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A flexible model with both bathtub and upside down bathtub shape hazard rates

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Abstract

In this article a new distribution called the beta type two half logistic Topp-Leone-G (BTIIHLTL-G) Distribution is presented. The new distribution is generated via the Beta-G (BG) family of distributions originally proposed by Jones [11]. We study three special cases of the BTIIHLTL-G Distribution. The new model's moments, conditional moments, moment generating function, order statistics, and entropy are among the statistical characteristics that are examined. Finally, we present an implementation of our novel model on actual data sets.

Subject Classification (2010): *60E05, 60E10.*

Keywords: *Half Logistic distribution, Type II Half Logistic-G distribution, Maximum likelihood estimation.*

1. Introduction

Component failure periods are frequently modeled using the Half Logistic (HL) distribution. The HL distribution is a single-parameter distribution which has an increasing hazard rate function (hrf) and a decreasing probability density function (pdf). Although the HL distribution is widely used in a variety of fields, its major disadvantage is that it is not suitable for data with non-monotone failure rates and

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unimodal density [17]. To address this limitation, several extensions have been proposed including, amongst others, the Type Two Half Logistic-G (TIIHL-G) distribution by Hassan et.al [9], Beta Half Logistic (BHL) by Jose and Manaharan [12], the power half logistic by Krishnarani [13], the Type I Generalized Half Logistic Distribution by Olapade [19] and the exponentiated odd exponential half logistic-G power series (EOEHL-GPS) class of distributions by Rannona et.al [22]. The TIIHL-G distribution is a tractable distribution that is commonly employed in analysing lifetime data sets [3, 17, 24].

In general, suppose the baseline cumulative distribution function (cdf) is $G(z;\xi)$, then the Type Two Half Logistic Topp-Leone-G (TIIHLTL-G) distribution has its cdf as

$$G_{TIIHLTL-G}(z)=\frac{2\Psi}{1+\Psi}, \tag{1}$$

where $\Psi=(1-\bar{G}^2(z;\xi))^b$, $b>0$ is the shape parameter, and ξ is a vector of unknown parameters.

In this article we extend the TIIHLTL-G distribution via the Beta-G (BG) method originally proposed by Jones [11]. The new model is known as the Beta Type Two Half Logistic Topp-Leone-G (BTIIHLTL-G) distribution. The new hrf exhibits a variety of shapes such as decreasing, increasing, bathtub, upside-down bathtub and uni-modal shapes. Moreover, the pdf of the order statistic of the new distribution can be written as a linear combination of the exponentiated-G (Exp-G) distribution. Estimation of the model parameters is done through maximum likelihood estimation [21].

If $G(z;\xi)$ is a continuous distribution function, then the BG family of distributions generated by $G(z;\xi)$, and the parameters $\{\alpha,\beta\}\in\mathbb{R}^+$, has its cdf as

$$F(z)=\{B(\alpha,\beta)\}^{-1}\int_0^{G(z;\xi)}u^{a-1}(1-u)^{b-1}du=\frac{B_{G(z;\xi)}(\alpha,\beta)}{B(\alpha,\beta)}. \tag{2}$$

Special cases of BG family of distributions have been proposed over the years [1, 4, 5, 6, 7, 14, 15]. The new model is obtained by replacing $G(z;\xi)$ in (2) with the cdf in (1). Thus the cdf and pdf of the BTIIHLTL-G are

$$\begin{aligned}F_{BTIIHLTL-G}(z;\alpha,\beta,b,\xi)&=\{B(\alpha,\beta)\}^{-1}\int_0^{G_{TIIHLTL-G}}u^{\alpha-1}(1-u)^{\beta-1}du\\&=\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\frac{\left[\frac{2(1-\bar{G}^2(z;\xi))^b}{1+(1-\bar{G}^2(z;\xi))^b}\right]^{\alpha+j}}{(\alpha+j)},\end{aligned} \tag{3}$$

and

$$f_{BTIIHLTL-G}(z;\alpha,\beta,b,\xi)=\Phi(z)\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\left[\frac{2(1-\bar{G}^{2}(z;\xi))^{b}}{1+(1-\bar{G}^{2}(z;\xi))^{b}}\right]^{\alpha+j-1}, \quad (4)$$

respectively, where

$$\Phi(z)=\frac{4bg(z;\xi)\bar{G}(z;\xi)(1-\bar{G}^{2}(z;\xi))^{b-1}}{(1+(1-\bar{G}^{2}(z;\xi))^{b})^{2}}.$$

Note that through the binomial expansions, Equation (4) can be expressed as

$$f_{BTIIHLTL-G}(z;\alpha,\beta,b,\xi)=\sum_{l=0}^{\infty}\eta_{m}g_{m}(z;\xi), \quad (5)$$

where $g_m(z;\xi)=(m+1)g(z;\xi)G^{m}(z;\xi)$ is an Exp-G pdf and

$$\eta_{m}=\frac{2^{\alpha+j-1}}{B(\alpha,\beta)}\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\sum_{k,l,m=0}^{\infty}\frac{(-1)^{l+m}4b}{m+1}\times\binom{k+\alpha+j}{k}\binom{b(\alpha+j+k)-1}{l}\binom{2l+1}{m}. \quad (6)$$

Therefore, the BTIIHLTL-G pdf can be written as an infinite linear combination of the Exp-G distributions. Thus, some important statistical properties of the BTIIHLTL-G distribution can be derived from those of the Exp-G distribution.

