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An adaptive run sum scheme for process ratio

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Abstract

Tracking the ratio of two normal variables (RTNV) is crucial in ensuring the quality regarding some manufacturing processes. In this study, the run sum and variable sampling interval procedures are combined to develop a more effective scheme for controlling the RTNV A comparison with the existing basic ratio and run sum ratio schemes, as per average time to signal criterion, shows that the new chart substantially surpasses its two existing counterparts. An example of application using factual data is given to showcase the operation of the new scheme in practice.

Subject Classification: *62P30, 93C83.*

Keywords: *Variable sampling interval, Run sum ratio scheme, Distribution of the ratio statistic, Markov chain, Average time to signal.*

1. Introduction

Maintaining process stability is regarded as the sole prime objective of Statistical Process Control (SPC), and can be accomplish through identifying variations that are capable of influencing process quality adversely, as timely as plausible. Fundamentally, control schemes are used to monitor the process effectiveness and to detect strange process behaviour. Control charts are arguably the ultimate essential SPC mechanism in process examination for unfolding quality issues and owing

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to this pretext the application of control schemes for modern manufacturing industries alongside non-manufacturing sectors is acquiring widespread recognition amid quality control specialists.

Tracking the ratio of two normal variables (RTNV) had been the subject of concern over the years. In some manufacturing scenarios, practitioners are fascinated to track the stability of the RTNV of a bivariate vector, instead of the vector average or matrix covariance of the said bivariate vector. [23] initiated a control technique for tracking the two random variables (TRV) that are normally distributed. Subsequently, [20] provided guidelines in implementing the Shewhart (SH) type charts for conducting a virtual tracking in the glass industry using the ratio scheme. [6] statistically studied the properties of the SH scheme for individual measurements in monitoring RTNV, called the SH ratio scheme. [4] considered the Phase-II SHR schemes, where each sample consists of $n > 1$ measurements and each sample size is entitled to swap. An investigation on the Phase-II synthetic scheme for tracking RTNV was conducted by [5] and the developed scheme was shown to be superior to the SHR scheme. [25] investigated the Phase-II runs rules SH schemes for monitoring RTVs. The exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) schemes for the ratio were developed by [26] and [28], respectively. An appraisal of the SHR scheme in the company of measurement errors utilizing linear covariate error model was carried out by [27]. [30] assessed the steady state performance of the EWMA ratio scheme for tracking the proportion of the TNVs, while the EWMA ratio and CUSUM ratio control schemes with adaptive (VSI) feature was advocated by [17] and [18], respectively. [1] proposed the run sum (RSR) schemes in favour of bivariate normal variables. In another evolution, [16] examined two one-sided SHR schemes just as measurement errors are present, whereas [19] investigated effect of estimating errors on the EWMA scheme for monitoring the fraction of TNVs. The one-sided SH scheme for tracking the fraction of TVs in tiny production runs was suggested by [29]. [2] propounded the RSR schemes even as measurement errors are available.

Common practice in employing control charts for process monitoring is by considering samples that are drawn from a system at fixed sampling intervals (FSI). To amplify the efficacy of control schemes accompanied by FSI feature, adaptable charts have been developed. In an adaptive chart, at least one of this parameter is varied, i.e., sample size, sampling interval and control limits' width. By permitting the chart's parameters to fluctuate, a control scheme provides more flexibility in its design. For example, if the current quality is poor, a shorter sampling interval and sufficient sample size, besides tighter control bounds are adopted for upcoming sample so that the process shift which results in the process having poor quality can be detected quicker.

