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NBINAR(1) process defined with Poisson-weighted exponential innovations

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Abstract

In this paper, we present a new integer-valued time series model based on Poisson-weighted exponential innovations and a negative binomial thinning operator. One of its notable characteristics is its ability to handle over-dispersed integer-valued time series. Two different approaches are employed for estimating the related parameters. Furthermore, simulation tests are carried out to demonstrate the accuracy of these approaches. Finally, we compare our model with fair competitors using some count time series data and show that it is statistically superior.

Subject Classification: *62M10, 62M05.*

Keywords: *Poisson weighted exponential distribution, Over-dispersion, Simulation, Negative binomial thinning, INAR(1) process.*

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1. Introduction

Discrete (probability) distributions are fundamental concepts in many fields, including statistics, engineering, economics and computer science. They are designed to describe phenomena that have a finite number or countable values. Mathematically, they are based on the distribution of a random variable (RV) and are defined by a probability mass function (pmf). A famous statistical model for analyzing count data with autocorrelation is the first-order integer-valued autoregressive (INAR(1)) process. In this case, it is assumed that the data are derived from a discrete distribution and that a given count is dependent on previous counts. The INAR(1) processes have been shown to model count data more accurately than standard models. Among these are the Poisson and negative binomial (NB) models. The basic INAR(1) process, which combines Poisson innovations with the binomial thinning operator (TO), was introduced by Al-Osh and Alzaid [2]. Since then, researchers have become increasingly interested in INAR(1) processes, resulting in studies by Aghababaei Jazi, Jones and Lai [1] Lívio et al. [9] and Maya et al. [10].

Through the introduction of TOs and the development of new INAR processes, the scope of the INAR(1) processes has been extended and revised. Exploiting the structure of the NB TO, Ristić and Nastić [14] showed that it can be more efficient than the binomial TO in modelling some practical data. In particular, it is known that Bernoulli counting sequences can be used to model the persistence or disappearance of random events over time. Aly and Bouzar [7] and Ristić et al. [16] introduced the NB TO to adapt to this situation. Also, Ristić et al. [16] presented a stationary INAR(1) process with a geometric marginal distribution using the NB TO algorithm. With this approach, count dependence in a time series can be captured, including the over-dispersion and other characteristics associated with count data. This solution addresses issues like over-dispersion and skewness, enhancing the performance of INAR(1) processes.

The exponential distribution's versatility, with extensions like the weighted exponential distribution by Oguntunde et al. [12], the truncated exponentiated-exponential distribution by Ribeiro-Reis [13], and the alpha power generalized exponential distribution by Eghwerido, Nzei and Zelibe [8], underscores the availability of diverse models for compounding. Consequently, Altun [3] proposed a new Poisson-weighted exponential (PWE) distribution and the corresponding INAR(1) process. This paper follows on from Altun [3] and proposes a new INAR(1) process that benefits from the NB TO. We then aim to compare its performance with that of an INAR(1) process using the binomial TO and PWE innovations. We also plan to compare it with other competitors, such as the NBINAR(1)-CP process introduced by Mohammadi et al. [11], and other INAR(1) processes using well-known discrete innovations.

The remainder of the paper is as follows: The PWE distribution and its important properties are provided in Section 2. The NBINAR(1) process with PWE innovations is presented in Section 3 along with a description of its properties. Section 4 deals with the conditional maximum likelihood (CML) and conditional least squares (CLS) estimations, followed by a simulation study in Section 5. A comparison of the proposed NBINAR(1) process with competing processes is presented in Section 6.

2. The PWE distribution

To begin, some important functions that define the PWE distribution must be recalled. Let us consider a RV Z, defined on a probability space with a probability operator P, that follows the PWE distribution. Then the corresponding pmf is given by

$$P(Z=x)=\lambda(1+\beta)(1+\lambda+\lambda\beta)^{-x-1}, \qquad \lambda>0, \beta>0, x=0,1,2,\ldots \tag{2.1}$$

