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The new continuous distribution : Statistical properties and modelling for Covid-19 data

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Abstract

Our research unveils a transformative distribution model: the Kumaraswamy power burr distribution. This innovative model emerges through the application of the Kumaraswamy transformation to the power burr distribution. We meticulously outline the probability and distribution functions, supported by compelling graphs and a rigorous mathematical foundation for the probability function of this novel distribution. In addition, we offer detailed equations and an extensive analysis of the statistical properties of this distribution. Parameter estimation is skillfully executed using the maximum likelihood estimation (MLE) method. Our compelling evidence shows that this newly created distribution not only enhances accuracy but also outperforms existing models significantly in analyzing COVID-19 data.

Subject Classification (2010): *62E10, 62E17, 62E20.*

Keywords: *Burr Type XII distribution, Kumaraswamy distribution, Hazard rate function, Entropy.*

1. Introduction

In [2], the authors skillfully present mathematical expressions for estimating the asymptotic variances and covariances of maximum likelihood estimates (MLE) for the Burr type distribution XII. Their rigorous mathematical research not only confirms the existence of MLEs for random sample data but also guarantees their uniqueness. This robust

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methodology is adeptly applied to progressively censored sample data from life test experiments, providing a vital framework for understanding reliability and failure times. The implications of this research are significant for enhancing reliability analysis in various fields. In study [18], the author explored innovative models featuring three extended parameters of the Burr XII type distribution, showcasing its practical application in modeling extreme events like flood frequency. This work builds upon the foundational definitions of the probability density function (pdf), as well as the two-parameter distribution and reliability function of type XII, initially established in the literature by [10]. This research not only advances our understanding but also offers valuable tools for predicting and managing significant environmental challenges. The study referenced as [17] investigates the flexibility of the Burr type XII distribution, which has gained significant attention across numerous scientific disciplines. This distribution is particularly prevalent in fields such as actuarial science, forestry, ecological science, and reliability and survival analysis. Additionally, the insights from [16] further highlight its applications and relevance in statistical modeling and analysis within these areas. The adaptability of the Burr type XII distribution makes it a valuable tool for researchers needing reliable data interpretation and predictive modeling. In [4], Numerous studies have highlighted the effectiveness of the Burr distribution as a reliable model for lifetime analysis. Its application in expedited life testing, particularly for investigating the reduction times of insulating variables, has proven beneficial. While some researchers have raised questions about the accuracy of results and forecasts tied to the Burr distribution, its capacity to handle both whole and censored samples makes it an essential tool in statistical analysis. Adopting this model can significantly enhance the understanding and prediction of life data in various contexts. In [9], the research conducted has successfully established robust Bayesian estimates and reliability functions derived from censored type 2 samples. Furthermore, studies [7] and [8] have provided valuable Bayesian estimates for two key parameters: the reliability and failure rate functions of the Burr XII distribution, showcasing the effectiveness of these statistical methods. [14] developed a robust empirical Bayesian parameter estimate along with a reliability function utilizing expedited censored type 2 data. Through comprehensive examples, [6] obtained Bayesian estimates for β and the reliability function, confidently assuming a common parameter α. Meanwhile, [15] derived proximate Bayes estimates for both the reliability function and the parameters of the Burr distribution (α, β), leveraging advancing censored samples of type 2 [19]

used generalized rank statistics to reproduce Bayesian inference for the Burr XII model. [1] developed Bayesian estimates utilizing both symmetric loss functions (such as square error) and asymmetric loss functions. [12] tackle the intriguing challenge of characterizing the shape-scale family of distributions alongside the exponentiated additive Weibull through truncated moments, while also exploring a variety of well-established lifespan distributions that emerge within these frameworks as distinct submodels. In this study, we innovatively applied the power transformation to the Burr XII distribution, subsequently employing the Kumarasvamy transform to develop the six-parameter Kumaraswamy Power Burr XII distribution. We thoroughly examined the statistical properties of this cutting-edge distribution, proposing the Kumaraswamy Power Burr XII as a transformative model in statistical analysis. Our research delves into essential aspects such as reliability functions, moment assertions, mean deviations, and the Lorenz and Bonferroni curves, while also evaluating Renyi and Shannon entropies, all computationally assessed. We utilize a maximum likelihood method for parameter estimation, enabling us to derive precise parameter estimates and simulate the effectiveness of these estimators. Through our analysis, we showcase the significance and applicability of this novel family of distributions, leveraging two critical Covid-19 data sets fatality counts and rates from Turkey and Italy. Ultimately, our findings illustrate the superior performance of our advanced distribution model, establishing its value over existing alternatives.

2. Material and Methods

Definition 2.1: In [13], the Kumaraswamy distribution which is defined on (0, 1) for two-parameter was presented. Also, for this distribution, cdf and pdf are given by

$$G(x;\alpha,\beta) = 1-(1-x^{\alpha})^{\beta},\ x \in (0,1) \tag{2.1}$$

and

$$g(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1},\ x \in (0,1) \tag{2.2}$$

respectively, where $\alpha > 0$ and $\beta > 0$ are shape parameters. The Kw-G distribution was later presented in [5] with the following pdf f(x) and cdf F(x),

$$f(x) = abg(x)G^{a-1}(x)\{(1-G(x)^{a})\}^{b-1} \tag{2.3}$$

and

$$F(x) = 1 - (1 - G(x)^a)^b \tag{2.4}$$

respectively, where $g(x) = dG(x)/dx$, for any baseline cumulative function $G(x)$. The Kw-G distribution has the same parameters as the G distribution, except for the two additional shape parameters a > 0 and b > 0. We write $X \sim Kw - G\ (a, b)$ for a random variable X with (2.3).