The structure of this article is as follows, in Section 2 we discussed some statitistical properties on the new model including moments, conditional moments, moment generating function, pdf of the order statistic and the Rényi Entropy. In Section 3, estimation of model parameters using method of maximum likelihood estimation is discussed. Three special cases of the new model are studied in Section 4. The Monte Carlo simulation study which examines the bias, mean square error of the maximum likelihood estimators is carried out for one special case in Section 5. The practical relevence of the new model will be demonstrated in Section 6 using two data sets, followed by conclusion in Section 7.

2. Statistical Properties

In this section, some important statistical properties of the BTIIHLTL-G such as moments, incomplete moments, generating functions, pdf of order statistics and Rényi entropy are discussed.

2.1 *Moments, Conditional Moments, and Generating Functions*

Suppose $Z \sim BTIIHLTL-G$ and $Z_{m+1} \sim Exp-G$ with power parameter $m+1$. The k^{th} moment and the mgf of Z are $\sum_{m=0}^{\infty} \eta_m E\, [Z_{m+1}^k]$ and $\sum_{m=0}^{\infty} \eta_m\, M_{Z_{m+1}}(t)$, respectively, where η_m is as given in equation (6), and $M_{Z_{m+1}}(t)$ is the mgf of Z_{m+1}. The k^{th} conditional moment of Z is

$$E(Z^k \mid Z > n) = \frac{1}{\overline{F}(n;\alpha,\beta,b,\xi)} \int_n^{\infty} z^k \cdot f_{BTIIHLTL-G}(z;\alpha,\beta,b,\xi)dz$$
$$= \frac{1}{\overline{F}(n;\alpha,\beta,b,\xi)} \sum_{m=0}^{\infty} \eta_m \cdot E[Z^k I_{(Z^k>n)}],$$

where $E[Z^k I_{(Z^k>n)}]$ is k^{th} conditional moment of Z_{m+1}. The average lifetime of X is given by

$$E(Z) = \sum_{m=0}^{\infty} \eta_m E[Z_{m+1}], \tag{7}$$

where $E[Z_{m+1}]$ is average lifetime of the random variable Z_{m+1}.

2.2 *Order Statistics and Rényi Entropy*

Let $Z_1, Z_2, ..., Z_n$ be a random sample from BTIIHLTL-G distribution. The i^{th} order statistic has its pdf as

$$f_{i:n}(z) = i\binom{n}{i} \sum_{p=0}^{n-i} (-1)^p \binom{n-i}{t} \left[\{B(\alpha,\beta)\}^{-1} \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} \frac{\left[\frac{2(1-\overline{G}^2(z;\xi))^b}{1+(1-\overline{G}^2(z;\xi))^b}\right]^{\alpha+j}}{(\alpha+j)} \right]^{p+i-1}$$
$$\times \Phi(z)\{B(\alpha,\beta)\}^{-1} \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} \left[\frac{2(1-\overline{G}^2(z;\xi))^b}{1+(1-\overline{G}^2(z;\xi))^b}\right]^{\alpha+j-1}$$

Using the series expansion representation we have:

$$[1+(1-\overline{G}^2(z;\xi))^b]^{-(i+\alpha+j+p)} = \sum_{q=0}^{\infty} \binom{i+\alpha+j+p+q-1}{q} (1-\overline{G}^2(z;\xi))^{bq},$$

$$[(1-\overline{G}^2(z;\xi))]^{b(i+\alpha+j+p+q-1)-1} = \sum_{r=0}^{\infty} (-1)^r \binom{b(i+\alpha+j+p+q-1)-1}{r} \overline{G}^{2r}(z;\xi),$$

and

$$[\overline{G}(z;\xi)]^{2r+1} = [1-G(z;\xi)]^{2r+1} = \sum_{s=0}^{\infty} (-1)^s \binom{2r+1}{s} G^s(z;\xi).$$