The corresponding cumulative distribution function is indicated as

$$F(x)=P(Z\leq x)=1-(1+\lambda+\lambda\beta)^{-x-1}. \tag{2.2}$$

In addition, the mean of Z is

$$E(Z)=\frac{1}{\lambda(\beta+1)}, \tag{2.3}$$

the variance of Z is

$$V(Z)=E(Z^2)-[E(Z)]^2=\frac{(\beta+1)\lambda+1}{\lambda^2(\beta+1)^2} \tag{2.4}$$

and the dispersion index (DI) of Z is

$$DI(Z)=\frac{V(Z)}{E(Z)}=1+\frac{1}{\lambda(\beta+1)}. \tag{2.5}$$

Figure 1 depicts this DI for various values of β and λ. From this figure, we can see that $DI(Z)\geq 1$. This means that the PWE distribution facilitates the analysis of over-dispersed datasets. This is also a motivation to use it in an integer-valued time series context, as performed in the next section.

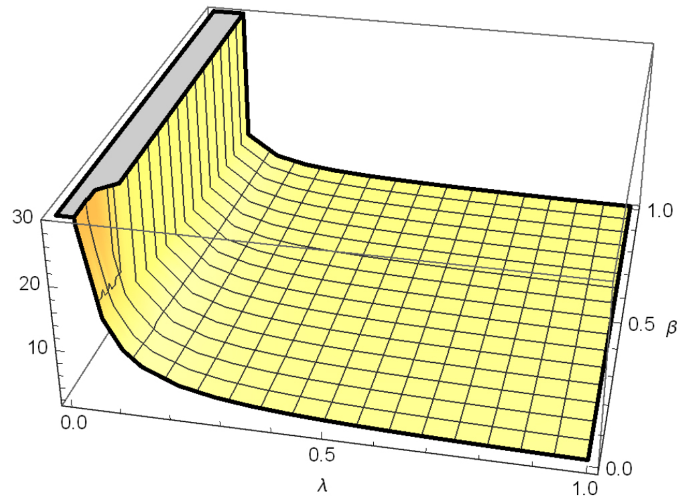

Figure 1

DI associated with the PWE distribution

3. New Process

The proposed INAR(1) process is developed using the NB TO and innovations based on the PWE distribution. It is logically referred to as the NBINAR(1)-PWE process.

3.1 *NBINAR(1)-PWE Process*

Formally, the NBINAR(1)-PWE process is defined as $(Z_t)_t$, where

$$Z_t = \alpha \circledast Z_{t-1} + \xi_t,$$

with $t \geq 1$, where "$\circledast$" represents the NBl TO, and $(\xi_t)_t$ is a process with independent and identically distributed (iid) RVs drawn from the PWE distribution. It is assumed that, for any $t \geq 1$, ξ_t is independent of $(G_j)_j$ (to be defined below) and Z_s for all $s \leq t$. The expression $\alpha \circledast Z$ includes independent geometrically distributed RVs in the following summation manner:

$$\alpha \circledast Z = \sum_{j=1}^{Z} G_j,$$

where Z denotes a non-negative integer-valued RV and $(G_j)_j$ is a sequence of iid geometric RVs with the common pmf

