, k_x = 4.0, \alpha = 0.05, h = 1, \varepsilon = 0.05, u_f = 0.02, u_p = 0.01, b_2 = 0.15, \beta = 0.05, k_y = 5.0, \mu = 1, u_c = 0.1, b_3 = 0.2, d = 4.0.$

Analysing the Stability of a System at the Equilibrium Point $(0, \boldsymbol{\frac{b_3}{d\mu}}, \boldsymbol{\frac{1}{\mu}}(\boldsymbol{b_2} - \boldsymbol{\frac{b_3}{d\mu K_y}} - \boldsymbol{u_c}))$ using the Routh-Hurwitz Criterion. The corresponding Eigenvalues of the Jacobian Matrix are

$$\lambda_1 = b_1 - \frac{\alpha b_3}{d\mu} - \varepsilon\left(\frac{b_2}{\mu} - \frac{b_3}{d\mu^2 K_y} - \frac{u_c}{\mu}\right) + u_f - u_p > 0.$$

$$\boldsymbol{\lambda_2} = \frac{\mathbf{1}}{\mathbf{2}}\left[\left(b_3 - \frac{b_3}{d\mu K_y} - u_c\right) + \sqrt{\left(b_3 - \frac{b_3}{d\mu K_y} - u_c\right)^2 - 4(b_3)(b_2 - \frac{b_3}{d\mu K_y} - u_c}\right] > \mathbf{0}.$$

$$\boldsymbol{\lambda_3} = \frac{\mathbf{1}}{\mathbf{2}}\left[\left(b_3 - \frac{b_3}{d\mu K_y} - u_c\right) - \sqrt{\left(b_3 - \frac{b_3}{d\mu K_y} - u_c\right)^2 - 4(b_3)(b_2 - \frac{b_3}{d\mu K_y} - u_c}\right] > \mathbf{0.}$$

Therefore, $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$. Since, all Eigenvalues are Positive, the Equilibrium Point $\boldsymbol{J_4}$ is **Unstable** (see Fig.10 and Fig.11) according to the Routh-Hurwitz Criterion.

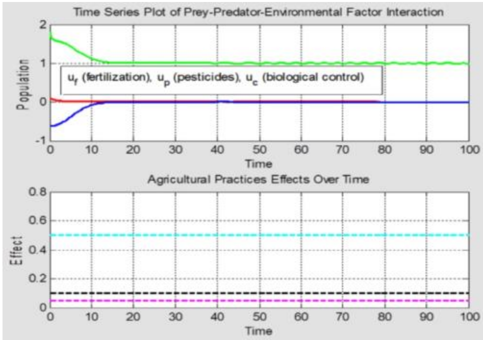

Figure 10

Subplot of Time Series Plot of Prey-Predator Environmental Factor Interaction with Time and Population and Agricultural Practices Effect Over Time with Time and Effect.

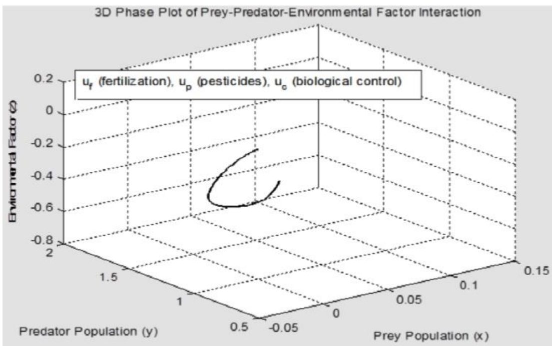

Figure 11

3D Phase Plot of Prey-Predator Environmental Factor Interaction

Interior Equilibrium Point: Assume the Interior Equilibrium Point $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$ exists. The stability of this Equilibrium Point is analyzed using the Jacobian Matrix of the System evaluated at (x^*, y^*, z^*). We consider the Parameter Values for $b_1 = 0.1, k_x = 3.0, \alpha = 0.05, h = 1, \varepsilon = 0.05, u_f = 0.02, u_p = 0.01, b_2 = 0.2, \beta = 0.05, k_y = 3.0, \mu = 1, u_c = 0.1, b_3 = 0.3, d = 3.0$. Analysing the Stability of a System at the Equilibrium Point $\left(\frac{\mu K_y}{\beta\mu K_y - \varepsilon}\left(b_2 - \frac{b_3}{K_y d} - \mu z^* - u_c\right), \frac{b_3}{d\mu} - \frac{\varepsilon x^*}{\mu}, \frac{(1+hx^*)(b_1 - \frac{x^*}{K_x}) - \alpha(\frac{b_3}{\mu d} - \frac{\varepsilon x^*}{\mu}) + (u_f - u_p)(1+hx^*)}{\varepsilon(1+hx^*)}\right)$ using the Routh-Hurwitz Criterion**.** The corresponding Eigenvalues of the Jacobian Matrix are $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, using MATLAB Code. Since, all the Eigenvalues have Negative Real Parts, the system is **Asymptotically Stable** (see Fig.12) at the Interior Equilibrium Point $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$**.** Perturbations from this Equilibrium Point will decay over time, and the System will return to the Equilibrium State, conforming that the System is **Stable.**