Examples of adaptable schemes are the VSI $\bar{X}$, variable sample size (VSS) $\bar{X}$, variable sample size and sampling interval (VSSI) $\bar{X}$, and variable parameters (VP) $\bar{X}$ schemes. The VSI $\bar{X}$, VSS $\bar{X}$, VSSI $\bar{X}$, and VP $\bar{X}$ charts allow the (i) sampling interval, (ii) sample size, (iii) sample size and sampling interval, and (iv) sampling interval, sample size and constant controlling the width of the control bounds, respectively, to fluctuate. VSI $\bar{X}$ scheme was developed by [21], where this scheme has visibly outstripped the FSI $\bar{X}$ scheme. In the same vein, [10] advocated the VSS $\bar{X}$ chart, which has transcended the fixed sample size $\bar{X}$ chart. Moreover, [11] fused the VSS and VSI techniques to come up with the VSSI $\bar{X}$ scheme, which has overtaken the VSS $\bar{X}$ and VSI $\bar{X}$ control schemes. [12] propounded the variable parameters (VP) $\bar{X}$ scheme, in which sampling interval, sample size, and width constant are entitle to fluctuate simultaneously, thereby allowing VP $\bar{X}$ scheme to supersede VSI $\bar{X}$, VSS $\bar{X}$ and VSSI $\bar{X}$ schemes. Other researches on adaptive schemes include the $\bar{X}$ scheme with variable control limits and warning limits [14], also the economic design of a modified VSSI scheme [15], to name some. [8] developed the single variable sampling plan with consumer's quality protection via an optimum profit model.

A run sum (RS) chart is fashioned by splitting the interval separating the upper and lower control limits of a scheme at hand to regions on top and beneath the center line of the scheme, followed by assigning scores to these regions. Periodically, samples being monitored are taken from the process and the scores of the regions where the sample means fall are accumulated. An out-of-control is signalled when the accumulated count reaches or outstrips a specified triggering score [7]. To the foremost of the authors' acquaintance with literature, there is no study on adaptive RSR scheme. In order to fill this research gap, the two-sided VSI RSR scheme in monitoring the RTNVs is developed as per this paper.

The second section furnishes a discussion on distribution of the RTNVs, whereas the third section explains the developed two-sided VSI RSR scheme. Fourth section enumerates the optimal design of the two-sided VSI RS scheme for process ratio in minimizing the average time to signal (ATS) value whilst the process is out-of-control. Fifth section is devoted to an evaluation of the ATS metrics of the two-sided VSI RSR scheme in comparison to its existing counterparts. Sixth section illustrates an application on a real quality control issue originating out of the food industry showcasing the implementation of the posit two-sided VSI RSR scheme. Finally, conclusions are extracted in the seventh section.

2. Distribution of the correlated ratio of two normal variables

Suppose that X and Y are variables from a bivariate normal vector $\boldsymbol{E}$, i.e., $\boldsymbol{E}=(X,Y)^T$ follows a $N_2(\mu_{\boldsymbol{E}},\Sigma_{\boldsymbol{E}})$ distribution. The vector means and matrix covariance for $\boldsymbol{E}$ are

$$\mu_{\boldsymbol{E}}=\begin{pmatrix}\mu_X\\ \mu_Y\end{pmatrix} \tag{1}$$

and

$$\Sigma_{\boldsymbol{E}}=\begin{pmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix}. \tag{2}$$

respectively. Here, μ_X, μ_Y are the average values of X and Y, distinctively, while the respective standard errors of X and Y are σ_X along with σ_Y, and ρ denotes the correlation coefficient between X and Y. Additionally, coefficient of variations (CVs) for X and Y are $\eta_X=\frac{\sigma_X}{\mu_X}$ and $\eta_Y=\frac{\sigma_Y}{\mu_Y}$, respectively, and the standard errors for the ratio statistic, $Z=\frac{X}{Y}$ is $\omega=\frac{\sigma_X}{\sigma_Y}$ ([4]).