Therefore,

$$f_{i:n}(z)=i\binom{n}{i}\sum_{p=0}^{\infty}(-1)^p\binom{n-i}{p}\sum_{j,q,r,s=0}^{\infty}\frac{(-1)^{2j}}{(\alpha+j)(B(\alpha,\beta))}$$
$$\times\frac{4b}{s+1}\binom{\beta-1}{j}\binom{\beta-1}{j}\binom{i+\alpha+j+p+q-1}{q}\binom{b(i+\alpha+j+p+q-1)-1}{r}\binom{2r+1}{s}$$
$$\times(s+1)g(z;\xi)G^s(z;\xi)$$
$$=\sum_{s=0}^{\infty}\eta_s g_s(z;\xi),$$

where

$$g_s(z;\xi)=(s+1)g(z;\xi)G^s(z;\xi),$$

is the pdf of the Exp-G distribution and

$$\eta_s=i\binom{n}{i}\sum_{p=0}(-1)^p\binom{n-i}{p}\sum_{j,q,r,s=0}^{\infty}\frac{(-1)^{2j}}{(\alpha+j)(B(\alpha,\beta))}$$
$$\times\frac{4b}{s+1}\binom{\beta-1}{j}\binom{\beta-1}{j}\binom{i+\alpha+j+p+q-1}{q}\binom{b(i+\alpha+j+p+q-1)-1}{r}\binom{2r+1}{s}.$$

The Rényi Entropy is evaluated as

$$E_{R(\upsilon)}=(1-\upsilon)^{-1}\log(\int_0^{\infty}f^{\upsilon}_{BTIIHLTL-G}(z;\alpha,\beta,b,\xi)dz)$$

Note that

$$f^{\upsilon}_{BTIIHLTL-G}(z;\alpha,\beta,b,\xi)=\frac{\Phi^{\upsilon}(z)}{B(\alpha,\beta)^{\upsilon}}\left[\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\left[\frac{2(1-\bar{G}^2(x;\xi))^b}{1+(1-\bar{G}^2(x;\xi))^b}\right]^{\alpha+j-1}\right]^{\upsilon}.$$

Using binomial expansion we have

$$[1+(1-\bar{G}^2(z;\xi))^b]^{-(3\upsilon)}=\sum_{t=0}^{\infty}\binom{3\upsilon-1}{t}(1-\bar{G}^2(z;\xi))^{bt},$$

$$[(1-\bar{G}^2(z;\xi)]^{b(t+2v)-v}=\sum_{u=0}^{\infty}(-1)^u\binom{b(t+2v)-v}{u}\bar{G}^{2u}(z;\xi),$$

and

$$[1-G(z;\xi)]^{2u+\upsilon}=\sum_{w=0}^{\infty}(-1)^w\binom{2u+\upsilon}{w}G^w(z;\xi).$$

Therefore,

$$f^{\upsilon}(z;\alpha,\beta,b,\xi)=\{B(\alpha,\beta)\}^{-\upsilon}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\sum_{t,u,w=0}^{\infty}(-1)^{u+w}(4b)^{\upsilon}$$
$$\times\binom{3\upsilon-1}{t}\binom{b(t+2v)-v}{u}\binom{2u+\upsilon}{w}\times g^{\upsilon}(z;\xi)G^w(z;\xi).$$

As a result, we have

$$I_{R(\upsilon)}=(1-\upsilon)^{-1}\log\Bigg(\int_0^\infty\Bigg[\{B(\alpha,\beta)\}^{-\upsilon}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\sum_{t,u,w=0}^{\infty}(-1)^{u+w}(4b)^{\upsilon}$$
$$\times\binom{3\upsilon-1}{t}\binom{b(t+2\upsilon)-\upsilon}{u}\binom{2u+\upsilon}{w}g^{\upsilon}(z;\xi)G^{w}(z;\xi)\Bigg]dz\Bigg)$$
$$=(1-\upsilon)^{-1}log(\sum_{w=0}^{\infty}s_w^{*}e^{(1-\upsilon)I_{REG}}),$$

where

$$I_{REG}=\frac{1}{1-\upsilon}log\left(\int_0^\infty\left(\frac{1}{\left[\frac{w}{\upsilon}+1\right]^{\upsilon}}g(z;\xi)G^{\frac{w}{\upsilon}}(z;\xi)\right)^{\upsilon}dz\right),$$

is the Rényi entropy of Exp-G distribution with power parameter $\frac{1}{\left[\frac{w}{\upsilon}+1\right]^{\upsilon}}$ and

$$s_w^{*}=\frac{1}{B(\alpha,\beta)^{\upsilon}}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\sum_{t,u,w=0}^{\infty}(-1)^{u+w}(4b)^{\upsilon}$$
$$\times\binom{3\upsilon-1}{t}\binom{b(t+2\upsilon)-\upsilon}{u}\binom{2u+\upsilon}{w}\frac{1}{\left[\frac{w}{\upsilon}+1\right]^{\upsilon}}.$$