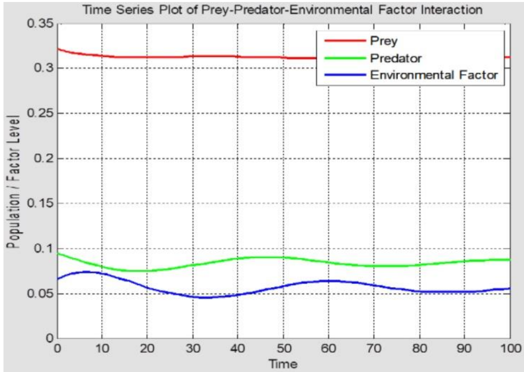

Figure 12
Time Series Plot of Prey-Predator Environmental Factor Interaction

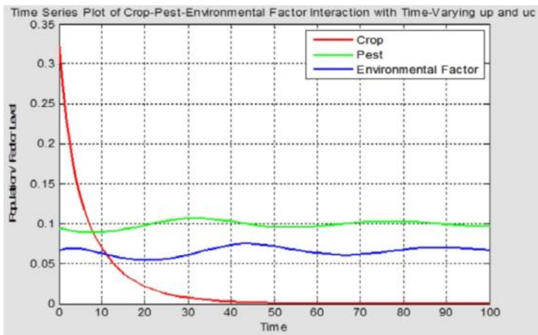

Figure 13
Time Series Plot of Crop-Pest Environmental Factor Interaction with Time- Varying u_p and u_c.

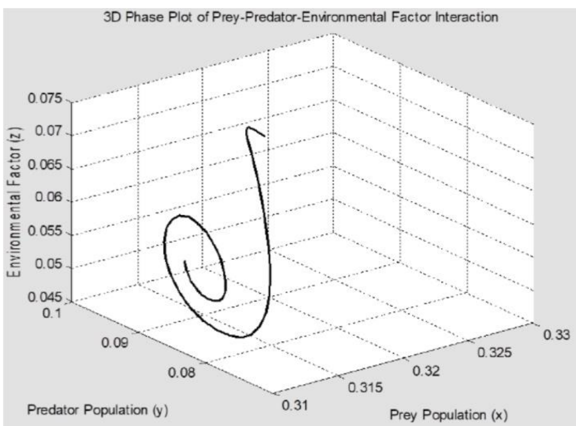

Figure 14
3D Phase Plot of Prey-Predator Environmental Factor Interaction

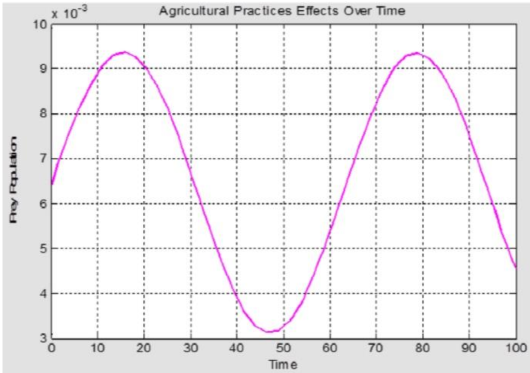

Figure 15
Agricultural Practices Effects Over Time with Time and Prey Population

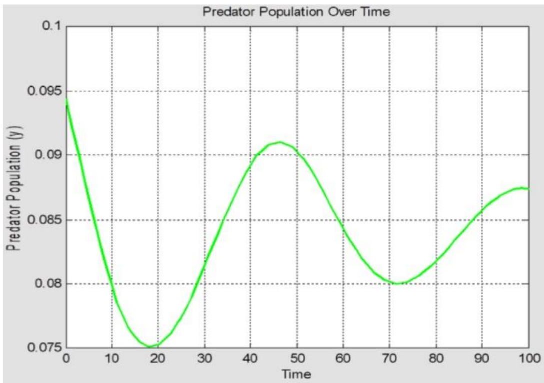

Figure 16
Predator Population Over Time with Time and Predator Population(y)