Some of the past literatures had discussed the distribution of the RTNVs, for example, see [13] and [3]. The distribution of the R statistic, $Z=\frac{X}{Y}$ corresponding to the approach presented in [13] or [4]. The cumulative distribution function (cdf) of Z resembled as [4]

$$F_Z(z\,|\,\eta_X,\eta_Y,\omega,\rho)\cong\Phi\left(\frac{A}{B}\right), \tag{3}$$

where Φ(·) represents the standard normal cdf, thus,

$$A=\frac{z}{\eta_Y}-\frac{\omega}{\eta_X} \tag{4}$$

and

$$B=\sqrt{\omega^2-2\rho\omega z+z^2}. \tag{5}$$

An inexact probability density function (pdf) of Z is given as [4]

$$f_Z(z \mid \eta_X, \eta_Y, \omega, \rho) \cong \left(\frac{1}{B\eta_Y} - \frac{(z-\rho\omega)A}{B^2} \right) \times \phi\left(\frac{A}{B} \right), \tag{6}$$

where ϕ(·) stands for standard normal pdf. Moreover, an inexact inverse cdf (icdf) of Z is [4]

$$F_Z^{-1}(\alpha \mid \eta_X, \eta_Y, \omega, \rho) \cong \begin{cases} \frac{-D_2 - \sqrt{D_2^2 - 4D_1D_3}}{2D_1}, & \text{if } \alpha \in (0, 1/2] \\ \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1}, & \text{if } \alpha \in (1/2, 1] \end{cases}, \tag{7}$$

where D_1, D_2 and D_3 are defined as follows:

$$D_1 = \frac{1}{\eta_Y^2} - (\Phi^{-1}(\alpha))^2, \tag{8a}$$

$$D_2 = 2\omega\left(\rho(\Phi^{-1}(\alpha))^2 - \frac{1}{\eta_X \eta_Y} \right) \tag{8b}$$

and

$$D_3 = \omega^2 \left(\frac{1}{\eta_X^2} - (\Phi^{-1}(\alpha))^2 \right). \tag{8c}$$

Here, $\Phi^{-1}(\cdot)$ stands for icdf of the standard normal random variable.

In tracking the RZ, independent bivariate vectors $\{\boldsymbol{E}_{i,1}, \boldsymbol{E}_{i,2}, \ldots, \boldsymbol{E}_{i,n}\}$ of size n are gathered at sample i (= 1, 2, …), whereas $\boldsymbol{E}_{i,j} = (X_{i,j}, Y_{i,j})^T \sim N_2(\mu_{\boldsymbol{E}_i}, \boldsymbol{\Sigma}_{\boldsymbol{E}_i})$ as in Equation (2) for $j = 1, 2, \ldots, n$. It is deduced that $\boldsymbol{E}_{i,j}$ follows a bivariate normal distribution. As adjudged by [4], the underline presumptions can be made.

(i) Sample units are at liberty to fluctuate from one sample to another, implying that $\boldsymbol{\mu}_{\boldsymbol{E}_i} \neq \boldsymbol{\mu}_{\boldsymbol{E}_l}$ and $\boldsymbol{\Sigma}_{\boldsymbol{E}_i} \neq \boldsymbol{\Sigma}_{\boldsymbol{E}_l}$, for $i \neq l$, where i and l denote the sample number.

(ii) For an in-control process, $z_0 = \frac{\mu_{X_i}}{\mu_{Y_i}}$, for $i = 1, 2, \ldots$, where z_0 symbolize the in-control value of the R, $Z = \frac{X}{Y}$.

The charting statistic pertaining to SH ratio scheme is [4]

$$\hat{Z}_i = \frac{\hat{\mu}_{X_i}}{\hat{\mu}_{Y_i}} = \frac{\overline{X}_i}{\overline{Y}_i} = \frac{\sum_{j=1}^{n} X_{i,j}}{\sum_{j=1}^{n} Y_{i,j}}, \; i = 1, 2, \ldots \text{ and } j = 1, 2, \ldots, n. \tag{9}$$

When the scheme for process R is in-control, the cdf and icdf of $\hat{Z}_i$ are [28]

$$F_{\hat{Z}_i}(z \mid n, \eta_X, \eta_Y, z_0, \rho) = F_Z\left(z \left| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{z_0 \eta_X}{\eta_Y}, \rho \right. \right) \tag{10}$$

likewise

$$F_{\hat{Z}_i}^{-1}(\alpha \mid n, \eta_X, \eta_Y, z_0, \rho) = F_Z^{-1}\left(\alpha \left| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{z_0 \eta_X}{\eta_Y}, \rho \right. \right), \tag{11}$$

respectively. The moment the process R is out-of-control, the cdf and icdf of $\hat{Z}_i$ are still obtained using Equations (10) and (11), respectively, but by replacing z_0 with $z_1 = \tau z_0$, where $\tau \neq 1$.