3. Maximum Likelihood Estimation

In this section, we consider parameter estimation using the method of maximum likelihood estimation. Let $Z\sim$ BTIIHLTL-G and $\Lambda=(\alpha,\beta,b,\xi)$ be the vector of model parameters. Based on a random sample of size n, the log-likelihood, denoted by $\mathcal{L}(\Lambda)$, from the BTIIHLTL-G distribution is:

$$\mathcal{L}(\Lambda)=(\alpha-1)\sum_{i=0}^{n}\ln\left[\frac{2(1-\overline{G}^2(z;\xi))^b}{1+(1-\overline{G}^2(z;\xi))^b}\right]+(\beta-1)\sum_{i=0}^{n}\ln\left[\frac{1-(1-\overline{G}^2(z;\xi))^b}{1+(1-\overline{G}^2(z;\xi))^b}\right]+n\ln(4b)$$
$$+\sum_{i=0}^{n}\ln(g(z;\xi))+\sum_{i=0}^{n}\ln(\overline{G}(z;\xi))+(b-1)\sum_{i=0}^{n}\ln[1-\overline{G}^2(z;\xi)]-\ln(B(\alpha,\beta))$$
$$-2\sum_{i=0}^{n}\ln[1+(1-\overline{G}^2(z;\xi))^b].$$

The maximum likelihood estimates (MLEs) are obtained numerically by solving the equation

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \alpha} \\ \frac{\partial \mathcal{L}}{\partial \beta} \\ \frac{\partial \mathcal{L}}{\partial b} \\ \frac{\partial \mathcal{L}}{\partial \xi_k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

We then estimate the standard errors using the observed Fisher information matrix [18] assessed at MLEs and derive the confidence intervals for the parameters.

4. Examples of the BTIIHLTL-G Distributions

Here, we give some examples of the BTIIHLTL-G distribution. For the baseline distributions, we considered the uniform, the exponential, and the log-logistic distribution. These examples illustrate the diverse shapes and characteristics exhibited by the BTIIHLTL-G family of distributions, including right or left skewed, reverse-J, J, almost symmetric shapes (for pdf), as well as decreasing, increasing, bathtub, and upside-down bathtub shapes (for hrf). All these shapes can be seen in Figures 1, 2 and 3.

4.1 *Beta type two half logistic Topp-Leone uniform distribution*

The beta type two half logistic Topp-Leone uniform (BTIIHLTL-U) distribution has its hrf and pdf as

$$h_{BTIIHLTL-U}(z) = \frac{\{B(\alpha,\beta)\}^{-1} \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} \frac{\left[\frac{2\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b}{1+\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b}\right]^{\alpha+j-1}}{\alpha+j} \times \frac{4b\left(\frac{1}{\theta}\right)\left(1-\frac{z}{\theta}\right)\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^{b-1}}{\left(1+\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b\right)^2}}{1-\{B(\alpha,\beta)\}^{-1} \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} \frac{\left[1-\frac{1-\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b}{1+\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b}\right]^{\alpha+j}}{\alpha+j}}.$$

and

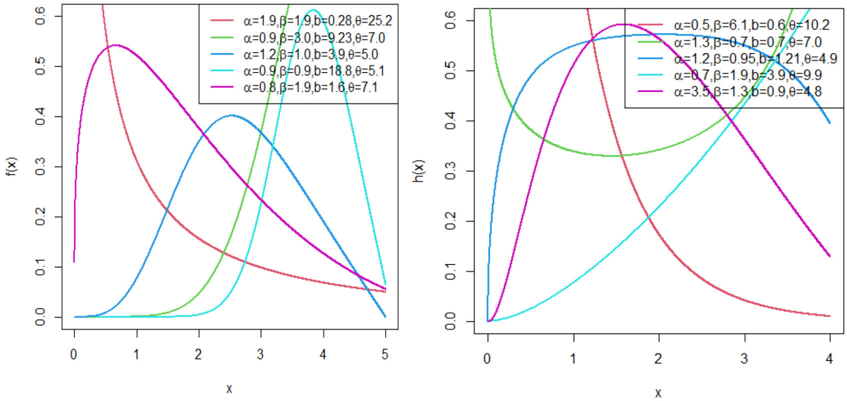

Figure 1

The pdf and hrf plots for the BTIIHLTL-U distribution

$$f_{BTIIHLTL-U}(z)=\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\frac{\left[\frac{2\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b}{1+\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b}\right]^{\alpha+j-1}}{\alpha+j}\times\frac{4b\left(\frac{1}{\theta}\right)\left(1-\frac{z}{\theta}\right)\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^{b-1}}{\left(1+\left(1-\left(1-\frac{z}{\theta}\right)^2\right)^b\right)^2},$$

respectively.