Definition 2.2: The Burr XII three-parameter distribution (BXII) is defined in [3] and [20], its cumulative distribution function and probability density function given by

$$G(x;s,k,c) = 1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-k} \quad s > 0,\ k > 0,\ c > 0. \tag{2.5}$$

and

$$g(x;s,k,c) = cks^{-c}x^{c-1}\left[1 + \left(\frac{x}{s}\right)^c\right]^{-k-1}, s > 0,\ k > 0,\ c > 0. \tag{2.6}$$

3. Results

3.1 *Power Burr XII distribution (PBXII)*

By considering the power transformation of the rvd, we define novel extension for Burr XII distribution, $Y = X^{1/b}$. For Y, the pdf is readily obtained to be power Burr XII distribution with parameters disribution given by

$$g(y;s,k,c,b) = \frac{bck}{s}y^{b-1}\left(\frac{y^b}{s}\right)^{c-1}\left(1 + \left(\frac{y^b}{s}\right)^c\right)^{-k-1}, s,k,c,b > 0 \tag{3.1}$$

The corresponding probability cumulative function for G(y) is given by

$$G(y;s,k,c,b) = \left\{1 - \left[1 + \left(\frac{y^b}{s}\right)^c\right]^{-k}\right\}, s,k,c,b > 0 \tag{3.2}$$

where $g(y) = dG(y)/dy$ is the probability density function.

3.2 *The New Kumaraswamy Power Burr XII*

We introduce the Kumaraswamy Power Burr XII (KPB XII) distribution, a powerful statistical tool characterized by its cumulative distribution function and probability density function, (for x > 0, *s*, *k*, *c*, *b*, *d*, *e* > 0) given by;

$$G(x;s,k,c,b,d,e)=\left[1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right] \tag{3.3}$$

$$g(x;s,k,c,b,d,e)=de\frac{bck}{s}x^{b-1}\left(\frac{x^b}{s}\right)^{c-1}$$
$$\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k}\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1}\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1} \tag{3.4}$$

Figures 1 and 2 depict some of the probable shapes of the KPBXII distribution's pdf and cdf, with *s*, *k*, *c*, *b*, *d*, *e* > 0parameters, respectively.

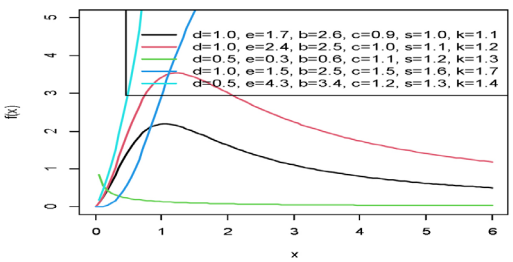


Figure 1

Probability function of KPBXII distribution

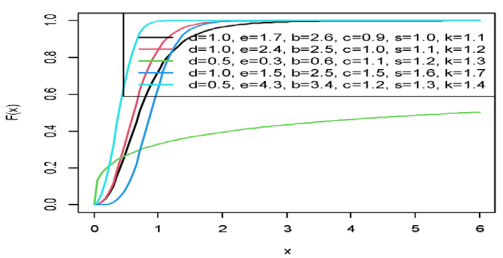


Figure 2

Cumulative Function of KPBXII distribution

Lemma 3.2.1: *If b = 1, the KPBXII distrubution in Equation (3.3) reduces to the Kumaraswamy Burr distribution.*

Lemma 3.2.2: *If d = 1, e = 1 and b = 1, KPBXII distrubution reduces to the generalized Burr distribution, with c, k and s parameters.*

3.3 *The density function's expansion*

If $|t|<1$, $b>0$ is real non-integer, we write the series,

$$(1-t)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} t^j \tag{3.5}$$

For $b > 0$, we obtain

$$(G(x))^{a-1}(1-G(x)^a)^{b-1} = \sum_{i=1}^{\infty} v_i G(x)^{a(i+1)-1} \tag{3.6}$$

where $v_i = (-1)^i \binom{b-1}{i}$. If b is an integer, the index i in the last sum stops at b − 1. If $a>0$ is real non-integer, an expansion for,

$$G(x)^a = \sum_{\varepsilon=0}^{\infty} \kappa_{\varepsilon}(a) G(x)^{\varepsilon}$$

where $\kappa_{\varepsilon}(a) = \sum_{n=\varepsilon}^{\infty} (-1)^{\varepsilon+n} \binom{a}{n}\binom{n}{\varepsilon}$. According to these expressions, we can write

$$G(x)^{a-1}\{1-G(x)^a\}^{b-1} = \sum_{\varepsilon=0}^{\infty} \delta_{\varepsilon} G(x)^{\varepsilon} \tag{3.7}$$

where, $\delta_{\varepsilon} = \delta_{\varepsilon}(a,b) = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \kappa_{\varepsilon}(a(i+1)-1)$. From equations (2.3), (3.1), (3.2) and (3.7), for *s, k, c, b, d, e* > 0

$$g(x;s,k,c,b,d,e) = de\frac{bck}{s} x^{b-1}\left(\frac{x^b}{s}\right)^{c-1}\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k} \sum_{\tau=}^{\infty} \delta_{\tau}\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{\tau}. \tag{3.8}$$

$$= de\sum_{\tau=0}^{\infty} \delta_{\tau} \sum_{j=0}^{\infty} \binom{\tau}{j} \frac{(-1)^j}{1+\tau-j} g(x;b,c,k(1+\tau-j),s) \tag{3.9}$$

and then interchanging the double sum $\sum_{\tau=0}^{\infty}\sum_{j=0}^{\tau}$ by $\sum_{j=0}^{\infty}\sum_{\tau=j}^{\infty}$ f(x) can be expressed as,

$$f(x) = \sum_{j=0}^{\infty} \upsilon_j g(x;b,c,k(1+\tau-j),s) \tag{3.10}$$

where, $\upsilon_j = de\sum_{\tau=j}^{\infty} \binom{\tau}{j} \frac{(-1)^j}{1+\tau-j}$ and $g(x;b,c,k(1+\tau-j),s)$ defined Power Burr XII density function. Equation (3.10) compellingly demonstrates that the Kumaraswamy Power Burr XII density is an infinite linear combination of Power Burr XII density functions.