			$UCL_k\ (=\infty)$
π_{+k}	$S(\hat{Z}_i) = +S_k$	C_{+k}	UCL_{k-1}
			UCL_2
π_{+2}	$S(\hat{Z}_i) = +S_2$	C_{+2}	UCL_1
π_{+1}	$S(\hat{Z}_i) = +S_1$	C_{+1}	CL
π_{-1}	$S(\hat{Z}_i) = -S_1$	C_{-1}	LCL_1
π_{-2}	$S(\hat{Z}_i) = -S_2$	C_{-2}	LCL_2
			LCL_{k-1}
π_{-k}	$S(\hat{Z}_i) = -S_k$	C_{-k}	$LCL_k\ (=-\infty)$

Figure 1

Two-sided VSI RS ratio chart with *k* regions, each above and below the *CL*

3. Two-sided VSI RSR scheme

The VSI RSR scheme comprises *k* regions, each above and beneath the center line (*CL*). The regions 'on top of' and 'beneath' the *CL* are used to detect increases and decreases in the process mean of the R statistic. The

t^{th} upper and lower control limits of the two-sided adaptive R scheme, i.e., UCL_t and LCL_t $(t = 1, 2, \ldots, k-1)$, respectively, are obtained as follows:

$$UCL_t = K \times z_0 F_Z^{-1}\left(\alpha_{u,t} \middle| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{\eta_X}{\eta_Y}, \rho\right), \tag{12}$$

where $\alpha_{u,t} = \Phi\left(\frac{3t}{k-1}\right)$, and

$$LCL_t = K \times z_0 F_Z^{-1}\left(\alpha_{l,t} \middle| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{\eta_X}{\eta_Y}, \rho\right), \tag{13}$$

where $\alpha_{l,t} = \Phi\left(\frac{-3t}{k-1}\right)$. Here, K pertains to the limits' constant selected to arrive at preferred in-control ATS (ATS_0) performance, $\Phi(\cdot)$ allude to the cdf of a standard normal random variable and $F_Z^{-1}(\cdot)$ is defined in Equation (7). The center line is defined as

$$CL = K \times z_0. \tag{14}$$

Regions 'on top of' the CL are designated as $C_{+1}, C_{+2}, \ldots, C_{+k}$, whereas regions 'beneath CL are represented as $C_{-1}, C_{-2}, \ldots, C_{-k}$, where C_{+t} = $[UCL_{t-1}, UCL_t)$ and $C_{-t} = (LCL_t, LCL_{t-1}]$. The control limits satisfy the constraint $UCL_0(=CL)$ < UCL_1 < UCL_2 < … < UCL_{k-1} < $UCL_k(=\infty)$ and $LCL_k(=-\infty)$ < LCL_{k-1} < … < LCL_2 < LCL_1 < $LCL_0(=CL)$. Each of the regions on top of the CL of the two-sided VSI RSR scheme has been assigned a positive whole number $+S_t$ (t = 1, 2, …, k), where $0 \le +S_1 \le +S_2 \le \ldots \le +S_k$. The hereunder score function is delineated for regions above the CL.

$$S(\hat{Z}_i) = +S_t, \text{ if } \hat{Z}_i \in C_{+t}, \text{ for } t = 1, 2, \ldots, k. \tag{15}$$

Similarly, each region below the CL regarding the two-sided VSI RSR scheme is associated with a negative whole number $-S_t$ (t = 1, 2, …, k), hence, $-S_k \le \ldots \le -S_2 \le -S_1 < 0$. The hereafter score function is designated for the regions beneath the CL.

$$S(\hat{Z}_i) = -S_t, \text{ if } \hat{Z}_i \in C_{-t}, \; t = 1, 2, \ldots, k. \tag{16}$$