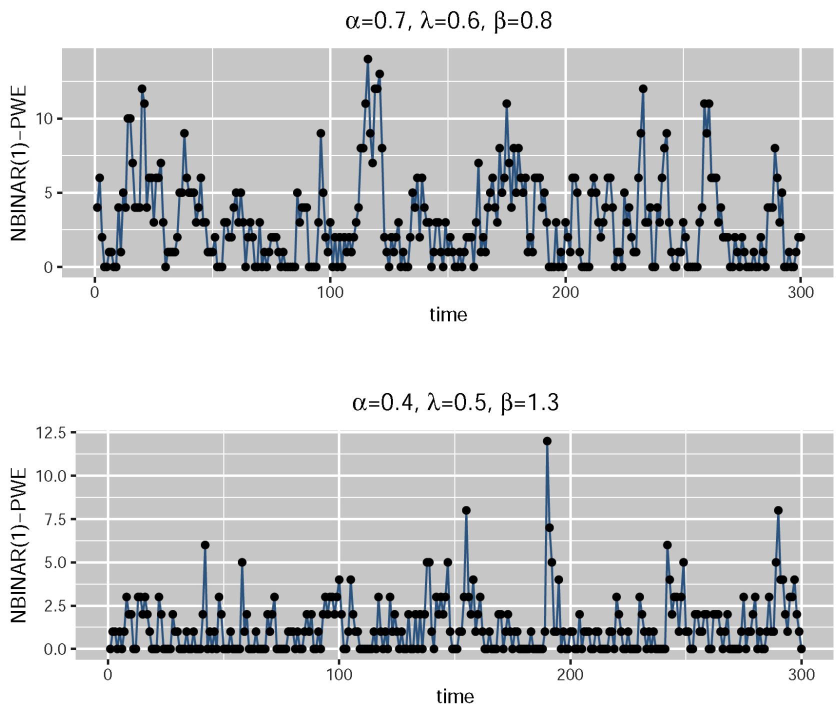

Figure 2

Some simulated paths associated with the NBINAR(1)-PWE process

$$P(G_1 = g) = \frac{\alpha^g}{(1+\alpha)^{g+1}}, \quad g = 0,1,2,\ldots, \quad \alpha \in (0,1),$$

also independent of Z. For a visual illustration, Figure 2 displays the simulated sample of 300 pathways of the NBINAR(1)-PWE process by taking different values of the parameters. The mean, variance and DI associated with the NBINAR(1)-PWE process are, respectively, given by

$$E(Z_t) = \frac{\mu_\xi}{1-\alpha}, \tag{3.1}$$

$$V(Z_t) = \frac{\alpha(1+\alpha)\mu_\xi + (1-\alpha)\sigma_\xi^2}{(1-\alpha)^2(1+\alpha)} \tag{3.2}$$

and

$$DI(Z_t) = \frac{\alpha(1+\alpha)+(1-\alpha)DI_\xi}{1-\alpha^2}, \tag{3.3}$$

where μ_ξ is the mean of ξ_1, and σ_ξ^2 is the corresponding variance. Mohammadi et al. [11] for more details. In our precise setting, we have $\mu_\xi = \frac{1}{(\beta+1)\lambda}$ and $\sigma_\xi^2 = \frac{(\beta+1)\lambda+1}{(\beta+1)^2\lambda^2}$. Hence, Equations (3.1), (3.2) and (3.3) give

$$E(Z_t) = \frac{1}{(\beta+1)\lambda(1-\alpha)}, \tag{3.4}$$

$$V(Z_t) = \frac{(\beta+1)\lambda(\alpha^2+1)-\alpha+1}{(\beta+1)^2\lambda^2(\alpha-1)^2(\alpha+1)} \tag{3.5}$$

and

$$DI(Z_t) = \frac{\alpha^2+\frac{1-\alpha}{\beta\lambda+\lambda}+1}{1-\alpha^2}, \tag{3.6}$$

respectively.

The DI associated with the NBINAR(1)-PWE process is shown in Table 1 for various value combinations of the parameters.

Table 1

DI associated with the NBINAR(1)-PWE process.

		α = 0.20							α = 0.40				
		λ							λ				
		0.01	0.23	0.47	0.87	0.97			0.01	0.23	0.47	0.87	0.97
β	0.10	76.8409	4.3771	2.6952	1.9541	1.8643	β	0.10	66.3160	4.2042	2.7625	2.1273	2.0504
	0.50	56.6389	3.4988	2.2654	1.7219	1.6561		0.50	49.0000	3.4513	2.3941	1.9283	1.8719
	0.75	48.7024	3.1537	2.0965	1.6307	1.5743		0.75	42.1973	3.1556	2.2494	1.8501	1.8017
	0.86	45.8862	3.0313	2.0366	1.5983	1.5452		0.86	39.7834	3.0506	2.1980	1.8224	1.7769
	0.98	43.1709	2.9132	1.9788	1.5671	1.5172		0.98	37.4560	2.9494	2.1485	1.7956	1.7529
	1.50	34.4167	2.5326	1.7926	1.4665	1.4270		1.50	29.9524	2.6232	1.9889	1.7094	1.6755
	2.50	24.8929	2.1185	1.5899	1.3570	1.3288		2.50	21.7891	2.2683	1.8152	1.6155	1.5913
	3.70	18.8138	1.8542	1.4606	1.2871	1.2661		3.70	16.5785	2.0417	1.7043	1.5556	1.5376
	5.00	14.9722	1.6872	1.3788	1.2430	1.2265		5.00	13.2857	1.8986	1.6342	1.5178	1.5037
	9.25	9.2134	1.4368	1.2563	1.1768	1.1671		9.25	8.3496	1.6839	1.5292	1.4611	1.4528
		α = 0.60							α = 0.97				
		λ							λ				
		0.01	0.23	0.47	0.87	0.97			0.01	0.23	0.47	0.87	0.97

Contd...