3.4 *Quantile Function, Median, Skewness and Kurtosis*

[11] state that the definition of the quantile function for any distribution is $Q(u) = G^{-1}(u),\ 0 < u < 1$. To define the quantile function of the KPBXII distribution, $G(x; s, k, c, b, d, e) = u$. From here,

$$Q(u) = -s^{1/b}\left\{1-\left(1-\left(1-\left(1-u\right)^{1/e}\right)^{1/d}\right)\right)^{-1/k}\right\}^{1/b}$$

such that u is a uniform variate on the unit interval (0, 1). The median (M) of X is then obtained by setting u to 0.5 in equality $Q(u)$;

$$M = -s^{1/b}\left\{1-\left(1-\left(1-\left(0.5\right)^{1/e}\right)^{1/d}\right)\right)^{-1/k}\right\}^{1/b}.$$

However, by setting u as 0.25 and 0.75 in equality $Q(u)$, for the random variable X, the 25th and 75th percentiles are determined as follows:

$$Q_1 = -s^{1/b}\left\{1-\left(1-\left(1-\left(0.75\right)^{1/e}\right)^{1/d}\right)\right)^{-1/k}\right\}^{1/b} \quad Q_3 = -s^{1/b}\left\{1-\left(1-\left(1-\left(0.25\right)^{1/e}\right)^{1/d}\right)\right)^{-1/k}\right\}^{1/b}.$$

The quantile function yields the Bowley's skewness and the Moor's kurtosis as:

$$S_k = \frac{Q_{0.75}-2Q_{0.50}+Q_{0.25}}{Q_{0.75}-Q_{0.25}},\ M_k = \frac{Q_{0.875}-Q_{0.025}-Q_{0.375}+Q_{0.125}}{Q_{0.75}-Q_{0.25}}.$$

We may readily get X by using u as a uniform random variable in the interval (0, 1).

3.5 *Reliability Analysis*

The probability that an element does not fail before time t is defined by the reliability function R (t) R (t) = 1 - F (t). The reliability or survival function for the KPBXII distribution is expressed as follows by placing it in (3.3) R (t), and the figure 3 are

$$R(t) = \left[1-\left\{1-\left(1-\left[1+\left(\frac{t^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right], s,k,c,b,d,e>0 \qquad (3.1)$$

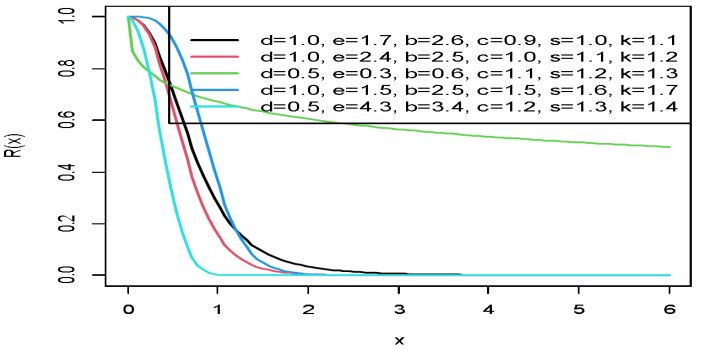


Figure 3

Reliability Function of KPBXII distribution

Another interesting property for a random variable is the risk ratio function, described as:

$$h(t) = \frac{f(t)}{R(t)} \tag{3.12}$$

This function and figure 4 can be obtained by substituting the hazard ratio for (3.3) and (3.11) in (3.12) for the distribution and can be defined as follows;

$$h(t) = \frac{de\frac{bck}{x}\left(\frac{t^b}{s}\right)^c\left(1+\left(\frac{t^b}{s}\right)^c\right)^{-1-k}\left(1-\left[1+\left(\frac{t^b}{s}\right)^c\right]^{-k}\right)^{d-1}\left\{1-\left(1-\left[1+\left(\frac{t^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1}}{\left[1-\left\{1-\left(1-\left[1+\left(\frac{t^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right]} \tag{3.13}$$

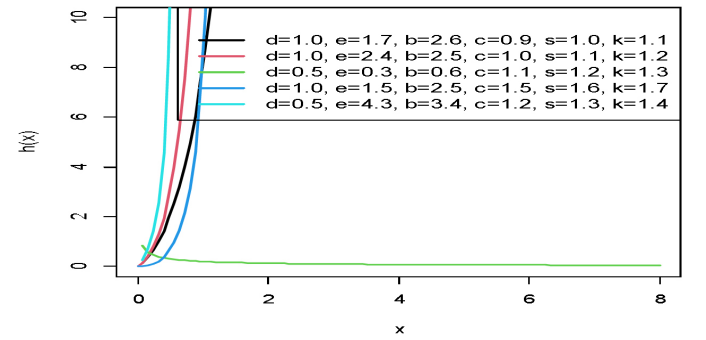


Figure 4

Hazard Rate Function of Kumarasvamy power Burr XII distribution

The cumulative hazard function is described as follows:

$$\Lambda(x) = -\log R(x) \tag{3.14}$$

This function for the KPBXII distribution can be obtained by setting (3.11) to (3.14) and takes the following format;

$$\Lambda(x) = -\log\left[1-\left\{1-\left(1-\left[1+\left(\frac{t^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right] \tag{3.15}$$

The connection between the probability density function and the cumulative distribution function is a fundamental concept in statistics that underscores the essence of probability;

$$r(x) = \frac{f(x)}{F(x)}. \tag{3.16}$$

This function and figure 5 of the KPBXII distrubution can be obtained by adjusting to (3.3) and (3.4) in (3.16) and has the following format;

$$r(x) = \frac{de\frac{bck}{x}\left(\frac{x^b}{s}\right)^c\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k}\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1}\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1}}{\left[1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right]} \tag{3.17}$$