The two-sided VSI RSR scheme operates based on the aggregate scores W_i and F_i elucidated as follows:

$$W_i = \begin{cases} W_{i-1} + S(\hat{Z}_i), & \text{if } \hat{Z}_i > CL \\ 0, & \text{if } \hat{Z}_i < CL \end{cases} \tag{17}$$

and

$$F_i = \begin{cases} F_{i-1} + S(\hat{Z}_i), & \text{if } \hat{Z}_i < CL \\ 0, & \text{if } \hat{Z}_i > CL \end{cases}, \tag{18}$$

where $i = 1, 2, \ldots$. The starting value for the two-sided VSI RSR scheme are $W_0 = F_0 = 0$. The VSI RSR scheme alert an out-of-control condition when the accumulative score $W_i \geq H \;/\; F_i \leq -H,$ where $H = S_k$.

The probability of $\hat{Z}_i$ that plots in the region C_{+t} (t = 1, 2, …, k) is calculated as

$$\pi_{+t} = \Pr(\hat{Z}_i \in C_{+t})$$
$$= F_Z\left(UCL_t \left| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{\tau z_0 \eta_X}{\eta_Y}, \rho \right.\right) - F_Z\left(UCL_{t-1} \left| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{\tau z_0 \eta_X}{\eta_Y}, \rho \right.\right) \tag{19}$$

and the probability of $\hat{Z}_i$ falling in the region C_{-t} $(t = 1, 2, \ldots, k)$ is calculated as

$$\pi_{-t} = \Pr(\hat{Z}_i \in C_{-t})$$
$$= F_Z\left(LCL_{t-1} \left| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{z_1 \eta_X}{\eta_Y}, \rho \right.\right) - F_Z\left(LCL_t \left| \frac{\eta_X}{\sqrt{n}}, \frac{\eta_Y}{\sqrt{n}}, \frac{z_1 \eta_X}{\eta_Y}, \rho \right.\right), \tag{20}$$

where $z_1 = \tau z_0$ is the out-of-control value of the process R. $\tau > 1$ indicates a rise in the in-control value of the R (z_0), while $\tau \in (0,1)$ reflects a decline in z_0. Figure 1 shows the regions, scores and probabilities of $\hat{Z}_i$ drifting in each of the regions of the VSI RSR scheme.

Let $T_{\hat{Z}_i}$ symbolizes the sampling interval separating samples i and i + 1, for the VSI RSR scheme. The span of the sampling interval is dictated by the values of the accumulative sums, W_i and F_i accordingly:

$$T_{\hat{Z}_i} = \begin{cases} d_1, & \text{if } H/G \leq W_i < H \text{ or} -H < F_i \leq -H/G \\ d_2, & \text{if } 0 \leq W_i < H/G \text{ or} -H/G < F_i \leq 0 \end{cases}, \tag{21}$$

whereas d_1 stands the short sampling interval and $d_2 (> d_1)$ signifies the long sampling interval. Note that G is a non-negative integer that controls the rate of fluctuation between d_1 and d_2. The ATS value of the two-sided VSI RSR scheme can be calculated as [9]

$$\text{ATS} = \boldsymbol{s}^T (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{b} - \boldsymbol{s}^T \boldsymbol{b}. \tag{22}$$

In Equation (22), $\boldsymbol{s}$ is the $m \times 1$ initial probability vector, $\boldsymbol{I}$ is the $m \times m$ identity matrix, $\boldsymbol{b}$ is the $m \times 1$ vector containing elements of either d_1 or d_2, and $\boldsymbol{Q}$ is the $m \times m$ transition probability matrix (tpm) for the transient states. More details on $\boldsymbol{s}$, $\boldsymbol{I}$ and $\boldsymbol{b}$ are given in [9], while additional explanation in constructing $\boldsymbol{Q}$, based on the Markov chain method, is provided in the Appendix of the article.

