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An alternative method for out-of-sample forecast of FIEGARCH model

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Abstract

Volatility is an inherent characteristic of a time series (TS). It is said to be asymmetric when negative and positive shocks of equal scale have different effects. If the volatility of one

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epoch is influenced by its distant counterpart, then volatility has a long memory structure. Nonlinear variance models such as the FIEGARCH model are used to deal with asymmetric volatility and long memory simultaneously. This article attempts to derive the direct formulae regarding the 'out-of-sample' forecast and the error variances associated with these forecasts for the ARMA(1, 0)- FIEGARCH(1, d, 1) model. Four real-life TS data are used for empirical purposes. It is observed that each TS has a significant long memory in volatility and its volatility is asymmetric.

Subject Classification: *91B84, 62M10.*

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1. Introduction

The modeling of time series (TS) is an important area of statistics. A TS can be defined as the set of data points of a variable over time. The data points may be collected at evenly spaced time intervals. The crucial feature of a TS is that the observations are correlated over time. This characteristic of a TS is the way forward for TS modeling. In a broader sense, a TS can be divided into components subjected to linear and nonlinear perspectives. The classical 'autoregressive integrated moving average' (ARIMA) model [1] is only suitable for the linear component. This necessitated the development of more complex models that could capture the nonlinear component of a TS. The volatility of a TS is considered to be the unexpected variation over time. It is the nonlinear component of a TS process. By relaxing the assumptions of the ARIMA model, it is possible to model several TS processes using nonlinear models. The most commonly used nonlinear TS models to capture volatility are the 'autoregressive conditional heteroscedastic' (ARCH) model [2] and the generalized ARCH (GARCH) model [3 and 4, independent of each other]. As a recent reference, let us mention that Maheshwari and Kapoor [5] formulated a prediction model to study the volatility of the National Stock Exchange (NSE) based on the GARCH model. This model is regarded as a symmetric model in the sense that negative and positive shocks of equal scale have the same effect on volatility. In many practical situations, this may not be the case.

Volatility can be classified as asymmetric when negative and positive shocks with similar scales have dissimilar reactions to volatility [6]. The asymmetric volatility of any TS can be effectively captured by using the exponential GARCH (EGARCH) model [7]. Again, it is often noticeable that the dependence between the observations of a TS is significant for a large lag. We refer to this phenomenon as the long-term persistence. This phenomenon can be identified through the visualization of autocorrelation among the realizations like ACF and PACF. A TS with a long memory will exhibit hyperbolic decay (slow decay) in its ACF and/or PACF. Long-term persistence can exist in both linear and nonlinear domains. The fractionally integrated ARIMA (ARFIMA) model [8] is applicable when long memory exists in the linear structure. In this model, the differencing parameter takes fractional values, as opposed to integer values in the ARIMA model. With the nonlinear structure's long memory present, the fractionally integrated version of the GARCH model, i.e., the FIGARCH model [9], is useful. The fractionally integrated EGARCH model, i.e., the FIEGARCH model [10], fulfils the needs of both the EGARCH and FIGARCH models.

Correct understanding regarding the volatility of a financial TS helps the stakeholders to make appropriate decisions in time to optimise their performance. Agricultural commodity prices also exhibit volatility. Weather abnormalities, seasonal variations in agro-meteorological parameters and changes in administrative policies have an immense impact on the volatility of agricultural commodity prices. Karali and Thurman [11] found that the month before the harvest is typically the most volatile. The seasonal production cycle of agricultural commodities is the main reason behind it. Prior knowledge of the possible price movements of different agricultural commodities can help farmers make relevant decisions throughout the crop calendar ranging from planning to harvesting. Agricultural practices are the livelihood for a major fraction of the Indian population. Applications of the ARFIMA [12], GARCH [13], FIGARCH [14 and 15] and ARFIMA-FIGARCH models [16] can be found in the literature using Indian agricultural TS data. Rakshit and Paul [17] modeled the onion price of the Delhi, Bengaluru and Lasalgaon markets and found that the log return series of these markets have long memory and asymmetric volatility. They found that the FIEGARCH model performed best for them.

The naive method can be used for out-of-sample forecasting. This method starts with a one-step-ahead forecast. This forecast realization is treated as an actual observation and is added as the latest realization to the model initialization set. In the next step, the parameters are re-estimated using the resulting model initialization set and the next one-step-ahead forecast is obtained. A multi-step ahead forecast can be obtained by this step-by-step process. However, as far as we are aware, the literature contains no formula to obtain the multi-step forecasts directly for the FIEGARCH model. An attempt has been made here to develop the formulae regarding the 'out-of-sample' forecasts and the error variances associated with these forecasts for the ARMA(1, 0)- FIEGARCH(1, d, 1) model. To derive the formulae, the conditional expectations and conditional variances have been applied recursively, as followed by Ghosh et al [18] and Rakshit and Paul [19]. For empirical illustration, three onion price series from three major Indian markets and one stock market index (S&P 500) are used. Onion is an integral part of Indian cuisine and is used in a wide variety of dishes across the country. There are several studies in the literature aimed at understanding the behaviour of onion price series in India [20, 21 and 22] as well as globally [23, 24 and 25]. The onion price series is of immense importance for food inflation in the Indian economy. Saxena et al [26] showed that the onion price series had the highest instability index (49.30%) among all the major cultivated vegetables in India.