	0.10	58.9432	4.5954	3.3339	2.7781	2.7108		0.10	78.9877	34.8473	33.8228	33.3714	33.3167
	0.50	43.7917	3.9366	3.0115	2.6039	2.5546		0.50	66.6819	34.3123	33.5610	33.2299	33.1898
	0.75	37.8393	3.6778	2.8849	2.5355	2.4932		0.75	61.8475	34.1021	33.4581	33.1744	33.1400
	0.86	35.7272	3.5860	2.8399	2.5112	2.4714		0.86	60.1320	34.0275	33.4216	33.1546	33.1223
	0.98	33.6907	3.4974	2.7966	2.4878	2.4504		0.98	58.4780	33.9556	33.3864	33.1356	33.1052
β	1.50	27.1250	3.2120	2.6569	2.4124	2.3827	β	1.50	53.1455	33.7238	33.2730	33.0743	33.0503
	2.50	19.9821	2.9014	2.5049	2.3303	2.3091		2.50	47.3442	33.4715	33.1495	33.0077	32.9905
	3.70	15.4229	2.7032	2.4079	2.2778	2.2621		3.70	43.6412	33.3105	33.0707	32.9651	32.9523
	5.00	12.5417	2.5779	2.3466	2.2447	2.2324		5.00	41.3012	33.2088	33.0210	32.9382	32.9282
	9.25	8.2226	2.3901	2.2547	2.1951	2.1879		9.25	37.7933	33.0563	32.9463	32.8979	32.8920

Every parameter combination has a DI greater than 1, which makes it easier to model datasets that are over-dispersed. The standard transition probability of the NBINAR(1)-PWE process is expressed as

$$P(Z_t = j \mid Z_{t-1} = 0) = P(\xi_t = j) = \lambda(1+\beta)(1+\lambda+\lambda\beta)^{-j-1} \tag{3.7}$$

and, for $i \neq 0$,

$$\begin{aligned} P(Z_t = j \mid Z_{t-1} = i) &= P(\alpha \circledast Z_{t-1} + \xi_t = j \mid Z_{t-1} = i) \\ &= \sum_{k=0}^{j} \binom{i+j-k-1}{i-1} \frac{\alpha^{j-k}}{(1+\alpha)^{i+j-k}} \lambda(1+\beta)(1+\lambda+\lambda\beta)^{-k-1}. \end{aligned} \tag{3.8}$$

Calculating the standard conditional mean and variance of the NBINAR(1)-PWE process can be accomplished using the structure of the NB TO. They are, respectively, given by

$$E(Z_t \mid Z_{t-1}) = \alpha Z_{t-1} + \frac{1}{\lambda(\beta+1)} \tag{3.9}$$

and

$$V(Z_t \mid Z_{t-1}) = \alpha(1+\alpha)Z_{t-1} + \frac{(\beta+1)\lambda+1}{(\lambda^2\beta+1)^2}. \tag{3.10}$$

Furthermore, for any positive integer h, we establish that

$$E(Z_{t+h} \mid Z_t) = \alpha^h Z_t + \frac{1-\alpha^h}{1-\alpha}\mu_\xi$$

and

$$V(Z_{t+h} \mid Z_t) = \frac{\alpha^h(1+\alpha)(1-\alpha^h)}{1-\alpha} Z_t + \left(\frac{1-\alpha^{2h}}{1-\alpha^2} - \frac{1-2\alpha^h+\alpha^{2h}}{(1-\alpha)^2} \right) \mu_\xi^2$$
$$+ \frac{1-\alpha^{2h}}{1-\alpha^2} \sigma_\xi^2 + \frac{\alpha(1+\alpha+2\mu\xi)}{1-\alpha} \left(\frac{1-\alpha^{2h}}{1-\alpha^2} - \alpha^{h-1} \frac{1-\alpha^h}{1-\alpha} \right) \mu_\xi.$$

The corresponding autocovariance and autocorrelation functions are given by $\gamma_h = \alpha^h \gamma_0$ and $\rho_h = \alpha^h$, respectively.