4. Optimal design of the two-sided VSI RSR scheme

An optimal design of the VSI RSR chart in minimizing the out-of-control ATS (ATS_1) value of the VSI RSR scheme is explained in this section. In this optimal design, the optimal scores $\{S_1, S_2, \ldots, S_k\}$, limits' constant ($K$) and long sampling interval (d_2) of the two-sided VSI RSR scheme that minimize the ATS_1 value are computed. The objective function in minimizing the ATS_1 value of the two-sided VSI RSR chart, for the shift size τ, is presented below.

$$\underset{(K,\, d_2, \{S_1, S_2, \ldots, S_k\})}{\text{Minimize}}\ \text{ATS}_1 \tag{23}$$

subject to specifying the desired values of the following parameters:

(i) $\text{ATS}_0 = \varepsilon$,
(ii) sample size = n,
(iii) CV of X is η_X,
(iv) CV of Y is η_Y,
(v) short sampling interval = d_1,
(vi) coefficient of correlation between X and Y is ρ,
(vii) in-control ratio = z_0

Optimal values $(K, d_2, \{S_1, S_2, \ldots, S_k\})$, computed based on the aforementioned optimal design model will yield the least ATS_1 value, for the shift size τ, among all $(K, d_2,\ \{S_1, S_2, \ldots, S_k\})$ combinations that produce $\text{ATS}_0 = \varepsilon$. The k = 4 and 7 regions VSI RSR schemes are considered. Additionally, n =5, $\rho \in$ {–0.8, –0.4, 0, 0.4, 0.8}, $\tau \in$ {0.9, 0.95, 0.98, 0.99, 1.01, 1.02, 1.05, 1.1}, $(\eta_X, \eta_Y) \in$ {(0.01, 0.01), (0.2, 0.2), (0.01, 0.2), (0.2, 0.01)}, $G \in$ {2, 3}, $d_1 \in$ {0.1, 0.3}, $S_k \leq 15$ and $\varepsilon = 200$ are considered.

The optimal $(K, d_2, \{S_1, S_2, \ldots, S_k\})$ combinations of the VSI RSR scheme which minimizes the ATS_1 values for the shift sizes $\tau \in$ {0.9, 0.95, 0.98, 0.99, 1.01, 1.02, 1.05, 1.1}, based on the various parameter combinations, are presented in Tables 1 – 4, for k = 4 and 7. These optimal $(K, d_2, \{S_1, S_2, \ldots, S_k\})$ combinations are computed using MATLAB. For instance, in Table 1, when $G = 2$, $d_1 = 0.1$, $(\eta_X, \eta_Y) = (0.01, 0.01)$, $n = 5$, $\tau = 0.99$ and ρ

= –0.8, the optimal $(K, d_2, \{S_1, S_2, S_3, S_4\})$ combination of the VSI RSR scheme with k = 4 is $(K, d_2, \{S_1, S_2, S_3, S_4\})$ = (0.9998, 1.0645, {1, 2, 6, 15}). This optimal $(K, d_2, \{S_1, S_2, S_3, S_4\})$ combination yields the smallest ATS_1 (= 5.9) value when G = 2, d_1 = 0.1, (η_X, η_Y) = (0.01, 0.01), n = 5, τ = 0.99 and ρ = –0.8 (see Table 5) among the ATS_1 values generated by all the $(K, d_2, \{S_1, S_2, S_3, S_4\})$ combinations that give ATS_0 = 200. Similarly, the optimal $(K, d_2, \{S_1, S_2, ..., S_7\})$ combination of the 7 regions VSI RSR scheme with the same G, d_1, (η_X, η_Y), n, τ and ρ values scrutinized above is $(K, d_2, \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\})$ = (0.9997, 1.0711, {0, 1, 2, 3, 5, 8, 14}) (see Table 3). This optimal $(K, d_2, \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\})$ combination gives ATS_1 = 5.2 (see Table 5), which is the minimum ATS_1 value among the ATS_1 values evaluated from all $(K, d_2, \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\})$ combinations that give ATS_0 = 200. For brevity, the optimal $(K, d_2, \{S_1, S_2, ..., S_k\})$ combinations of the VSI RSR scheme with k = 4 and 7 regions, for $n \in$ {1, 7, 10, 15}, $d_1 \in$ {0.3, 0.5) are not given in Tables 1 – 4 but they can be requested from the 1st author.