2. Some nonlinear models

2.1 *The ARCH & GARCH models*

The former model is formulated based on the assumption that at time t, the error term's variance is not constant but depends on the squared error terms from previous time epochs. Let $\{\varepsilon_t\}$ be a TS process and I_{t-1} be the existing information up to the previous lag. Then the conditional distribution of $\{\varepsilon_t\}$ given (I_{t-1}) follows normal with zero mean and conditional variance h_t. Here, this h_t follows the ARCH (q) model and it can be written as

$$\varepsilon_t = \sqrt{h_t}\nu_t \tag{1}$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 \tag{2}$$

where α_0 is non-zero non-negative, α_i's are non-negative, $\sum_{i=1}^{q} \alpha_i < 1$ and innovation $\nu_t \sim$ IID $(0, 1)$.

The GARCH model requires fewer parameters to be estimated. The h_t of the GARCH model is a linear function of both lagged shocks and its own lags. For the GARCH (p, q) process, it is expressed as

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \tag{3}$$

where α_0 is non-zero non-negative, α_i's and β_j's are non-negative.

In the above equation, the unknown parameters α_i and β_j measure the effect of previous shocks and volatilities on current volatility, respectively. The necessary and sufficient condition for this model to be weakly stationary is

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1 \tag{4}$$

The GARCH methodology assumes that the consequence of volatility is solely determined by the level of the shocks and is unaffected by the sign of the shocks. This gap can be filled by the EGARCH model.

2.2 *The EGARCH model*

This is a robust model with a logarithmic expression of the conditional variance. In addition to dealing with asymmetric volatility, this model has the advantage that there are no restrictions on the parameters of the model. The h_t is expressed as

$$\ln h_t = \alpha_0 + \sum_{j=1}^{p} \beta_j \ln h_{t-j} + \sum_{i=1}^{q} \left(\alpha_i \left| \frac{\varepsilon_{t-i}}{\sqrt{h_{t-i}}} \right| + \gamma_i \frac{\varepsilon_{t-i}}{\sqrt{h_{t-i}}} \right) \tag{5}$$

where γ_i is considered as the asymmetric parameter which helps to capture the asymmetric effects.

2.3 *The FIEGARCH Model*

In the presence of asymmetric volatility and long-term persistence, the FIEGARCH model is useful. By adding the fractional differentiation term and carrying out a few algebraic operations on the EGARCH model's h_t, this model is produced. Like the EGARCH model, the FIEGARCH model also has no restrictions on the model parameters. The h_t of the FIEGARCH (p, d, q) model is described as

$$\beta(L)(1-L)^d(\ln h_t - \alpha_0) = \alpha(L)g(z_{t-1})$$

$$\Rightarrow \ln h_t = \alpha_0 + \frac{\alpha(L)}{\beta(L)}(1-L)^{-d}g(z_{t-1}) \quad (6)$$

where $g(z_{t-1}) = \theta z_{t-1} + \gamma(|z_{t-1}| - E|z_{t-1}|)$ and $z_{t-1} = \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}}$ is the normalized innovation. $\beta(L)$ and $\alpha(L)$ are the lagged polynomials, of order p and q, respectively.

3. The derivation of the formulae for out-of-sample forecasts of ARMA(1, 0)- FIEGARCH(1, d, 1) model

Suppose, there are $T + l$ realizations in a time series data, out of which the first T realizations are employed in model building and the latest l realizations are employed for validation purposes.

Let the ARMA (1,0) be applied as the mean model. Therefore, we have

$$y_T = \mu + \phi_1 y_{T-1} + \varepsilon_T \quad (7)$$

where $\varepsilon_T|I_{T-1} \sim N(0, h_T)$ and $\varepsilon_T = \sqrt{h_T}\nu_T$

I_{T-1} and ν_T are defined in the same line as the ARCH process.

The formula for the out-of-sample forecast for **one-step ahead** is obtained as

$$\begin{aligned}\hat{y}_{T+l+1} &= \hat{y}_{T+l+1|1,2,\dots,T+l}\\ &= E(y_{T+l+1}|y_1, y_2, \dots, y_{T+l})\\ &= \hat{\mu} + \hat{\phi}_1 y_{T+l} \quad (8)\end{aligned}$$

In a similar way, the **two-step ahead** forecast is obtained using the conditional expectation recursively, as follows:

$$\begin{aligned}\hat{y}_{T+l+2} &= \hat{y}_{T+l+2|1,2,\dots,T+l} = E(y_{T+l+2}|y_1, y_2, \dots, y_{T+l})\\ &= E\left(\hat{y}_{T+l+2|1,2,\dots,T+l+1}\middle|y_1, y_2, \dots, y_{T+l}\right)\\ &= \hat{\mu} + \hat{\phi}_1\hat{y}_{T+l+1}\\ &= \hat{\mu} + \hat{\phi}_1(\hat{\mu} + \hat{\phi}_1 y_{T+l})\\ &= \hat{\mu}(1 + \hat{\phi}_1) + \hat{\phi}_1^{\,2} y_{T+l} \quad (9)\end{aligned}$$

4. The derivation of the forecast error variances

In equation (6), h_T is defined as follows:

$$\ln h_T = \alpha_0 + \frac{\alpha(L)}{\beta(L)}(1-L)^{-d}g(z_{T-1})$$

$$= \alpha_0 + \left(\frac{1+\alpha_1 L}{1-\beta_1 L}\right)\left(1 + dL + \frac{d(d+1)}{2}L^2\right)\left[\theta\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}} + \gamma\left(\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right| - E\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right|\right)\right]$$