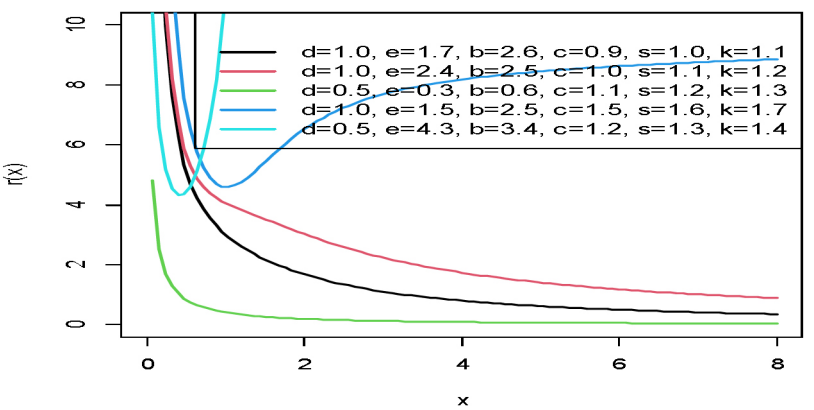


Figure 5

Reserved Hazard Function of KPBXII distribution

The odds function defined by is the subsequent format;

$$O(x)=\frac{F(x)}{R(x)}. \tag{3.18}$$

This function and figure 6 for the Kumarasvamy Power Burr XII (KPBXII) distrubution can be acquired by setting (3.4) and (3.5) in (3.18) and subsequent statement is acquired.

$$O(x)=\left[1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right]\left[1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right]^{-1} \tag{3.19}$$

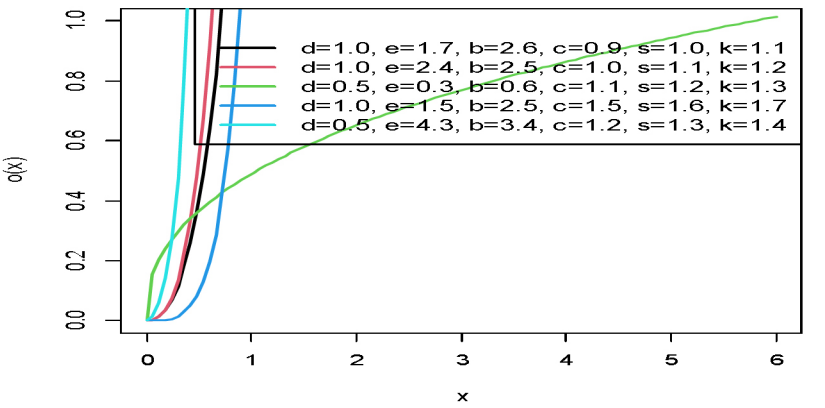


Figure 6

Odd Function of KPBXII distribution

3.6 *Order Statistics*

If $X_{(1)}\le X_{(2)}\le\ldots\le X_{(n)}$ represents the order statistic of a random sample $X_1,\ldots,X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, we know that the pdf is provided $X_{(j)}$ by the following formula:

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}f_X(x)(F(x))^{j-1}(1-F_X(x))^{n-j}$$

for j = 1, . . . , n. The pdf of the jth order statistic for a the KPBXII distrubution is given by:

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}de\frac{bck}{s}x^{b-1}\left(\frac{x^b}{s}\right)^{c-1}\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k}\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1}\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1}$$

$$\left(1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right)^{j-1}\left(1-\left(1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right)\right)^{n-j} \tag{3.20}$$

Therefore, the pdf of $X_{(n)}$ is;

$$f_{X_{(n)}}(x)=nde\frac{bck}{s}x^{b-1}\left(\frac{x^b}{s}\right)^{c-1}\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k}\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1}$$

$$\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1}\left(1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right)^{n-1} \tag{3.21}$$

and the pdf of $X_{(1)}$ is;

$$f_{X_{(j)}}(x)=nde\frac{bck}{s}x^{b-1}\left(\frac{x^b}{s}\right)^{c-1}\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k}\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1}\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1}$$

$$\left(1-\left(1-\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^e\right)\right)^{n-1}$$

3.7 *Maximum Likelihood Estimation*

We employ the maximum likelihood estimation (MLE) method to accurately estimate our model parameters. This powerful technique allows us to identify the parameter values that maximize the likelihood function, providing a precise understanding of how well our model fits the observed data. Let $x=(x_1,...,x_n)^T$ be a random sample of size n from the MOG distribution with unknown parameter vector, $\gamma=(b,c,d,k,e,s)^T$ The log-likelihood function for γ is obtained as;

$$l_n = n\log de^{\frac{bck}{s}} + (b-1)\sum_{i=1}^{n}\log x_i + (c-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{s}^b\right) - (1+k)\sum_{i=}^{n}\log\left(1+\left(\frac{x_i}{s}^b\right)^c\right) +$$

$$(d-1)\sum_{i=1}^{n}\log\left(1-\left(1+\left(\frac{x_i}{s}^b\right)^c\right)^{-k}\right) + (e-1)\sum_{i=1}^{n}\log\left(1-\left(1-\left(1+\left(\frac{x_i}{s}^b\right)^c\right)^{-k}\right)^d\right) \quad (3.22)$$