4. Estimation

4.1 *CML estimation*

Transition probabilities are sufficient to formulate the likelihood (L) function for the CML technique. Let T be a positive integer, $Z_2, Z_3, \ldots, Z_T$ be RVs drawn from the NBINAR(1)-PWE process, and $z_2, z_3, \ldots, z_T$ be some observations of them. Then the conditional log-L function is given by

$$l(\alpha, \lambda, \beta) = \sum_{t=2}^{T} \ln[P(Z_t = z_t \mid Z_{t-1} = z_{t-1})],$$

where Z_1 is supposed to be fixed and $P(Z_t = z_t \mid Z_{t-1} = z_{t-1})$ is given by Equations (3.7) and (3.8), respectively, depending on the value of z_{t-1}. The CML estimates (CMLEs) of α, λ and β are obtained by maximizing $l(\alpha, \lambda, \beta)$. We will denote them as $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\beta}$, respectively. In practice, we use the R software, and the *optim* function in particular.

4.2 *CLS estimation*

Basically, by minimizing the following equation, we get the CLS estimates (CLSEs) of α, λ and β:

$$Q(\alpha, \lambda, \beta) = \sum_{t=2}^{T} (z_t - E(Z_t \mid Z_{t-1} = z_{t-1}))^2. \tag{4.1}$$

5. Simulation

In this section, a simulation study under the NBINAR(1)-PWE process demonstrates the acceptable performance of the CMLEs and CLSEs. For different parameter combinations, 1,001 replications of the simulation were used for samples of sizes 50, 100, 200, 300, and 500. Tables 2 and 3 collect the biases and mean square errors (MSEs) associated with the estimates. We observe that they are decreasing as the sample size increases. So both the CML and CLS methods provide consistent estimates.

Table 2

NBINAR(1)-PWE process simulation results.

$\alpha = 0.45, \lambda = 0.75, \beta = 0.92$							
Parameter	***T***	**CML**			**CLS**		
		CMLEs	**Bias**	**MSE**	**CLSEs**	**Bias**	**MSE**
α	50	0.4007	0.1317	0.0272	0.3705	0.1426	0.0323
	100	0.4237	0.0898	0.0125	0.4061	0.0971	0.0148
	200	0.4365	0.0604	0.0058	0.4278	0.0671	0.0072
	300	0.4429	0.0511	0.0041	0.4382	0.0551	0.0048
	500	0.4462	0.0379	0.0023	0.4431	0.0425	0.0029
λ	50	0.7899	0.1570	0.0483	0.7595	0.1559	0.0481
	100	0.7672	0.1043	0.0195	0.7492	0.1101	0.0202
	200	0.7572	0.0729	0.0086	0.7485	0.0779	0.0100
	300	0.7651	0.0603	0.0060	0.7599	0.0646	0.0072
	500	0.7554	0.0478	0.0036	0.7519	0.0513	0.0042
β	50	0.8908	0.0934	0.0159	0.8722	0.0983	0.0182
	100	0.8959	0.0694	0.0084	0.8889	0.0744	0.0101
	200	0.8991	0.0508	0.0046	0.8975	0.0532	0.0053
	300	0.9092	0.0393	0.0028	0.9101	0.0416	0.0032
	500	0.9063	0.0340	0.0021	0.9071	0.0357	0.0023
$\alpha = 0.75, \lambda = 1.5, \beta = 0.4$							
Parameter	***T***	**CML**			**CLS**		
		CMLEs	**Bias**	**MSE**	**CLSEs**	**Bias**	**MSE**
α	50	0.6686	0.1329	0.0321	0.6183	0.1594	0.0431
	100	0.7029	0.0919	0.0141	0.6678	0.1126	0.0209
	200	0.7276	0.0580	0.0057	0.7072	0.0735	0.0090
	300	0.7368	0.0460	0.0035	0.7229	0.0577	0.0055
	500	0.7434	0.0345	0.0019	0.7329	0.0433	0.0030
λ	50	1.4142	0.2788	0.1289	1.3119	0.3042	0.1479
	100	1.4537	0.1891	0.0579	1.3899	0.2374	0.0893
	200	1.4620	0.1385	0.0315	1.4298	0.1844	0.0563
	300	1.4767	0.1059	0.0189	1.4575	0.1510	0.0362
	500	1.4826	0.0827	0.0114	1.4667	0.1214	0.0253
β	50	0.4358	0.1708	0.0604	0.3257	0.1810	0.0536
	100	0.4337	0.1195	0.0265	0.3556	0.1505	0.0356
	200	0.4243	0.0827	0.0130	0.3785	0.1151	0.0249
	300	0.4226	0.0654	0.0081	0.3882	0.0955	0.0153
	500	0.4184	0.0504	0.0047	0.3912	0.0732	0.0096

Table 3

NBINAR(1)-PWE process simulation results (continue).