(considering the binomial expansion and taking approximation up to the second order)

$$= \alpha_0 + \left[(1+\alpha_1 L)(1-\beta_1 L)^{-1}\left(1 + dL + \frac{d(d+1)}{2}L^2\right)\right] \times \left[\theta \frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}} + \gamma\left(\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right| - E\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right|\right)\right]$$

$$= \alpha_0 + \left[1 + (\alpha_1+\beta_1+d)L + \left\{(\alpha_1+\beta_1)(\beta_1+d) + \frac{d(d+1)}{2}\right\}L^2\right] \times \left[\theta \frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}} + \gamma\left(\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right| - E\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right|\right)\right]$$

$$= \alpha_0 + \left[\theta \frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}} + \gamma\left(\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right| - E\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right|\right)\right] + (\alpha_1+\beta_1+d)\left[\theta \frac{\varepsilon_{T-2}}{\sqrt{h_{T-2}}} + \gamma\left(\left|\frac{\varepsilon_{T-2}}{\sqrt{h_{T-2}}}\right| - E\left|\frac{\varepsilon_{T-2}}{\sqrt{h_{T-2}}}\right|\right)\right] + \left\{(\alpha_1+\beta_1)(\beta_1+d) + \frac{d(d+1)}{2}\right\}\left[\theta \frac{\varepsilon_{T-3}}{\sqrt{h_{T-3}}} + \gamma\left(\left|\frac{\varepsilon_{T-3}}{\sqrt{h_{T-3}}}\right| - E\left|\frac{\varepsilon_{T-3}}{\sqrt{h_{T-3}}}\right|\right)\right]$$

$$\Rightarrow h_T = exp\left[\alpha_0 + \left[\theta \frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}} + \gamma\left(\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right| - E\left|\frac{\varepsilon_{T-1}}{\sqrt{h_{T-1}}}\right|\right)\right] + (\alpha_1+\beta_1+d)\left[\theta \frac{\varepsilon_{T-2}}{\sqrt{h_{T-2}}} + \gamma\left(\left|\frac{\varepsilon_{T-2}}{\sqrt{h_{T-2}}}\right| - E\left|\frac{\varepsilon_{T-2}}{\sqrt{h_{T-2}}}\right|\right)\right] + \left\{(\alpha_1+\beta_1)(\beta_1+d) + \frac{d(d+1)}{2}\right\}\left[\theta \frac{\varepsilon_{T-3}}{\sqrt{h_{T-3}}} + \gamma\left(\left|\frac{\varepsilon_{T-3}}{\sqrt{h_{T-3}}}\right| - E\left|\frac{\varepsilon_{T-3}}{\sqrt{h_{T-3}}}\right|\right)\right]\right] \quad (10)$$

For the ARMA(1,0)- FIEGARCH(1, d, 1) model, the forecast error variance corresponding to **one-step ahead** forecast is computed as

$$\sigma^2_{T+l+1|1,2,\ldots,T+l} = E\left[\left\{y_{T+l+1} - \hat{y}_{T+l+1|1,2,\ldots,T+l}\right\}^2 | y_1, y_2, \ldots, y_{T+l}\right]$$
$$= E\left[\hat{\varepsilon}^2_{T+l+1|1,2,\ldots,T+l}\right] = \hat{h}_{T+l+1} \quad (11)$$

where

$$\hat{h}_{T+l+1} = exp\left[\hat{\alpha}_0 + \left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right|\right)\right] + \left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)\left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right|\right)\right] + \left\{\left(\hat{\alpha}_1 + \hat{\beta}_1\right)\left(\hat{\beta}_1 + \hat{d}\right) + \frac{\hat{d}(\hat{d}+1)}{2}\right\}\left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l-2}}{\sqrt{\hat{h}_{T+l-2}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l-2}}{\sqrt{\hat{h}_{T+l-2}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l-2}}{\sqrt{\hat{h}_{T+l-2}}}\right|\right)\right]\right] \quad (12)$$

Now, according to the assumption

$$\hat{\varepsilon}_{T+l} \sim N\left(0, \hat{h}_{T+l}\right)$$
$$\Rightarrow \frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} \sim N(0,1)$$

$$\Rightarrow E\left(\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right) = 0 \;\; and \;\; V\left(\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right) = E\left(\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right)^2 = 1$$

Once again, the mean deviation from the mean of this standard normal variate is given by the following equation:

$$E\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| = \sqrt{\frac{2}{\pi}}$$

Hence,

$$\hat{h}_{T+l+1} = exp\left[\hat{\alpha}_0 + \left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right] + \left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)\left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right] + \left\{\left(\hat{\alpha}_1 + \hat{\beta}_1\right)\left(\hat{\beta}_1 + \hat{d}\right) + \frac{\hat{d}(\hat{d}+1)}{2}\right\}\left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l-2}}{\sqrt{\hat{h}_{T+l-2}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l-2}}{\sqrt{\hat{h}_{T+l-2}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right]\right] \quad (13)$$

The forecast error variance for **two-step ahead** forecast is

$$\sigma^2_{T+l+2|1,2,\ldots,T+l} = E\left[\left\{y_{T+l+2} - \hat{y}_{T+l+2|1,2,\ldots,T+l}\right\}^2 | y_1, y_2, \ldots, y_{T+l}\right]$$

$$= E\left[E\left\{y_{T+l+2} - \hat{y}_{T+l+2|1,2,\ldots,T+l+1}\right\}^2 | y_1, y_2, \ldots, y_{T+l}\right] + V\left[\hat{y}_{T+l+2|1,2,\ldots,T+l+1} | y_1, y_2, \ldots, y_{T+l}\right]$$

$$= E\left[E\left\{\hat{\varepsilon}^2_{T+l+2|1,2,\ldots,T+l}\right\}\right] + V(\hat{y}_{T+l+2})$$

$$= E\left(\hat{h}_{T+l+2}\right) + V(\hat{y}_{T+l+2}) \quad (14)$$

$$= E\left(\hat{h}_{T+l+2}\right) + V\left[\hat{\mu} + \hat{\phi}_1\hat{y}_{T+l+1}\right]$$

$$= E\left(\hat{h}_{T+l+2}\right) + \hat{\phi}_1^{\,2}\, V(\hat{y}_{T+l+1}) \quad (15)$$