4.2 *Beta type two half logistic Topp-Leone exponential distribution*

The beta type two half logistic Topp-Leone exponential (BTIIHLTL-E) is a member of the BTIIHLTL-G with hrf and pdf

$$h_{BTIIHLTL-E}(z)=\frac{\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\left[\frac{2(1-e^{-2\lambda z})^b}{1+(1-e^{-2\lambda z})^b}\right]^{\alpha+j-1}\times\frac{4b\lambda e^{-2\lambda z}(1-e^{-2\lambda z})^{b-1}}{(1+(1-e^{-2\lambda z})^b)^2}}{1-\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\frac{\left[1-\frac{1-(1-e^{-2\lambda z})^b}{1+(1-e^{-2\lambda z})^b}\right]^{\alpha+j}}{\alpha+j}},$$

and

$$f_{BTIIHLTL-E}(z)=\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\left[\frac{2(1-e^{-2\lambda z})^b}{1+(1-e^{-2\lambda z})^b}\right]^{\alpha+j-1}\times\frac{4b\lambda e^{-2\lambda z}(1-e^{-2\lambda z})^{b-1}}{(1+(1-e^{-2\lambda z})^b)^2}.$$

respectively.

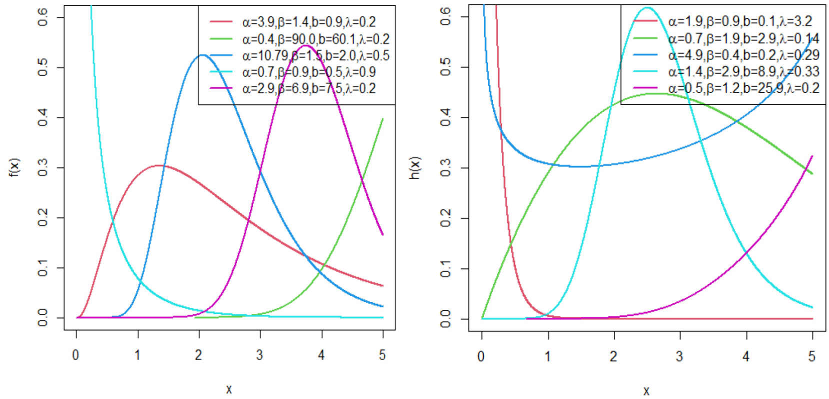

Figure 2

The pdf and hrf plots for the BTIIHLTL-E distribution

4.3 *Beta type two half logistic Topp-Leone log logistic distribution*

The beta type two half logistic Topp-Leone log logistic (BTIIHLTL-LLoG) has its hrf and pdf as

$$h_{BTIIHLTL-LLoG}(z) = \frac{\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\left[\frac{2(1-(1+z^d)^{-2})^b}{1+(1-(1+z^d)^{-2})^b}\right]^{\alpha+j-1}}{1-\{B(\alpha,\beta)\}^{-1}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\frac{\left[1-\frac{1-(1-(1+z^d)^{-2})^b}{1+(1-(1+z^d)^{-2})^b}\right]^{\alpha+j}}{\alpha+j}}$$
$$\times\frac{4b(dz^{d-1}(1+z^d)^{-2}(1-(1+z^d)^{-2})^{b-1}}{(1+(1-(1+z^d)^{-2})^b)^2},$$

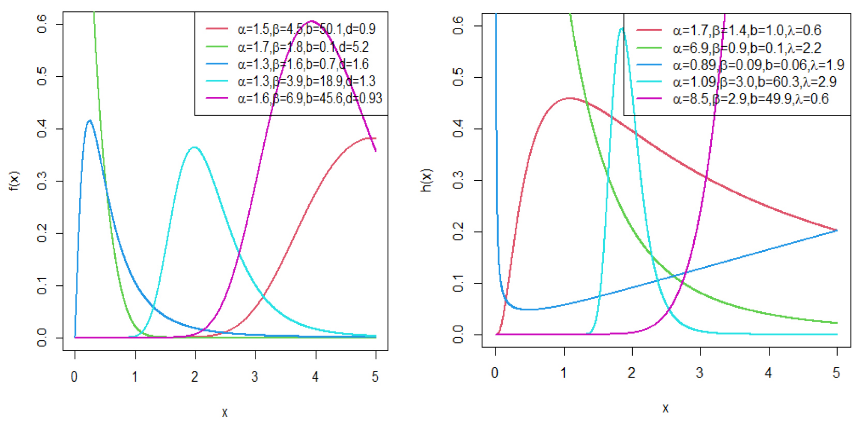

Figure 3

The pdf and hrf plots for the BTIIHLTL-LLoG distribution

and

$$f_{BTIIHLTL-LLoG}(z) = \frac{1}{B(\alpha,\beta)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} \left[\frac{2(1-(1+z^d)^{-2})^b}{1+(1-(1+z^d)^{-2})^b} \right]^{\alpha+j-1} \times \frac{4b(dz^{d-1}(1+z^d)^{-2}(1-(1+z^d)^{-2})^{b-1}}{(1+(1-(1+z^d)^{-2})^b)^2}.$$

respectively.