From here,

$$\frac{dl_n}{d} = \frac{n}{d} + \sum_{i=1}^{n}\log\left[1-\left(1+\left(\frac{x_i}{s}^b\right)^c\right)^{-k}\right] + (-1+e)\sum_{i=1}^{n} - \frac{\log\left[1-\left(1+\left(\frac{x_i}{s}^b\right)^c\right)^{-k}\right]\left(1-\left(1+\left(\frac{x_i}{s}^b\right)^c\right)^{-k}\right)^d}{1-\left(1-\left(1+\left(\frac{x_i}{s}^b\right)^c\right)^{-k}\right)^d}$$

$$\frac{dl_n}{de} = \frac{n}{e} + \sum_{i=1}^{n}\log\left[1-\left(1-\left(1+\left(\frac{x_i}{s}^b\right)^c\right)^{-k}\right)^d\right]$$

$$\frac{dl_n}{db} = \frac{n}{b} + \sum_{i=1}^{n}\log[x_i] + (-1+c)\sum_{i=1}^{n}\log[x_i] - (1+k)\sum_{i=1}^{n}\frac{c\log[x_i]x_i^b\left(\frac{x_i^b}{s}\right)^{-1+c}}{s\left(1+\left(\frac{x_i^b}{s}\right)^c\right)}$$

$$+(-1+d)\sum_{i=1}^{n}\frac{ck\log[x_i]x_i^b\left(\frac{x_i^b}{s}\right)^{-1+c}\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-1-k}}{s\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)}$$

$$+(-1+e)\sum_{i=1}^{n} - \frac{cdk\log[x_i]x_i^b\left(\frac{x_i^b}{s}\right)^{-1+c}\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-1-k}\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^{-1+d}}{s\left(1-\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^d\right)}$$

$$\frac{dl_n}{dc}=\frac{n}{c}+\sum_{i=1}^{n}\log\left[\frac{x_i^b}{s}\right]-(1+k)\sum_{i=1}^{n}\frac{\log\left[\frac{x_i^b}{s}\right]\left(\frac{x_i^b}{s}\right)^c}{1+\left(\frac{x_i^b}{s}\right)^c}+-1+d)\sum_{i=1}^{n}\frac{k\log\left[\frac{x_i^b}{s}\right]\left(\frac{x_i^b}{s}\right)^c\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-1-k}}{1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}}$$

$$+(-1+e)\sum_{i=1}^{n}-\frac{dk\log\left[\frac{x_i^b}{s}\right]\left(\frac{x_i^b}{s}\right)^c\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-1-k}\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^{-1+d}}{1-\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^{d}}$$

$$\frac{dl_n}{dk}=\frac{n}{k}-\sum_{i=1}^{n}\log\left[1+\left(\frac{x_i^b}{s}\right)^c\right]+(-1+d)\sum_{i=1}^{n}\frac{\log\left[1+\left(\frac{x_i^b}{s}\right)^c\right]\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}}{1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}}$$

$$+(-1+e)\sum_{i=1}^{n}-\frac{d\log\left[1+\left(\frac{x_i^b}{s}\right)^c\right]\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^{-1+d}}{1-\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^{d}}$$

$$\frac{dl_n}{ds}=-\frac{n}{s}-\frac{(-1+c)n}{s}-(1+k)\sum_{i=1}^{n}-\frac{cx_i^b\left(\frac{x_i^b}{s}\right)^{-1+c}}{s^2\left(1+\left(\frac{x_i^b}{s}\right)^c\right)}+(-1+d)\sum_{i=1}^{n}-\frac{ckx_i^b\left(\frac{x_i^b}{s}\right)^{-1+c}\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-1-k}}{s^2\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)}$$

$$+(-1+e)\sum_{i=1}^{n}\frac{cdkx_i^b\left(\frac{x_i^b}{s}\right)^{-1+c}\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-1-k}\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^{-1+d}}{s^2\left(1-\left(1-\left(1+\left(\frac{x_i^b}{s}\right)^c\right)^{-k}\right)^{d}\right)} \tag{3.23}$$

Partially differentiating the equation (3.23) with respect to *d*, *e*, *s*, *k*, *c* and *b*, the estimations are achieved, although the EM algorithm is the best method for estimating the parameters. The MLE This non-linear system of equations is solved to provide the MLE $(\hat{e},\hat{s},\hat{k},\hat{d},\hat{b},\hat{c})$ of (e,s,k,d,b,c) correspondingly. To effectively maximize the sample likelihood function,

nonlinear optimization techniques, such as the quasi-Newton algorithm, are typically more suitable. These methods are commonly favored for their practicality in achieving the maximization of the sample likelihood function. The MLE $\hat{\gamma} = (\hat{e}, \hat{s}, \hat{k}, \hat{d}, \hat{b}, \hat{c})$ can be treated as being approximately tri-variate normal with mean $\hat{\eta}$, variance-covariance matrix equal to the inverse of the expected information matrix $\sqrt{n}(\hat{\gamma} - \gamma) \to N_6(0, nI^{-1}(\gamma))$ Where, $I^{-1}(\gamma)$ is the variance-covariance matrix of $\hat{\gamma}$. The elements of the 6x6 matrix $I(\gamma)$ can be estimated by