$\alpha = 0.81, \lambda = 1.4, \beta = 2.3$							
Parameter	T	**CML**			**CLS**		
		CMLEs	**Bias**	**MSE**	**CLSEs**	**Bias**	**MSE**
α	50	0.6696	0.1873	0.0655	0.5980	0.2292	0.0862
	100	0.7315	0.1208	0.0283	0.6794	0.1498	0.0403
	200	0.7718	0.0778	0.0110	0.7339	0.0977	0.0167
	300	0.7860	0.0589	0.0063	0.7571	0.0749	0.0102
	500	0.7978	0.0432	0.0032	0.7784	0.0565	0.0054
λ	50	1.4752	0.4354	0.4178	1.2130	0.4627	0.3662
	100	1.4561	0.2914	0.1606	1.2504	0.3549	0.1895
	200	1.4495	0.1961	0.0702	1.2892	0.2781	0.1167
	300	1.4205	0.1489	0.0370	1.3013	0.2430	0.0861
	500	1.4248	0.1156	0.0223	1.3429	0.1953	0.0589
β	50	2.2942	0.2164	0.1062	2.1126	0.2984	0.1986
	100	2.2729	0.1623	0.0530	2.1493	0.2358	0.1090
	200	2.2731	0.1232	0.0311	2.1858	0.2008	0.0814
	300	2.2582	0.1118	0.0247	2.1858	0.1887	0.0663
	500	2.2697	0.0885	0.0158	2.2327	0.1568	0.0471
$\alpha = 0.94, \lambda = 0.98, \beta = 1.1$							
Parameter	T	**CML**			**CLS**		
		CMLEs	**Bias**	**MSE**	**CLSEs**	**Bias**	**MSE**
α	50	0.8320	0.1239	0.0319	0.7782	0.1678	0.0478
	100	0.8878	0.0667	0.0090	0.8474	0.0974	0.0167
	200	0.9141	0.0391	0.0031	0.8913	0.0552	0.0055
	300	0.9245	0.0271	0.0014	0.9056	0.0403	0.0029
	500	0.9302	0.0206	0.0008	0.9163	0.0309	0.0017
λ	50	0.9312	0.2904	0.1421	0.7169	0.3603	0.1749
	100	0.9586	0.2211	0.0828	0.7383	0.3150	0.1509
	200	0.9613	0.1480	0.0353	0.8029	0.2457	0.0833
	300	0.9706	0.1145	0.0212	0.8297	0.2129	0.0625
	500	0.9774	0.0900	0.0131	0.8703	0.1885	0.0508
β	50	1.0111	0.1951	0.0725	0.8760	0.2585	0.1084
	100	1.0338	0.1533	0.0433	0.9072	0.2228	0.0810
	200	1.0502	0.1088	0.0214	0.9653	0.1735	0.0494
	300	1.0645	0.0841	0.0127	0.9889	0.1501	0.0361
	500	1.0748	0.0657	0.0079	1.0191	0.1304	0.0270

6. Data Analysis

A comparison is now carried out to corroborate our results. More precisely, the INAR(1) processes with binomial TOs are compared with some NBINAR(1) processes with known discrete innovations. The competing models are listed below with their references.

- INAR(1)-PWE process, see Altun [3].
- NBINAR(1)-CP process, see Mohammadi et al. [11].
- INAR(1)-P process, see Al-Osh and Alzaid [2].
- INAR(1)-G process, see Aghababaei Jazi, Jones and Lai [1]
- INAR(1)-PL process, see Lívio et al. [9].
- INAR(1)-PEE process, see Maya et al. [10].
- INAR(1)-PTE process, see Altun and Khan [4].
- INAR(1)-DBL process, see Altun et al. [6].
- INAR(1)-PQZ process, see Altun et al. [5].