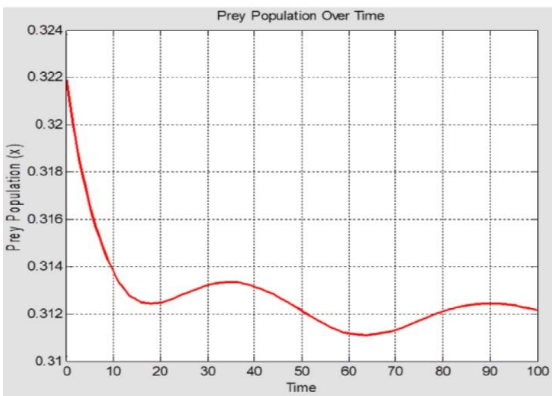

Figure 17
Prey Population Over Time with Time and Prey Population(x)

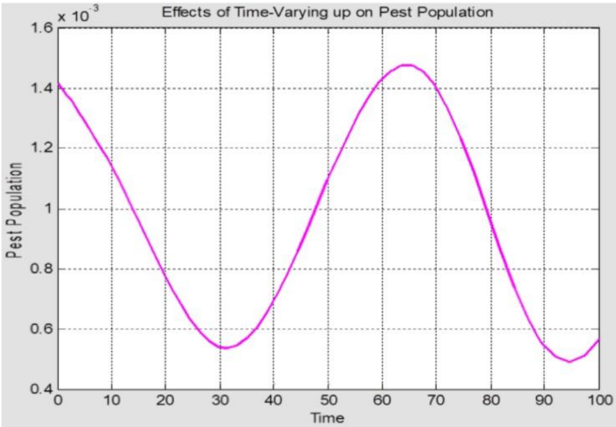

Figure 18
Effect of Time-Varying u_p on Pest Population with Time and Pest Population

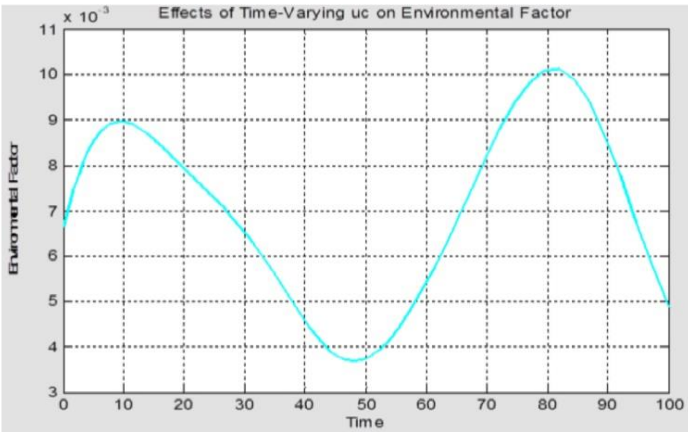

Figure 19
Effect of Time-Varying u_c on Environmental Factor with Time and Environmental Factor

Conclusions

This Three-Species Interaction Model provides a Framework for understanding the Dynamics of Crop-Pest-Environment Interactions in Agricultural Ecosystems. It integrates classical Ecological Theories, including Resource Competition, Predator-Prey Relationships, and Population Dynamics, as established by foundational work such as Lotka and Volterra [4], [9], and further developed by May [3] and Holling [5]. The Model incorporates the Effects of Agricultural Practices, Environmental Stressors, and Species Interactions, helping to explore Sustainable Agricultural Management and Pest Control Strategies. Prey (Crops) Dynamics: The Crop Population (Prey) is influenced by its Intrinsic Growth Rate and is limited by Predation from Pests, Modeled with a Holling Type II Functional Response [3]. Agricultural Interventions such as Fertilization and Pesticide use are incorporated as Control Mechanisms, allowing for both Positive and Negative impacts on Crop Growth. Environmental Factors, represented by the Third Species (z), exert pressure on Crop Populations, reflecting the complex balance between Agricultural Productivity and Ecological Resilience, as discussed in [7] and [8]. The Time Series Plot of Prey-Predator-Environmental Factor Interaction (Figure 1) illustrates these Population Changes over Time. Predator (Pests) Dynamics: The Pest Population (Predator) depends on the Crop Population for Growth and is limited by its own Carrying Capacity.