5. Simulation Results

To investigate the consistency property of the maximum likelihood estimators, we present a Monte Carlo (MC) simulation analysis. Six random samples, each selected 1000 times, with sizes n = 80, 100, 200, 400, 800, and 1000 from the BTIIHLTL-E distribution are taken into consideration. The simulation results are shown in Table 1. For the estimated parameter, the following formulas are used to calculate the root mean square error (RMSE) and average bias (Bias)

$$Bias(\hat{\Lambda}) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\Lambda}_i - \Lambda), \quad \text{and} \quad RMSE(\hat{\Lambda}) = \sqrt{\frac{\Sigma_{i=1}^{N} (\hat{\Lambda}_i - \Lambda)^2}{1000}},$$

respectively. In general, from Table 1, we observe that the sample mean values approximate the true parameter value. The results get better as the sample size increases with both the average bias and root mean square error decreasing towards zero.

Table 1

Results of the Monte Carlo Simulation for the BTIIHLTL-E

		$\alpha = 1.6, \beta = 0.4, \lambda = 1.3, b = 1.9$			$\alpha = 1.6, \beta = 1.3, \lambda = 0.4, b = 1.9$		
	Samplesize	**Mean**	**RMSE**	**Bias**	**Mean**	**RMSE**	**Bias**
α	80	6.2642140	14.90337870	4.66421402	3.1205368	4.4744893	1.52053679
	100	6.3064700	13.75671548	4.70646997	3.0603971	4.4947804	1.46039710
	200	4.3409803	8.43252568	2.74098028	2.5745864	3.1713529	0.97458638
	400	2.8854429	3.05864680	1.28544288	2.5492740	2.8339159	0.94927402
	800	2.1721010	1.28736477	0.57210101	2.0615333	2.2836728	0.46153333
β	80	0.7516888	1.26799206	0.35168880	3.0835651	5.6123375	1.78356512
	100	0.7879010	1.35185473	0.38790102	2.5841549	4.6060726	1.28415487
	200	0.6916808	1.05853076	0.29168081	2.0482354	3.1184255	0.74823544
	400	0.5302681	0.92291957	0.13026810	1.6048946	1.8848559	0.30489462

Contd...

	800	0.3715273	0.11857123	-0.02847273	1.4445795	1.0980904	0.14457949
λ	80	1.6397614	1.12670736	0.33976138	0.7327673	0.7728058	0.33276725
	100	1.6085294	1.10203200	0.30852945	0.7670219	0.7612674	0.36702192
	200	1.5089639	0.88596008	0.20896389	0.6204864	0.5107088	0.22048636
	400	1.6105099	0.75911915	0.31050994	0.5599121	0.3572264	0.15991208
	800	1.5511518	0.56457277	0.25115180	0.4763063	0.2247356	0.07630631
b	80	6.3285334	14.13741969	4.42853342	7.6725142	21.8641365	5.77251422
	100	4.1982740	7.83300616	2.29827397	7.4690233	19.4348939	5.56902330
	200	4.5435763	8.99200556	2.64357626	5.7799990	11.3671324	3.87999903
	400	3.2263888	4.36238150	1.32638881	4.9347211	8.5917698	3.03472113
	800	1.8893938	0.63594880	-0.01060620	3.7049965	5.1436859	1.80499651

6. Applications

Here, we apply our new model to real-world data sets. The BTIIHLTL-E is compared to the following models: Type II Exponentiated Half Logistic Weibull (TIIEHLW)[2], Beta Weibull (BW) [8], Transmuted Topp- Leone Weibull (TTLW)[10] and the Beta Generalized Lindley (BGL) [20]. We performed assessment of the model using goodness-of-fit (GoF) statistics which include; -2 log likelihood (-2 log L), Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICC), Kolmogorov-Sminorv (K-S) statistic and its p-value, the Anderson-Darling (A*) and Cramer-von Mises (W*) statistics. The model that provides the best fit to the given data is considered to be the one with the least value of the GoF statistics with a bigger p-value.

6.1 *The yarn sample*

The first data set considered, presented in Table 2, was taken from [23] where an experiment was carried out to test the breaking strength of yarn at 2.3% strain level.

Table 2

The yarn sample at 2.3% Strain Level Data

86 146 251 653 98 249 400 292 131 169 175 176 76 264 15 364 195 262 88 264 157 220 42 321 180 198 38 20 61 121 282 224 149 180 325 250 196 90 229 166 38 337 65 151 341 40 40 135 597 246 211 180 93 315 353 571 124 279 81 186 497 182 423 185 229 400 338 290 398 71 246 185 188 568 55 55 61 244 20 284 393 396 203 829 239 236 286 194 277 143 198 264 105 203 124 137 135 350 193 188

Table 3

MLE's and GOF statistics for the yarn sample at 2.3% strain level !