As initial values, the CLSEs are used to obtain the CMLEs of the parameters. For all the models discussed above, the standard errors (SEs) of the series are calculated along with the maximized log-L (ln L), and two standard information criteria (IC): the Akaike IC (AIC) and Bayesian IC (BIC).

The best fit to the data comes from a model with a minimum AIC and BIC.

Following the method in Ristic and Popovic [15], for a given dataset, the standardized Pearson residuals are calculated to check the accuracy of the NBINAR(1)-PWE process. They are obtained by the following formula:

$$w_t = \frac{z_t - E(Z_t|Z_{t-1}=z_{t-1})}{V(Z_t|Z_{t-1}=z_{t-1})^{\frac{1}{2}}}. \tag{6.1}$$

Fitted models are appropriate if the corresponding mean and variance are close to 0 and 1, respectively. Using the residuals of the autocorrelation function (ACF), we can determine if they are uncorrelated.

Downloads data

In this section we consider the time series dataset presented by Weiβ [17], which expresses the daily number of downloads of a certain TEX editor from June 2006 to February 2007. This dataset has a mean of 2.4007 and a variance of 7.5343. It is clear that there is an over-dispersion, since the DI is 3.1383.

For this downloads dataset, Figure 3 shows time series, barplots, ACF, and partial ACF plots. The first lag is significant in the PACF plot, while the exponential decay is observed in the ACF plot, making the data suitable to be modeled by INAR(1) processes.

For different processes, Table 4 shows the CMLEs of the parameters, with their SEs, AICs, BICs, theoretical means (μ_x) and variances (σ_x^2). The NBINAR(1)-PWE process provides a better fit than competing processes based on the ln L value and the minimum AIC and BIC values. In addition, the NBINAR(1)-PWE process has a close match between its theoretical mean and variance. It is concluded that the NBINAR(1)-PWE process can effectively explain the characteristics of the dataset.

The Pearson residuals are considered in Figure 4. According to this figure, they do not have autocorrelation based on their ACF. This is supported by the Ljung box test with 10 degrees of freedom. Indeed, it gives a p value of 0.4111 (so greater than 0.05). This confirms that there is no correlation between them, as suggested by the graph. In addition to that, the Pearson residuals have a mean and variance of -0.0037 and 1.0818, which are close to 0 and 1, respectively. As a result, we conclude that the NBINAR(1)-PWE process fits the download data quite well and is accurate.

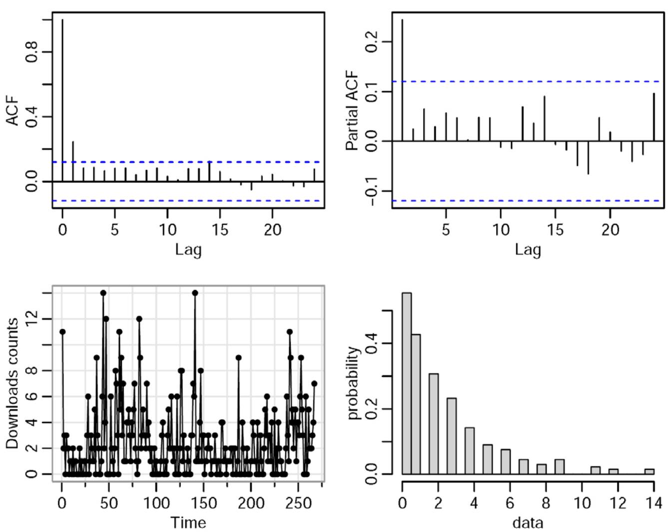

Figure 3

Plots for the downloads dataset.

Table 4

Downloads data analysis results.