Now, we have

$$E\left(\hat{h}_{T+l+2}\right) = E\left[exp\left[\hat{\alpha}_0 + \left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right|\right)\right] + \left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)\left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right|\right)\right] + \left\{\left(\hat{\alpha}_1 + \hat{\beta}_1\right)\left(\hat{\beta}_1 + \hat{d}\right) + \frac{\hat{d}(\hat{d}+1)}{2}\right\}\left[\hat{\theta}\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right|\right)\right]\right]\right]$$

$$= E(\exp\hat{\alpha}_0) \times E\left[exp\left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right|\right)\right\}\right] \times$$
$$E\left[exp\left[\left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)\left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right|\right)\right\}\right]\right] \times$$
$$E\left[exp\left[\left\{\left(\hat{\alpha}_1 + \hat{\beta}_1\right)\left(\hat{\beta}_1 + \hat{d}\right) + \frac{\hat{d}(\hat{d}+1)}{2}\right\}\left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right| - \right.\right.\right.\right.$$
$$\left.\left.\left.\left. E\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right|\right)\right\}\right]\right] \tag{16}$$

Now, from the assumption

$$\hat{\varepsilon}_{T+l+1} \sim N\left(0, \hat{h}_{T+l+1}\right)$$
$$\Rightarrow \frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} \sim N(0,1)$$
$$\Rightarrow E\left(\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right) = 0 \text{ and } V\left(\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right) = E\left(\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right)^2 = 1$$

Again, the mean deviation about the mean of this standard normal variate is given by

$$E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| = \sqrt{\frac{2}{\pi}}$$

Now, expressing the expectations separately and after putting these values, we get

$$E\left[exp\left\{\theta\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \gamma\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right|\right)\right\}\right]$$
$$= E\left[1 + \left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right|\right)\right\}\right.$$
$$\left. + \frac{1}{2!}\left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right|\right)\right\}^2 + \cdots\right]$$

(let's consider up to the second order of approximation of exponential expansion)

$$= E\left[1 + \left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right\} + \frac{1}{2}\left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right\}^2\right]$$

$$= E\left[1 + \left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right\} + \frac{1}{2}\left\{\hat{\theta}^2\frac{\hat{\varepsilon}^2_{T+l+1}}{\hat{h}_{T+l+1}} + \hat{\gamma}^2\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)^2 + 2\hat{\theta}\hat{\gamma}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right\}\right]$$

$$= E\left[1 + \left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right\} + \frac{1}{2}\left\{\hat{\theta}^2\frac{\hat{\varepsilon}^2_{T+l+1}}{\hat{h}_{T+l+1}} + \hat{\gamma}^2\left(\frac{\hat{\varepsilon}^2_{T+l+1}}{\hat{h}_{T+l+1}} + \frac{2}{\pi} - \frac{2\sqrt{2}}{\sqrt{\pi}}\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right|\right) + 2\hat{\theta}\hat{\gamma}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right\}\right]$$

$$= E\left[1 + \left\{\hat{\theta}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right)\right\} + \frac{\left(\hat{\theta}^2 + \hat{\gamma}^2\right)}{2}\frac{\hat{\varepsilon}^2_{T+l+1}}{\hat{h}_{T+l+1}} + \frac{\hat{\gamma}^2}{\pi} - \hat{\gamma}^2\sqrt{\frac{2}{\pi}}\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| + \hat{\theta}\hat{\gamma}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \hat{\theta}\hat{\gamma}\sqrt{\frac{2}{\pi}}\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right]$$

$$= 1 + \hat{\theta}\, E\left(\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right) + \hat{\gamma}\left(E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| - \sqrt{\frac{2}{\pi}}\right) + \frac{\left(\hat{\theta}^2 + \hat{\gamma}^2\right)}{2}\, E\left(\frac{\hat{\varepsilon}^2_{T+l+1}}{\hat{h}_{T+l+1}}\right) + \frac{\hat{\gamma}^2}{\pi}$$

$$- \hat{\gamma}^2 \sqrt{\frac{2}{\pi}}\, E\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right| + \hat{\theta}\hat{\gamma}\, E\left(\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\left|\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right|\right)$$

$$- \hat{\theta}\hat{\gamma}\sqrt{\frac{2}{\pi}}\, E\left(\frac{\hat{\varepsilon}_{T+l+1}}{\sqrt{\hat{h}_{T+l+1}}}\right)$$

$$= 1 + \hat{\gamma}\left(\sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{\pi}}\right) + \frac{\left(\hat{\theta}^2 + \hat{\gamma}^2\right)}{2} + \frac{\hat{\gamma}^2}{\pi} - \frac{2}{\pi}\hat{\gamma}^2$$

(as for $X \sim N(0,1), E(X|X|) = 0$)

$$= 1 + \frac{\left(\hat{\theta}^2 + \hat{\gamma}^2\right)}{2} - \frac{\hat{\gamma}^2}{\pi} \tag{17}$$