The Interaction between Pests and the Third Species or Environmental Factor is critical, with Agricultural Interventions like Biological Control influencing Pest Dynamics [12]. The Model shows that a balanced application of control measures can Stabilize Pest Populations, as supported by Theories on Population Management in Fluctuating Environments [1]. These Dynamics are clearly depicted in the 3D Phase Plot of Prey-Predator-Environmental Factor Interaction (Figure 2) and further explained in Subplot of Time Series Plot that includes Agricultural Practices over Time (Figure 3). Third Species Environmental Factor Dynamics: The Third Species or Environmental Factor (z) plays a significant role in the System, potentially representing Beneficial Organisms or Environmental Stressors. This Species interacts with both Crops and Pests, affecting the overall Stability of the System. The inclusion of Ecological Pressures like Climate or Habitat Changes mirrors real-world challenges in Ecosystem Management [6]. The Intrinsic Growth and Decay of this Species, along with its Interactions with Crops and Pests, emphasize the complexity of balancing Agricultural Practices with Ecological Conservation [7]. Figures 10 and 13 represent these interactions graphically in the 3D Phase Plot of Prey-Predator-Environmental Factor Interaction and Time Series Plot of Crop-Pest-Environmental Factor Interaction with Time-Varying u_p *and* u_c. Equilibrium and Stability: The Model features multiple Equilibrium Points that represent different states of balance between Crops, Pests, and Environmental Factors.

The Stability of these Equilibrium Points was evaluated using Jacobian Matrices and Eigenvalue Techniques. [5], [6] For example, at the Equilibrium Point $\boldsymbol{J_0(0,0,0)}$, the Eigenvalues $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 > 0)$ indicate that the System is Unstable (Figure 5). However, for $\boldsymbol{J_1\left(K_x\left(b_1 + u_f - u_p\right), 0, 0\right)}$, the Eigenvalues $(\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0)$ indicate that the System is Stable (Figure 6). These Equilibrium Points and their Stability properties offer insights into how different Factors $-$ such as Growth Rates, Interaction Coefficients, and Agricultural Interventions $-$ affect System Stability. The 3D Phase Plot (Figure 7) and Time Series Plot of Agricultural Practices Effect Over Time (Figure 15) highlight the impact of Agricultural Interventions on these Equilibrium Points.

Interior Equilibrium Point: An Interior Equilibrium Point (x^*, y^*, z^*) was analyzed using the Jacobian Matrix of the System, evaluated with the Routh-Hurwitz Criterion. The Eigenvalues $(\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0)$ suggest that the

System is Asymptotically Stable at this Point, Perturbations from this Equilibrium State will decay over time, and the System will return to the Equilibrium State, confirming Stability. This is shown in Figures 12, 17, and 18, which demonstrate the Dynamics of Prey, Predator, and Environmental Factors under various Agricultural Interventions. Agricultural Practices and Sustainability: The Model underscores the importance of Sustainable Agricultural Practices, including careful Management of Pesticide Use, Fertilization, and Biological Control, to maintain Ecosystem Balance. As noted by Beddington and May [1], over-reliance on Chemical Interventions may lead to short-term Pest Suppression but could destabilize the broader Ecosystem. The Model supports the argument for Integrated Pest Management (IPM) and other Ecologically Sound Practices, which align with long-term Sustainability and Productivity goals [8]. The Subplot of Agricultural Practices Effect Over Time (Figures 19) and Prey - Predator Population Over Time (Figures 16 and 17) emphasize these findings, illustrating the importance of time-varying Agricultural Practices such as u_p and u_c in Stabilizing Populations.

In Conclusion, this Three-Species Interaction Model provides a robust tool for exploring the intricate Dynamics between Crops, Pests, and Environmental Factors. By simulating various Agricultural Practices and their Ecological Impacts, the Model offers valuable insights for Decision-Making in Sustainable Farming, Pest Management, and Ecosystem Conservation. The findings build on classical Ecological Theories and offer practical guidance for maintaining long-term Stability and Productivity in Agricultural Systems [4], [5], [9]. Through the use of MATLAB Software for numerical illustration, the Model has provided comprehensive simulations and insights, as shown in the 3D Phase Plot (Figure 14) and Time Series Plots (Figures 12, 13).

The Time Series and Phase Plots (Figures 1–19) illustrate the complex interactions between Prey, Predators, and the Environmental Factor, showing how these populations evolve over time. These Dynamics are also influenced by Agricultural Practices such as Fertilization and Pesticide Application, consistent with studies on Ecosystem Management and Sustainability. [8] The findings underscore the importance of managing these practices to maintain Ecological Balance and avoid destabilizing populations. [1], [2] This study extends classical Predator-Prey Models by incorporating additional Environmental and Agricultural Factors. The results provide valuable insights into how Species Interactions and External Practices influence Stability, offering a Framework for improving Sustainable Farming Systems and Ecosystem Conservation. [7] In this Mathematical Modeling and using MATLAB(R2007b) Software version as used for numerical illustrations.

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