$$I^{-1} = -E\begin{bmatrix} \frac{\partial^2 \log L}{\partial d^2} & \frac{\partial^2 \log L}{\partial d \partial e} & \frac{\partial^2 \log L}{\partial d \partial k} & \frac{\partial^2 \log L}{\partial d \partial s} & \frac{\partial^2 \log L}{\partial d \partial c} & \frac{\partial^2 \log L}{\partial d \partial b} \\ \frac{\partial^2 \log L}{\partial e \partial d} & \frac{\partial^2 \log L}{\partial e^2} & \frac{\partial^2 \log L}{\partial e \partial k} & \frac{\partial^2 \log L}{\partial e \partial s} & \frac{\partial^2 \log L}{\partial e \partial c} & \frac{\partial^2 \log L}{\partial e \partial b} \\ \frac{\partial^2 \log L}{\partial k \partial d} & \frac{\partial^2 \log L}{\partial k \partial e} & \frac{\partial^2 \log L}{\partial k^2} & \frac{\partial^2 \log L}{\partial k \partial s} & \frac{\partial^2 \log L}{\partial k \partial c} & \frac{\partial^2 \log L}{\partial k \partial b} \\ \frac{\partial^2 \log L}{\partial s \partial d} & \frac{\partial^2 \log L}{\partial s \partial e} & \frac{\partial^2 \log L}{\partial s \partial k} & \frac{\partial^2 \log L}{\partial s^2} & \frac{\partial^2 \log L}{\partial s \partial c} & \frac{\partial^2 \log L}{\partial s \partial b} \\ \frac{\partial^2 \log L}{\partial c \partial d} & \frac{\partial^2 \log L}{\partial c \partial e} & \frac{\partial^2 \log L}{\partial c \partial k} & \frac{\partial^2 \log L}{\partial c \partial s} & \frac{\partial^2 \log L}{\partial c^2} & \frac{\partial^2 \log L}{\partial c \partial b} \\ \frac{\partial^2 \log L}{\partial b \partial d} & \frac{\partial^2 \log L}{\partial b \partial e} & \frac{\partial^2 \log L}{\partial b \partial k} & \frac{\partial^2 \log L}{\partial b \partial s} & \frac{\partial^2 \log L}{\partial b \partial c} & \frac{\partial^2 \log L}{\partial b^2} \end{bmatrix} = \begin{bmatrix} I_{11}^{-1} & I_{12}^{-1} & I_{13}^{-1} & I_{14}^{-1} & I_{15}^{-1} & I_{16}^{-1} \\ I_{21}^{-1} & I_{22}^{-1} & I_{23}^{-1} & I_{24}^{-1} & I_{25}^{-1} & I_{26}^{-1} \\ I_{31}^{-1} & I_{32}^{-1} & I_{33}^{-1} & I_{34}^{-1} & I_{35}^{-1} & I_{36}^{-1} \\ I_{41}^{-1} & I_{42}^{-1} & I_{43}^{-1} & I_{44}^{-1} & I_{45}^{-1} & I_{46}^{-1} \\ I_{51}^{-1} & I_{52}^{-1} & I_{53}^{-1} & I_{54}^{-1} & I_{55}^{-1} & I_{56}^{-1} \\ I_{61}^{-1} & I_{62}^{-1} & I_{63}^{-1} & I_{64}^{-1} & I_{65}^{-1} & I_{66}^{-1} \end{bmatrix}.$$

3.8 *Moment Generating Function and Characteristic Function*

We shall talk about the many structural characteristics of the suggested KPBXII distribution. Suppose *X* is a random variable that follows the KPBXII distrubution with parameters *d,e,s,b, c,k* then for a given probability distribution r. the moment of degree E (x^r) is given by;

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x) dx \tag{3.24}$$

$$= \int_0^\infty x^r \left\{ de \frac{bck}{s} x^{b-1} \left(\frac{x^b}{s}\right)^{c-1} \left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k} \left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1} \left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1} \right\} dx \tag{3.25}$$

Using binomial expansion of $\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1} = \sum_{i=0}^{d-1} \binom{d-1}{i} \left(1+\left(\frac{x^b}{s}\right)^c\right)^{-ki} (-1)^{d-1-i}$

and $\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1} = \sum_{j=0}^{e-1} \sum_{t=0}^{dj} \binom{e-1}{j} \binom{dj}{t} (-1)^{e-1-j+dj-t} \left(\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-k}\right)^t$, equation (3.25) will become,

$$= debck\sum_{i=0}^{d-1}\sum_{j=0}^{e-1}\sum_{t=0}^{dj}\binom{d-1}{i}\binom{e-1}{j}\binom{dj}{t}(-1)^{d-2-i+e-j+dj-t}\int_0^{\infty}x^{r-1}\left\{\left(\frac{x^b}{s}\right)^c\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k-ki-kt}\right\}dx.$$

From here

$$debck\sum_{i=0}^{d-1}\sum_{j=0}^{e-1}\sum_{t=0}^{dj}\binom{d-1}{i}\binom{e-1}{j}\binom{dj}{t}(-1)^{d-2-i+e-j+dj-t}\frac{\left(\left(\frac{1}{s}\right)^c\right)^{-\frac{r}{bc}}\Gamma\left[1+\frac{r}{bc}\right]\Gamma\left[-\frac{r}{bc}+k(1+i+t)\right]}{bc\Gamma[1+k(1+i+t)]} \quad (3.26)$$

is obtained. Since equation (3.26) is a convergent series , therefore all the moments exist:

$$debck\sum_{i=0}^{d-1}\sum_{j=0}^{e-1}\sum_{t=0}^{dj}\binom{d-1}{i}\binom{e-1}{j}\binom{dj}{t}(-1)^{d-2-i+e-j+dj-t}\frac{\left(\left(\frac{1}{s}\right)^c\right)^{-\frac{1}{bc}}\Gamma\left[1+\frac{1}{bc}\right]\Gamma\left[-\frac{1}{bc}+k(1+i+t)\right]}{bc\Gamma[1+k(1+i+t)]} \quad (3.27)$$

and

$$debck\sum_{i=0}^{d-1}\sum_{j=0}^{e-1}\sum_{t=0}^{dj}\binom{d-1}{i}\binom{e-1}{j}\binom{dj}{t}(-1)^{d-2-i+e-j+dj-t}\frac{\left(\left(\frac{1}{s}\right)^c\right)^{-\frac{2}{bc}}\Gamma\left[1+\frac{2}{bc}\right]\Gamma\left[-\frac{2}{bc}+k(1+i+t)\right]}{bc\Gamma[1+k(1+i+t)]}. \quad (3.28)$$

Therefore, the variance of X can be obtained as

$$V(X) = E(X^2) - (E(X))^2. \quad (3.29)$$

Let X have KPBXII distribution. Then the moment generating function of X is obtained as;

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx \quad (3.30)$$

In equation (3.30), we use taylor's series as follows:

$$M_X(t) = \int_0^{\infty}\left(1 + tx + \frac{(tx)^2}{2!} + ...\right)f(x)dx \quad (3.31)$$