Similarly,

$$E\left[exp\left[\left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)\left\{\hat{\theta}\,\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right|\right)\right\}\right]\right]$$

$$= E\left[1 + \left[\left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)\left\{\hat{\theta}\,\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right|\right)\right\}\right] + \right.$$

$$\left. \frac{1}{2!}\left[\left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)\left\{\hat{\theta}\,\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}} + \hat{\gamma}\left(\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l}}{\sqrt{\hat{h}_{T+l}}}\right|\right)\right\}\right]^2 + \cdots\right]$$

$$= 1 + \left(\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d}\right)^2\left\{\frac{\left(\hat{\theta}^2 + \hat{\gamma}^2\right)}{2} - \frac{\hat{\gamma}^2}{\pi}\right\} \tag{18}$$

Furthermore,

$$E\left[exp\left[\left\{\left(\hat{\alpha}_1 + \hat{\beta}_1\right)\left(\hat{\beta}_1 + \hat{d}\right) + \frac{\hat{d}\left(\hat{d} + 1\right)}{2}\right\}\left\{\theta\,\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right.\right.\right.$$

$$\left.\left.\left. + \gamma\left(\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right| - E\left|\frac{\hat{\varepsilon}_{T+l-1}}{\sqrt{\hat{h}_{T+l-1}}}\right|\right)\right\}\right]\right]$$

$$= 1 + \left\{(\hat{\alpha}_1 + \hat{\beta}_1)(\hat{\beta}_1 + \hat{d}) + \frac{\hat{d}(\hat{d}+1)}{2}\right\}^2 \left\{\frac{(\hat{\theta}^2+\hat{\gamma}^2)}{2} - \frac{\hat{\gamma}^2}{\pi}\right\} \quad (19)$$

Hence, from equations (16), (17), (18) and (19),

$$E(\hat{h}_{T+l+2}) = c \times \left[1 + \frac{(\hat{\theta}^2+\hat{\gamma}^2)}{2} - \frac{\hat{\gamma}^2}{\pi}\right] \times \left[1 + (\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d})^2 \left\{\frac{(\hat{\theta}^2+\hat{\gamma}^2)}{2} - \frac{\hat{\gamma}^2}{\pi}\right\}\right] \times \left[1 + \left\{(\hat{\alpha}_1 + \hat{\beta}_1)(\hat{\beta}_1 + \hat{d}) + \frac{\hat{d}(\hat{d}+1)}{2}\right\}^2 \left\{\frac{(\hat{\theta}^2+\hat{\gamma}^2)}{2} - \frac{\hat{\gamma}^2}{\pi}\right\}\right] \quad (20)$$

(where $c = exp(\hat{\alpha}_0)$)

As a result, the formula for forecast error variance is

$$\sigma^2_{T+l+2|1,2,\ldots,T+l} = E(\hat{h}_{T+l+2}) + \hat{\phi}_1^{\,2}\, V(\hat{y}_{T+l+1})$$

$$= c \times \left[1 + \frac{(\hat{\theta}^2+\hat{\gamma}^2)}{2} - \frac{\hat{\gamma}^2}{\pi}\right] \times \left[1 + (\hat{\alpha}_1 + \hat{\beta}_1 + \hat{d})^2 \left\{\frac{(\hat{\theta}^2+\hat{\gamma}^2)}{2} - \frac{\hat{\gamma}^2}{\pi}\right\}\right] \times \left[1 + \left\{(\hat{\alpha}_1 + \hat{\beta}_1)(\hat{\beta}_1 + \hat{d}) + \frac{\hat{d}(\hat{d}+1)}{2}\right\}^2 \left\{\frac{(\hat{\theta}^2+\hat{\gamma}^2)}{2} - \frac{\hat{\gamma}^2}{\pi}\right\}\right] + \hat{\phi}_1^{\,2}\hat{h}_{T+l+1} \quad (21)$$

5. Empirical Illustration

5.1 *Data Description*

Four TS have been selected for empirical illustration. Of these, three are the daily prices series in Rs./q of onion from three different locations in India, namely Delhi, Lasalgaon and Bengaluru, and the other is one USA-based share market index, the S&P 500. The TS is considered for the duration of 1 January 2008 to 28 February 2022. The onion price data are sourced from the Government of India. The index data are taken from the Yahoo Finance website. The data series are transformed to log return (LR) series and the analysis is performed on the transformed series. For a TS $\{y_t\}$ the LR series $\{r_t\}$ is calculated as

$$r_t = \ln\frac{y_t}{y_{t-1}} \quad (22)$$

The daily series was chosen because it has a greater chance of demonstrating long-term persistence due to the large number of realizations over a longer period.

5.2 *Descriptive Statistics*

The main purpose is to provide a quick overview of the main characteristics of a dataset. The basic statistical measures of the selected TS are given in Table 1. For the onion price series data sets, there are 5173 realizations, while for the index data, there are 3565 realizations. All four selected TS show a high degree of variation. The index data set has significantly less C.V. (%) than the other three onion price series. It also has significantly lower skewness and kurtosis. It is also noticeable that the price series attributable to the Bengaluru market has much higher kurtosis as comparable to the other series. The last 250 realizations of each selected TS serve as the holdout/test set, while the remaining earlier realizations serve as the model initialization set. Parameter estimation is performed using the model initialization data set. The holdout set is used to determine the in-sample forecasting efficacy.