Model	CMLEs / SE $\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	ln L	AIC	BIC	μ_x / σ_x^2
NBINAR(1)-PWE	0.1790	0.4845	0.0629	-537.2395	1080.4790	1091.2408	2.3651
	0.0519	11.5646	25.3717				9.2998
INAR(1)-PWE	0.1790	0.4845	0.0629	-538.2830	1082.5661	1093.3278	2.3408
	0.0519	11.5646	25.3717				3.3212
NBINAR(1)-CP	0.4490	2.1658	1.5113	-588.3632	1182.7263	1193.4881	2.3562
	0.0464	0.0119	0.0890				4.4343
INAR(1)-CP	0.2582	1.9372	1.8416	-646.7558	1299.5116	1310.2733	2.3631
	0.0302	0.0070	0.0918				2.7975
INAR(1)-PTE	0.2652	0.4303	0.1473	-538.0072	1082.0140	1092.7760	2.3653
	0.3919	0.0998	0.0393				2.3653
INAR(1)-PL	0.1179	0.7554		-541.0565	1086.1130	1093.2875	2.3557
	0.0527	0.0400					5.5807
INAR(1)-P	0.1718	1.9589		-634.1096	1272.2193	1279.3938	2.3653
	0.1096	0.0323					2.3653
INAR(1)-PEE	0.1384	0.4911	0.0005	-538.2830	1082.5661	1093.3278	2.3657
	0.0383	0.1726	0.1680				6.6014
INAR(1)-DBL	0.7337	0.0713		-562.5290	1129.0579	1136.2324	2.3221
	0.0129	0.0406					4.5801
INAR(1)-PQZ	4.6456	0.6697	0.1452	-537.8765	1081.7530	1092.5148	2.1573
	5.3627	0.1705	0.0380				6.3285

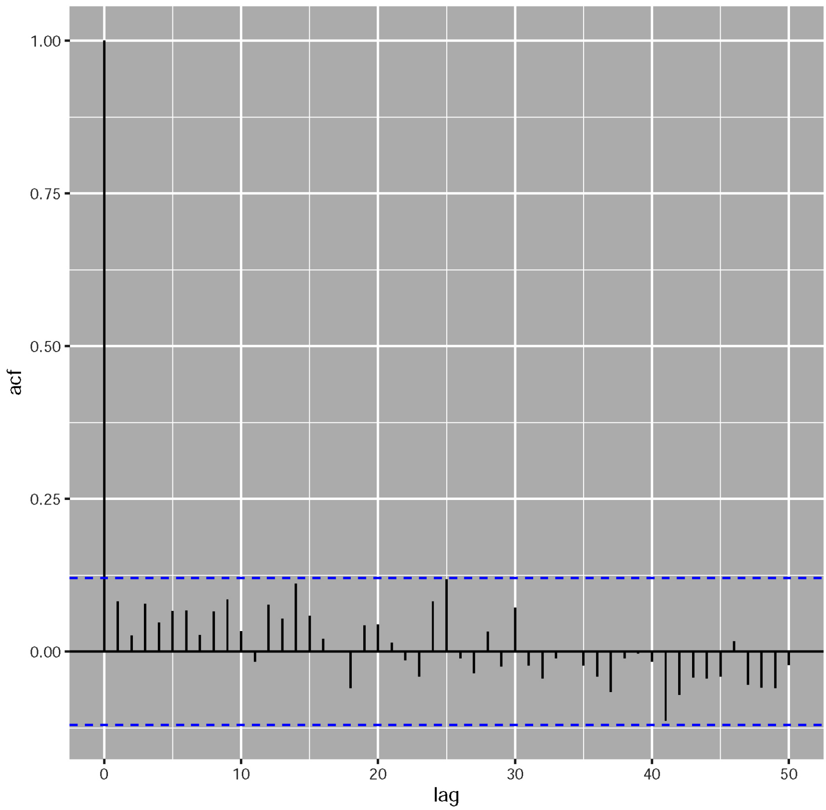

Figure 4

Plot of the ACF based on the Pearson residuals.

The NBINAR(1)-PWE process for the downloads dataset is given by

$$Z_t = 0.1790 Z_{t-1} + \xi_t, \tag{6.2}$$

where ξ_t follows the PWE distribution with estimated parameters $\hat{\lambda} = 0.4845$ and $\hat{\beta} = 0.0629$. Predicted values can be

$$z_1 = E(Z_t)_{\hat{\theta}_{CML}},$$

and

$$z_t = E(Z_t \mid Z_{t-1} = z_{t-1})_{\hat{\theta}_{CML}}, t = 2, 3, \ldots, T,$$

that is

$$z_1 = 2.3651$$

and

$$z_t = 0.1790 z_{t-1} + 1.9418.$$

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