From equation (3.31)

$$M_X(t) = \int_0^{\infty}\sum_{l=0}^{\infty}\frac{t^l}{l!}x^l f(x)dx \quad (3.32)$$

can be written. In equation (3.32), we write

$$debck\sum_{i=0}^{d-1}\sum_{j=0}^{e-1}\sum_{t=0}^{dj}\sum_{l=0}^{\infty}\frac{t^l}{l!}\binom{d-1}{i}\binom{e-1}{j}\binom{dj}{t}(-1)^{d-2-i+e-j+dj-t}\frac{\left(\left(\frac{1}{s}\right)^c\right)^{-\frac{r}{bc}}\Gamma\left[1+\frac{l}{bc}\right]\Gamma\left[-\frac{l}{bc}+k(1+i+t)\right]}{bc\Gamma[1+k(1+i+l)]} \tag{3.33}$$

Let X have KPBXII distribution, then the characteristic $\phi(e^{mt})$ function of X is obtained as;

$$debck\sum_{i=0}^{d-1}\sum_{j=0}^{e-1}\sum_{t=0}^{dj}\sum_{l=0}^{\infty}\frac{t^l}{l!}\binom{d-1}{i}\binom{e-1}{j}\binom{dj}{t}(-1)^{d-2-i+e-j+dj-t}\frac{\left(\left(\frac{1}{s}\right)^c\right)^{-\frac{r}{bc}}\Gamma\left[1+\frac{ml}{bc}\right]\Gamma\left[-\frac{ml}{bc}+k(1+i+t)\right]}{bc\Gamma[1+k(1+i+ml)]}. \tag{3.34}$$

3.9 *Information Measures of Kumarasvamy Power Burr XII (KPBXII) Distribution*

3.9.1. Renyi Entropy

Entropies quantify a system's diversity, unpredictability, or randomness. Named for Alfred Renyi, Renyi's entropy serves as the foundation for the idea of generalized dimensions in the context of fractional size estimation. In statistics and ecology, the Renyi entropy is a crucial indicator of diversity. Because it can be used as a gauge of entanglement, Renyi entropy is also significant in quantum information. The Renyi entropy for a particular probability distribution can be found using:

$$e(\beta)=\frac{1}{1-\beta}\log\left(\int_0^{\infty}f^{\beta}(x)dx\right),\ \beta>0,\ \beta\neq 1 \tag{3.35}$$

$$\frac{1}{1-\beta}\log\left(\int_0^{\infty}\left\{de\frac{bck}{x}\left(\frac{x^b}{s}\right)^c\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k}\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1}\left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1}\right\}^{\beta}dx\right) \tag{3.36}$$

$$(debck)^{\beta}\sum_{i=0}^{\beta(d-1)}\sum_{j=0}^{\beta(e-1)}\sum_{t=0}^{dj}\binom{\beta(d-1)}{i}\binom{\beta(e-1)}{j}\binom{dj}{t}(-1)^{\beta(d-1)-i+\beta(e-1)-j+j-t}\int_0^{\infty}x^{(b-1)\beta}\left(1+\left(\frac{x^b}{s}\right)^c\right)^{-ki-t-\beta-\beta k}dx \tag{3.37}$$

and From equation (3.37), we obtain

$$= (debck)^{\beta} \sum_{i=0}^{\beta(d-1)} \sum_{j=0}^{\beta(e-1)} \sum_{t=0}^{dj} \binom{\beta(d-1)}{i} \binom{\beta(e-1)}{j} \binom{dj}{t} (-1)^{\beta(d-1)-i+\beta(e-1)-j+j-t} \frac{\left(\left(\frac{1}{s}\right)^c\right)^{\frac{-1+\beta-b\beta}{bc}} \Gamma\left[\frac{1+(-1+b)\beta}{bc}\right] \Gamma\left[ik+kt+\beta+k\beta+\frac{-1+\beta-b\beta}{bc}\right]}{bc\Gamma[ik+\beta+k(t+\beta)]}.$$

3.9.2. Tsallis Entropy

The extension of Boltzmann-Gibbs (B-G) statistical mechanics introduced by Tsallis has sparked considerable interest across multiple disciplines. This groundbreaking generalization begins by defining the mathematical form of Tsallis entropy for continuous random variables, offering a fresh perspective on statistical behavior. By incorporating a parameter that enables a wider spectrum of statistical outcomes, Tsallis entropy enhances the traditional B-G framework. This adaptability makes it exceptionally pertinent for systems with non-extensive characteristics. Its implications reach far beyond physics, positively impacting fields such as economics and biology, by providing a robust methodology for understanding diverse phenomena that conventional statistical mechanics may overlook,this equation;

$$S_{\lambda} = \frac{1}{1-\lambda}\left(1 - \int_0^{\infty} f^{\lambda}(x)dx\right) \tag{3.38}$$

$$\frac{1}{1-\lambda}\left(1 - \int_0^{\infty} \left\{ de\frac{bck}{x}\left(\frac{x^b}{s}\right)^c \left(1+\left(\frac{x^b}{s}\right)^c\right)^{-1-k} \left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^{d-1} \left\{1-\left(1-\left[1+\left(\frac{x^b}{s}\right)^c\right]^{-k}\right)^d\right\}^{e-1} \right\}^{\lambda} dx\right) \tag{3.39}$$

and from equation (3.39)

$$= \frac{1}{1-\lambda}\left(1-\left(\frac{(debck)^{\lambda} \sum_{i=0}^{\lambda(d-1)} \sum_{j=0}^{\lambda(e-1)} \sum_{t=0}^{dj} \binom{\lambda(d-1)}{i} \binom{\lambda(e-1)}{j} \binom{dj}{t} (-1)^{\lambda(d-1)-i+\lambda(e-1)-j+\lambda(e-1)-j} \left(\left(\frac{1}{s}\right)^c\right)^{\frac{-1+\lambda-b\lambda}{bc}} \Gamma\left[\frac{1+(-1+b)\lambda}{bc}\right] \Gamma\left[ik+kt+\lambda+k\lambda+\frac{-1+\lambda-b\lambda}{bc}\right]}{bc\Gamma\left[ik+\lambda+k(t+\lambda)\right]}\right)\right)$$

is obtained.