Table 1
Descriptive statistics of the TS used

Statistics	Bengaluru	Delhi	Lasalgaon	S&P 500
Minimum	300.00	275.00	230.50	676.53
Mean	1366.58	1358.99	1332.41	2118.45
Median	1016.67	1069.00	991.00	1994.99
Maximum	12500.00	7650.00	8625.00	4796.56
S.D.	1081.14	918.44	1059.31	953.62
CV (%)	79.11	67.58	79.50	45.01
Skewness	3.28	2.04	2.04	0.86
Kurtosis	18.77	5.25	5.32	0.12

A time plot is a simple and sophisticated tool to identify the overall trend, seasonal effect and cyclic effect over time on a TS through visualization. Time plot provides some crucial insights into a TS. Figure 1 characterizes the temporal variation of the selected TS. Figure 1 shows that a similar pattern of price movement exists for all the price series. The strongest price increase can be seen towards the end of 2019. Although the locations are geographically far apart, the price series of these locations are well connected. The final subplot shows the index data, which, with the exception of two notches, shows an upward trend.

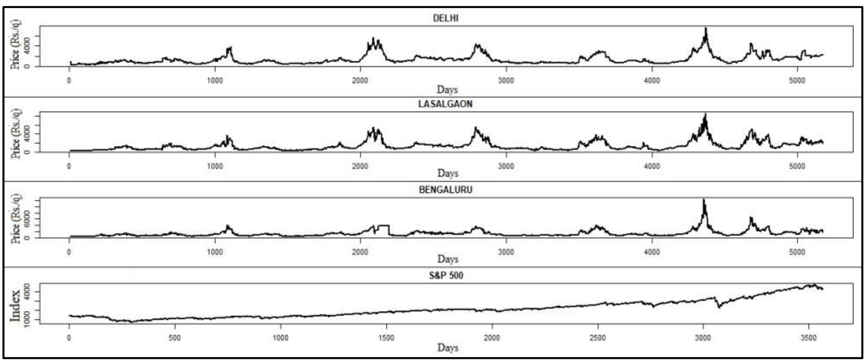

Figure 1
Time plots (actual series)

5.3 *Stationarity Test*

Before applying the GARCH model, the data sets must be stationary. Deviations from stationarity can lead to unreliable and misleading forecasts. If the series is non-stationary, the model might capture noise or transient patterns rather than stable underlying dynamics. The augmented Dickey-Fuller (ADF) test [27] and the PP test [28] are used to examine the possible existence of non-stationarity. Table 2 shows the results and all tests are significant with $p < 0.01$.

Table 2
Stationarity test

TS	Bengaluru	Delhi	Lasalgaon	S&P 500
LR				
ADF	-16.24	-17.40	-16.21	-15.04
PP	-77.82	-81.63	-80.88	-69.05
Squared LR				
ADF	-15.85	-14.39	-15.98	-8.53
PP	-70.33	-64.54	-61.02	-50.45

5.3 *Long Memory Test*

The GPH test [29] is a statistical method used to estimate the long memory parameter d in TS data. This test is performed, and Table 3 displays the findings. It shows that in the LR series, the estimates of d are not significant here. However, they are significant for the corresponding squared series. This suggests that while long memory is absent in the LR, it is present in the variance structure. It can therefore be concluded that volatility exhibits long-term persistence.

Table 3
Long memory test

TS	Bengaluru	Delhi	Lasalgaon	S&P 500
LR				
d	-0.019	-0.125	-0.037	0.057
s.e.	0.053	0.111	0.034	0.090
Z	-0.347	-1.125	-1.097	0.634
Squared LR				
d	0.089	0.301	0.067	0.275
s.e.	0.029	0.065	0.031	0.089
Z	3.051	4.637	2.184	3.093

5.4 *ACF and PACF*

ACF and PACF plots (Figure 2) are visual tools used in TS analysis to understand the dependencies within the realizations. They help to identify the nature and extent of correlations at different lags, which is crucial for building appropriate TS models. The critical values of the test statistic at a 5% significance level are indicated by the dotted lines in the plots. It is evident that the LR series's ACF and PACF decrease exponentially. This indicates that the mean model lacks long memory. However, for the last column of Figure 2, the situation is different. Here, hyperbolic decay and significant autocorrelations are visible at long lags. This implies the possible existence of long memory in volatility. The same conclusions are also supported by the results of the GPH test.