	Estimates			Statistics								
Model	α	β	λ	b	$-2 \log L$	AIC	AICC	BIC	W^*	A^*	KS	P–value
BTIIHLTL-E	0.30168 (0.05403)	1.90987 (0.80245)	0.002055 (0.0004369)	6.28726 (0.15126)	1248.623	1256.623	1257.044	1267.043	0.0785288	0.4421	0.080008	0.544
TIIEHLW	a 1.8361×10^{3} 3.4250×10^{-7}	λ 2.0530×10^{1} 9.2446×10^{-4}	δ 5.0765×10^{-1} 3.1303×10^{-2}	γ 1.4238×10^{-1} 1.0735×10^{-2}	1249.801	1257.801	1258.222	1268.221	0.099084	0.5538403	0.08365	0.4861
BW	a 1.362321 (0.5727940)	B 0.97612 (0.083167)	α 0.0049185 (0.0016294)	β 1.3643917 (0.3115990)	1249.825	1257.825	1258.246	1268.246	0.0959686	0.5397216	0.08231	0.507
TTLW	a 0.0029129 (-)	b 1.3625192 (0.1416454)	α 1.36456 (0.50312)	λ 0 (-)	1249.825	1257.825	1258.246	1268.246	0.090643	0.518964	0.19002	0.001461
BGL	α 1.0049×10^{-1} (4.1620×10^{-3})	θ 5.3027×10^{-9} (5.5710×10^{-7})	a 3.0009×10^{-4} $2.1014 \times 10^{-4})$	c 1.0100×10^{1} (1.2597×10^{-5})	1717.726	1725.882	1726.303	1736.303	0.4053357	2.213041	0.49964	2.2×10^{-1}

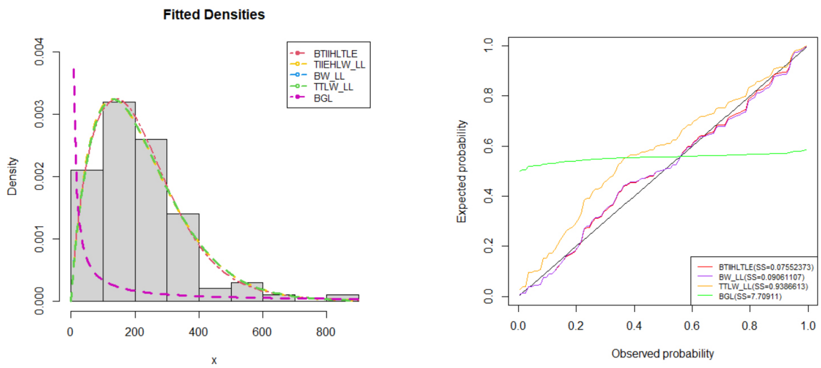

Figure 4

Probability and density plots for the Yarn Data

Out of all the models this research has looked at, the BTIIHLTL-E model offers the best fit for the yarn data. The results are presented in Table 3.

Figure 4's probability plots and fitted densities further demonstrate how well the BTIIHLTL-E model fits the the Yarn at 2.3% Level Strain data set in comparison to the other models that were considered.

For the BTIIHLTL-E model on Yarn at 2.3% Level Strain data, the estimated variance-covariance matrix is given by:

$$\begin{bmatrix} 2.920010\times10^{-3} & -0.026919578 & 1.899832\times10^{-5} & -4.998344\times10^{-3} \\ -2.691958\times10^{-2} & 0.6439287802 & -3.15984\times10^{-4} & 1.213777\times10^{-1} \\ 1.899832\times10^{-5} & -0.000315984 & 1.90954\times10^{-7} & -5.931686\times10^{-5} \\ -4.998344\times10^{-3} & 0.121377749 & -5.931686\times10^{-5} & 2.288238\times10^{-2} \end{bmatrix}.$$

Consequently, the model parameters' 95% confidence intervals are provided by:

$$\alpha \in [0.30168084 \pm 0.1059127439],\ \beta \in [1.90987965 \pm 1.5728053919],$$
$$\lambda \in [0.00205500 \pm 0.0008564864],\ b \in [6.28726198\times10^{1} \pm 0.2964876994].$$

6.2 *The monthly actual taxes revenue sample*

The second piece of data that we employed was the monthly actual taxes revenue from Egypt. The data is from January 2006 to November 2010 and was taken from Mead [16]. The data set is given in Table 4.

Table 4

Monthly actual taxes revenue in 1000 million Egyptian pounds

5.9 20.4 14.9 16.2 17.2 7.8 6.1 9.2 10.2 9.6
13.3 8.5 21.6 18.5 5.1 6.7 17 8.6 9.7 39.2
35.7 15.7 9.7 10 4.1 36 8.5 8 9.2 26.2 21.9
16.7 21.3 35.4 14.3 8.5 10.6 19.1 20.5 7.1
7.7 18.1 16.5 11.9 7 8.6 12.5 10.3 11.2 6.1
8.4 11 11.6 11.9 5.2 6.8 8.9 7.1 10.8

Out of all the models this work has considered, the BTIIHLTL-E model offers the best fit for the Egyptian monthly taxes sample (Table 5).