4. Application

This section compares the model to the Burr distribution and the Kumaraswamy Burr distribution using four real lifetime data sets in the Kumaraswamy Power Burr distribution. We can clearly observe from Tables 1 and 2 for two Covid 19 data sets (the number and rate of fatalities in Turkey and Italy) that the Kumaraswamy Power Burr distribution has values for the AIC and BIC when compared to the Burr distribution and the Kumaraswamy Burr distribution. The Kumaraswamy Power Burr distribution thus appears to suit the data better than both the Burr and Kumaraswamy Burr distributions.

Data set 1: To define, the new conclusions submitted in this work, we apply the KPBXII distribution to used the undermentioned COVID-19 data of Turkey between 3 April and 3 July 2021

186, 185, 193, 211, 276, 258, 253, 248, 237, 243, 273, 279, 297, 289, 288, 318, 341, 346, 362, 354, 343, 339, 347, 353, 346, 341, 339, 394, 373, 340, 347, 336, 356, 304, 278, 281, 283, 282, 278, 232, 238, 242, 236, 223, 223, 203, 233, 207, 214, 231, 197,178, 175, 166, 183, 164, 137, 134, 122, 129, 112, 114, 94, 92, 96, 91, 86, 87, 96, 69, 75, 53, 74, 84, 71, 62, 59, 51, 63, 51, 57, 65, 59, 56, 51, 52, 58, 53, 45, 42, 55, 45.

Table 1
For Data Set 1 Model Fitting

	d	e	b	c	k	s	AIC	BIC	W	A	K
Burr	-	-	-	2.266244	1.332534	185.473652	1138.468	1146.033	0.6354358	3.868411	0.1753
K Burr	31.58398	118.67	-	0.2400362	2.2242934	48.835064	1129.84	1142.45	0.514977	3.20000	0.1392
K Power Burr	0.13302	0.974	0.49271	17.8104684	12.8613966	20.6075052	1112.095	1113.084	0.2104825	1.643565	0.10802

Data Set 2: The 111 days of COVID-19 data that are taken into consideration are from Italy and were captured from 1 April to 20 July 2020. This information was created by dividing daily new cases by daily new deaths.

The data are as follows: 0.2070, 0.1520, 0.1628, 0.1666, 0.1417, 0.1221, 0.1767, 0.1987, 0.1408, 0.1456, 0.1443, 0.1319, 0.1053, 0.1789, 0.2032, 0.2167, 0.1387, 0.1646, 0.1375, 0.1421, 0.2012, 0.1957, 0.1297, 0.1754, 0.1390, 0.1761, 0.1119, 0.1915, 0.1827, 0.1548, 0.1522, 0.1369, 0.2495, 0.1253, 0.1597, 0.2195, 0.2555, 0.1956, 0.1831, 0.1791, 0.2057, 0.2406, 0.1227, 0.2196, 0.2641, 0.3067, 0.1749, 0.2148, 0.2195, 0.1993, 0.2421, 0.2430, 0.1994, 0.1779, 0.0942, 0.3067, 0.1965, 0.2003, 0.1180, 0.1686, 0.2668, 0.2113, 0.3371, 0.1730, 0.2212, 0.4972, 0.1641, 0.2667, 0.2690, 0.2321, 0.2792, 0.3515, 0.1398, 0.3436, 0.2254, 0.1302, 0.0864, 0.1619, 0.1311, 0.1994, 0.3176, 0.1856, 0.1071, 0.1041, 0.1593, 0.0537, 0.1149, 0.1176, 0.0457, 0.1264, 0.0476, 0.1620, 0.1154, 0.1493, 0.0673, 0.0894, 0.0365, 0.0385, 0.2190, 0.0777, 0.0561, 0.0435, 0.0372, 0.0385, 0.0769, 0.1491, 0.0802, 0.0870, 0.0476, 0.0562, 0.0138.

Table 2
For Data Set 2 Model Fitting

	d	e	b	c	k	s	AIC	BIC	W	A	K
Burr	-	-	-	2.47855	6.638743	0.3853288	252.6845	244.5559	0.113451	0.7085325	0.064272
K Burr	0.2035667	0.4823547	-	8.218951	1.273159	0.192100	253.649	240.101	0.020008	0.178851	0.04088
K Power Burr	0.273961	0.262996	2.3558	2.359005	3.860721	0.019659	251.315	235.057	0.0203526	0.186671	0.04203

5. Conclusion

We have successfully developed the Kumarasvamy Power Burr XII (KPBXII), a solid continuous probability distribution for non-negative random variables, recognized in the realms of probability theory, statistics, and econometrics as the Burr Type XII distribution, or simply the Burr distribution. Among the prominent distributions is the Singh-Maddala distribution, which is often classified as a generalized log-logistic

distribution. Our research has allowed us to accurately determine both the cumulative distribution function and the probability function for this distribution. Furthermore, we have derived essential reliability functions, rth moments, mean and variance, as well as the Bonferroni and Lorenz curves, command statistics, maximum likelihood estimators (MLE), mean time between failures (MTBF), and the characteristic function of the moment-generating function. By employing R software for visualization and Mathematica for precise computations, we confidently estimate the parameters of this novel model using maximum likelihood estimation. Additionally, we have constructed the information matrix to support our findings. Our distribution has effectively validated the outcomes of asymptotic theory through simulation research. We rigorously tested our new model on four real data sets, revealing its superior performance compared to existing models. The Kumarasvamy Power Burr XII (KPBXII) distribution distinctly outperforms others in the field.

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