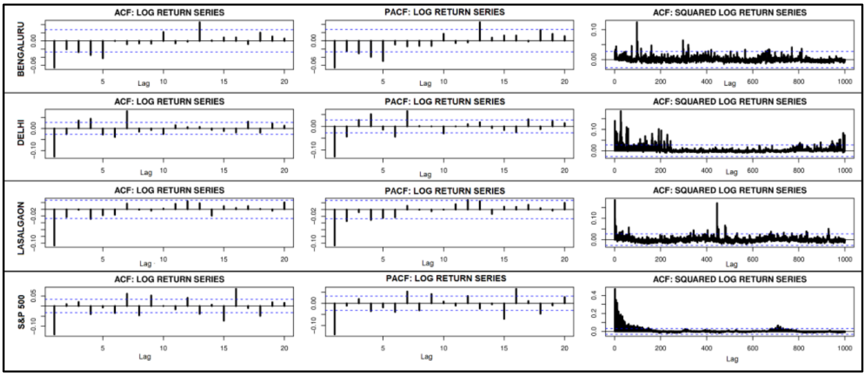

Figure 2
ACF and PACF plots

5.5 *Fitting of Models*

Since the stationary assumption is fulfilled for all the LR series, the first step is to apply the ARMA(1,0) model to each data set. The ARCH effects indicate the non-constant variance of the residuals over time which depends on past errors and it has been tested by carrying out the ARCH-LM test under H_0 that the residual series has no ARCH effect. The test was significant for all residuals. Next, the FIEGARCH(1, d, 1) model is applied to all the residuals. The estimated parameters are derived using the MLE procedure under the normality assumption and they are provided in Table 4. The parameter α_1 is highly significant for the Bengaluru, Delhi, and Lasalgaon, markets ($p < 0.01$). For the index data, it is significant with $p < 0.10$. From the estimated values of the parameter α_1, it can be inferred that for the index series, the dependency of present volatility on the past shock is less significant than for the other three selected TS. The parameter β_1 is highly significant ($p < 0.01$) for each series. This implies that for each series the current volatility is highly influenced by the past volatility. The asymmetric parameter γ is significant with $p < 0.01$ for the Delhi and Lasalgaon markets and also for the index series. On the other hand, it is significant with $p < 0.05$ for the Bengaluru market. The d parameter is significant for the index series with $p < 0.10$ and also highly significant ($p < 0.01$) for the remaining series. The volatility of the selected TS shows an asymmetric nature along with long memory, as indicated by the significant estimate of the asymmetric parameter and the fractional differencing parameter. The residual series are examined to ensure the model's adequacy. The fitted ARMA(1, 0)- FIEGARCH(1, d, 1) model's in-sample forecasting ability is assessed using the RMSE and MAE. Their respective formulae are given by

$$RMSE = \left[\frac{1}{l}\sum_{t=1}^{l}(y_t - \hat{y}_t)^2\right]^{1/2} \tag{23}$$

$$MAE = \frac{1}{l}\sum_{t=1}^{l}|y_t - \hat{y}_t| \quad (24)$$

where y_t indicates the actual value and $\hat{y}_t$ is its forecasted value, l is the number of realizations used for in-sample forecasting. Table 5 provides the computed values for these two error functions.

Table 6 shows the forecasts accompanied by their respective forecast error variances for the next two horizons of the selected TS obtained from the resultant formula for the ARMA(1,0)- FIEGARCH(1, d, 1) model. It has been observed that as the forecast horizons increase, so do the forecast error variances. This is an acceptable phenomenon because it is expected that the variability will increase with the higher length of forecasting horizons.

Table 4
Estimate of parameters

Parameter	Bengaluru	Delhi	Lasalgaon	S&P 500
ARMA(1, 0)				
Constant	0.005 (0.000)***	-0.003 (0.000)***	-0.001 (0.001)	0.000 (0.000)***
AR(1)	-0.071 (0.012)***	-0.041 (0.016)**	-0.073 (0.014)***	-0.062 (0.019)***
FIEGARCH(1, d, 1)				
Constant	-4.515 (0.001)***	-2.840 (0.065)***	-4.455 (0.068)***	-8.739 (0.175)***
α_1	-1.647 (0.275)***	-0.318 (0.056)***	-0.497 (0.032)***	-0.167 (0.100)*
β_1	0.972 (0.000)***	0.787 (0.018)***	0.957 (0.003)***	0.746 (0.034)***
θ	0.001 (0.000)**	0.179 (0.009)***	0.255 (0.011)***	0.197 (0.017)***
γ	-0.007 (0.003)**	0.030 (0.003)***	-0.074 (0.006)***	-0.164 (0.013)***
d	0.962 (0.074)***	-0.326 (0.035)***	-0.894 (0.057)***	-0.199 (0.115)*

***$p < 0.01$, **$p < 0.05$, *$p < 0.10$

Table 5
In-sample forecasting performance

Series	RMSE	MAE
Delhi	0.049	0.014
Lasalgaon	0.071	0.042
Bengaluru	0.087	0.043
S&P 500	0.009	0.007

Table 6
Out-of-sample forecasts

Series	Horizon	Forecasted values	Forecast error variances
Delhi	1	-0.0026	0.0542
	2	-0.0025	0.0598
Lasalgaon	1	0.0141	0.0083
	2	-0.0018	0.0118
Bengaluru	1	0.0188	0.0109
	2	0.0037	0.0110
S&P 500	1	0.0006	0.0001
	2	0.0004	0.0002

6. Conclusions

To the best of our knowledge, there is no direct formula available to get the out-of-sample forecasts for the ARMA(1,0)- FIEGARCH(1, d, 1) model and the error variances for these forecasts. In this article, an attempt has been made to derive these formulae by using the conditional forms of both expectations and variances recursively. Real-world TS have been used for empirical purposes. Every TS under consideration exhibits asymmetric volatility with long memory. Together with their forecast error variances, using the resulting formulae the forecasts of the selected series for the next two horizons have been obtained. It has been observed that as the forecast horizons increase, so do the forecast error variances. This study can be extended to obtain the same aspects of the formulae under non-normal innovation. Again, for the situation where long memory exists in both the mean and variance models.

Conflicts of interest

There is no conflict of interest.

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References

[1] G. E. P. Box, G. M. Jenkins, G. C. Reinsel, and G. M. Ljung, *Time series analysis: forecasting and control*, John Wiley & Sons, New Jersey, USA (2015).

[2] R. F. Engle, "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation", *Econometrica*, vol. 50, issue 4, pp. 987-1007 (1982).

[3] T. Bollerslev, "Generalized autoregressive conditional heteroskedasticity", *Journal of Econometrics*, vol. 31, issue 3, pp. 307-327 (1986).

[4] S. J. Taylor, *Modelling financial time-series,* John Wiley & Sons, Ltd., Chichester, Great Britain (1986).

[5] R. Maheshwari and V. Kapoor, "Estimating the volatility of stock price index for Indian market using GARCH model", *Journal of Statistics and Management Systems,* vol. 25, issue 7, pp. 1523-1530 (2022).

[6] D. Rakshit, R. K. Paul, M. Yeasin, W. Emam, Y. Tashkandy, and C. Chesneau, "Modeling asymmetric volatility: A news impact curve approach", *Mathematics,* vol. 11, issue 13, 2793 (2023).