The BTIIHLTL-E model's estimated variance-covariance matrix is provided by:

$$\begin{bmatrix} 2.5705\times10^{-2} & -7.6398\times10^{-3} & 7.7850\times10^{-3} & 5.6948\times10^{-5} \\ -7.6398\times10^{-3} & 5.4858\times10^{-3} & -2.8187\times10^{-3} & -1.4001\times10^{-5} \\ 7.7850\times10^{-3} & -2.8187\times10^{-3} & 2.5623\times10^{-3} & 1.6754\times10^{-5} \\ 5.6948\times10^{-5} & -1.4001\times10^{-5} & 1.6754\times10^{-5} & 1.2883\times10^{-7} \end{bmatrix}.$$

As such, the model parameters' 95% confidence intervals are provided by:

$$\alpha \in [2.8040\times10^{-1} \pm 0.3142],\ \beta \in [2.9689\times10^{-1} \pm 0.1451],$$
$$\lambda \in [1.9223\times10^{-1} \pm 0.0992],\ \mathrm{b} \in [7.6388\times10^{1} \pm 0.0007].$$

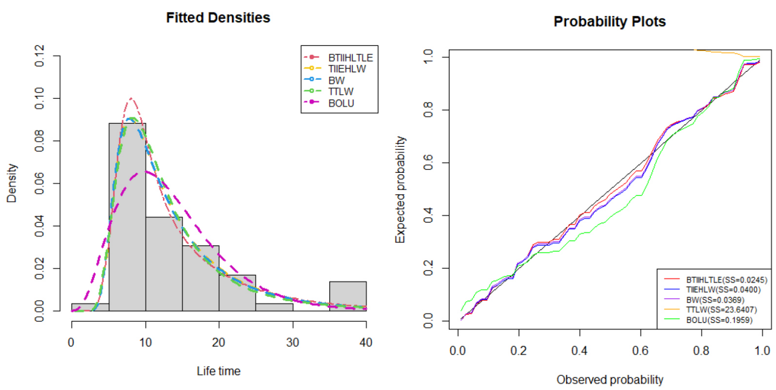

Figure 5

Fitted Densities and Probability Plots for the monthly actual taxes revenue Data

Table 5

MLE's and GOF statistics for the monthly actual taxes revenue

	Estimates				Statistics							
Model	**α**	**β**	**λ**	**b**	**−2 log *L***	**AIC**	**AICC**	**BIC**	***W******	***A******	**KS**	**P−value**
BTIIHLTL-E	2.8040×10^{-1} (1.6033×10^{-1})	2.9689×10^{-1} (7.4067×10^{-2})	1.9223×10^{-1} (5.0620×10^{-2})	7.6389×10^{1} (3.5892×10^{-2})	374.9	382.9	383.7	391.3	0.0260	0.1916	0.0550	0.9941
TIIEHLW	a 0.3626 (0.2741)	λ 27.4562 (32.9144)	δ 0.4956 (0.4548)	γ 0.9159 (0.3553)	375.6	383.7	384.5	391.9	0.0385	0.2500	0.0654	0.9625
BW	a 90.7890 (0.0042)	b 0.2360 (0.2298)	α 1.0548 (0.5718)	β 0.8346 (0.3353)	375.7	383.7	384.5	392.0	0.0378	0.2463	0.0612	0.9801
TTLW	a 8.7261×10^{00} (4.0828×10^{00})	b 2.9503×10^{-1} (3.7468×10^{-2})	α 1.6253×10^{3} (2.1944×10^{-2})	λ 7.7947×10^{-2} (6.2240×10^{-1})	376.5	384.5	385.2	392.8	-	-	1.0745	$<2.2 \times 10^{-16}$
BOLU	a 3.8800 (6.9603×10^{-1})	b 3.4012 (6.0467×10^{-1})	λ 3.2368×10^{5} (6.8236×10^{-1})	θ 5.1664×10^{6} (4.2752×10^{-7})	385.5	393.5	394.2	401.8	0.1862	1.1507	0.1321	0.2547

From the results shown in Table 5, we observe that the BTIIHLTL-E model outperforms the other models as it has the least value of goodness of fit statistics with a larger value of the K-S p-value. Also by observing plots of the fitted densities and probability in Figure 5, we arrive at the same conclusion.

7. Concluding Remarks

A brand-new distribution class was created, named the beta type two half logistic Topp-Leone-G. We presented three examples of our new distribution. We obtained mathematical properties of our new distribution such as maximum likelihood estimates, moments, entropy, and order statistics. To evaluate the consistency of the estimators, a simulation was performed. Two examples with actual data were given to show how the new family of distributions may be beneficial. Based on the two real-data sets used, our new model outperforms all the models this article has considered.

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