[7] D. B. Nelson, "Conditional Heteroskedasticity in Asset Returns: A New Approach", *Econometrica*, vol. 59, issue 2, pp. 347-370 (1991).

[8] C. W. J. Granger and R. Joyeux, "An introduction to long-memory time-series models and fractional differencing", *Journal of Time-series Analysis,* vol. 1, issue 1, pp. 15-29 (1980).

[9] R. T. Baillie, T. Bollerslev, and H. O. Mikkelsen, "Fractionally integrated generalized autoregressive conditional heteroskedasticity", *Journal of Econometrics,* vol. 74, issue 1, pp. 3-30 (1996).

[10] T. Bollerslev and H. O. Mikkelsen, "Modeling and pricing long memory in stock market volatility", *Journal of Econometrics*, vol. 73, issue 1, pp. 151-184 (1996).

[11] B. Karali and W. N. Thurman, "Components of grain futures price volatility", *Journal of Agricultural and Resource Economics,* vol. 35, issue 2, pp. 167–182 (2010).

[12] R. K. Paul, "Forecasting wholesale price of pigeon pea using long memory time-series models", *Agricultural Economics Research Review,* vol. 27, issue 2, pp. 167-176 (2014).

[13] R. K. Paul, Prajneshu, and H. Ghosh, "GARCH Nonlinear Time-series Analysis for Modeling and Forecasting of India's Volatile Spices Export Data", *Journal of the Indian Society of Agricultural Statistics,* vol. 63, issue 2, pp. 123-131 (2009).

[14] R. K. Paul, B. Gurung, A. K. Paul, and S. Samanta, "Long memory in conditional variance", *Journal of the Indian Society of Agricultural Statistics,* vol. 70, issue 3, pp. 243-254 (2016).

[15] D. Rakshit and R. K. Paul, "Long Memory in Volatility: Application of Fractionally Integrated GARCH Model", in Special Proceedings of the 25th (Silver Jubilee) International Annual Conference of the Society of Statistics, Computer and Applications, V. K. Gupta, B. N.

Mandal, R. V. Vardhan, R. K. Paul, R. Parsad, and D. R. Choudhury, Eds., pp. 107-118 (2023).

[16] D. Mitra, R. K. Paul, and A. K. Paul, "Statistical modelling for forecasting volatility in potato prices using ARFIMA-FIGARCH model", *Indian Journal of Agricultural Sciences,* vol. 88, issue 2, pp. 268-272 (2018).

[17] D. Rakshit and R. K. Paul, "Modeling Time Series with Asymmetric Volatility and Long Memory", *Indian Journal of Agricultural Economics*, vol. 79, issue 2, pp. 231-244 (2024).

[18] H. Ghosh, R. K. Paul, and Prajneshu, "Nonlinear time-series modeling and forecasting for periodic and ARCH effects", *Journal of Statistics Theory and Practice,* vol. 4, issue 1, pp. 27-44 (2010).

[19] D. Rakshit and R. K. Paul, "Development of out-of-sample forecast formulae for the FIGARCH model", *Model Assisted Statistics and Applications,* vol. 19, issue 2, pp. 133-143 (2024).

[20] R. K. Paul, R. Saxena, S. Chaurasia, Zeeshan, and S. Rana, "Examining Export Volatility, Structural Breaks in Price Volatility and Linkages between Domestic and Export Prices of Onion in India", *Agricultural Economics Research Review,* vol. 28, pp. 101-116 (2015).

[21] R. K. Paul, S. Rana, and R. Saxena, "Effectiveness of price forecasting techniques for capturing asymmetric volatility for onion in selected markets of Delhi", *Indian Journal of Agricultural Sciences,* vol. 86, issue 3, pp. 303-309 (2016).

[22] D. Rakshit, R. K. Paul, and S. Panwar, "Asymmetric Price Volatility of Onion in India", *Indian Journal of Agricultural Economics,* vol. 76, issue 2, pp. 245-260 (2021).

[23] D. Susanto, C. P. Rosson III, and F. J. Adcock, "Market integration and convergence to the law of one price in the North American onion markets", *Agribusiness: An International Journal*, vol. 24, issue 2, pp. 177-191 (2008).

[24] B. S. Kim and J. H. Cho, "A Review on the Volatility of Green Onion Wholesale Price", *Korean Journal of Agricultural Management and Policy*, vol. 37, issue 4, pp. 747-758 (2010).

[25] K. H. Nam and Y. C. Choe, "A study on onion wholesale price forecasting model", *Journal of Agricultural Extension & Community Development,* vol. 22, issue 4, pp. 423-434 (2015).

[26] R. Saxena, N. P. Singh, S. J. Balaji, U. Ahuja, R. Kumar, and D. Joshi, "Doubling Farmers' Income in India by 2022-23: Sources of Growth and Approaches", *Agricultural Economics Research Review*, vol. 30, issue 2, pp. 265-277 (2017).

[27] D. A. Dickey and W. A. Fuller, "Distribution of the Estimators for Autoregressive Time-series with a Unit Root", *Journal of the American Statistical Association*, vol. 74, pp. 427-431 (1979).

[28] P. C. B. Phillips and P. Perron, "Testing for a unit root in time-series regression", *Biometrika*, vol. 75, issue 2, pp. 335-346 (1988).

[29] J. Geweke and S. Porter-Hudak, "The estimation and application of long memory time-series models", *Journal of Time-series Analysis*, vol. 4, issue 4, pp. 221-238 (1983).

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