Table 1

Optimal $(K, d_2, \{S_1, S_2, S_3, S_4\})$ combination of the 4 regions VSI RSR scheme in minimizing the ATS_1 for τ, where n = 5, d_1= 0.1, $(\eta_X, \eta_Y) \in$ (0.01, 0.01), $\rho \in$ {–0.8, –0.4, 0, 0.4, 0.8}, $G \in$ {2, 3} and ATS_0 = 200

ρ	***G* = 2**			
	τ = 0.9	τ = 0.95	τ = 0.98	τ = 0.99
–0.8	0.9966,1.0034,0,0,0,1	0.9993,1.0864.0,2,7,14	0.9993,1.0864,0,2,7,14	0.9998,1.0645,1,2,6,15
–0.4	0.9970,1.0046,0,0,0,1	0.9994,1.0864,0,2,7,14	0.9994,1.0864,0,2,7,14	0.9996,1.0486,0,2,5,11
0	0.9975,1.0046,0,0,0,1	0.9995,1.0899,0,2,7,14	0.9995,1.0889,0,2,7,14	0.9995,1.0889,0,2,7,14
0.4	0.9980,1.0059,0,0,0,1	0.9996,1.0864,0,1,4,8	0.9996,1.0889,0,2,7,14	0.9996,1.0889,0,2,7,14
0.8	0.9989,1.0034,0,0,0,1	0.9989,1.0034,0,0,0,1	0.9998,1.0889,0,2,7,14	0.9998,1.0889,0,2,7,14
	τ = 1.01	τ = 1.02	τ = 1.05	τ = 1.1
–0.8	0.9983,1.0669,0,1,2,6	0.9989,1.0864,0,0,1,2	0.9989,1.0864,0,0,1,2	0.9966,1.0034,0,0,0,1
–0.4	0.9985,1.0669,0,1,2,6	0.9990,1.0889,0,0,1,2	0.9990,1.0889,0,0,1,2	0.9970,1.00460,0,0,0,1
0	0.9992,1.0889,0,1,5,10	0.9992,1.0889,0,0,1,2	0.9992,1.0889,0,0,1,2	0.9975,1.0046,0,0,0,1
0.4	0.99994,1.0864,0,1,7,14	0.9993,1.0864,0,0,1,2	0.9993,1.0864,0,0,1,2	0.9980,1.0059,0,0,0,1
0.8	0.9996,1.0864,0,0,1,2	0.9996,1.0864,0,0,1,2	0.9989,1.0034,0,0,0,1	0.9989,1.0034,0,0,0,1

Contd...

	G = 3			
	$\tau = 0.9$	$\tau = 0.95$	$\tau = 0.98$	$\tau = 0.99$
−0.8	0.9966,1.0034,0,0,0,1	0.9999,1.2012,1,1,5,12	0.9995,1.1792,0,2,5,11	0.9995,1.1792,0,2,5,11
−0.4	0.9970,1.0046,0,0,0,1	0.9999,1.2012,1,1,5,12	0.9996,1.1792,0,2,5,11	0.9996,1.1792,0,2,5,11
0	0.9975,1.0046,0,0,0,1	0.9999,1.2012,1,1,5,12	0.9996,1.1792,0,2,5,11	0.9996,1.1792,0,2,5,11
0.4	0.9980,1.0059,0,0,0,1	1.0000,1.2012,1,1,5,12	1.0000,1.2012,1,1,5,12	0.9997,1.1792,0,2,5,11
0.8	0.9989,1.0034,0,0,0,1	0.9989,1.0034,0,0,0,1	1.0000,1.2012,1,1,5,12	1.0000,1.2012,1,1,5,12
	$\tau = 1.01$	$\tau = 1.02$	$\tau = 1.05$	$\tau = 1.1$
−0.8	0.9983,12109,0,1,2,6	0.9970,1.1328,0,0,1,3	0.9970,1.1328,0,0,1,3	0.9966,1.0034,0,0,0,1
−0.4	0.9985,1.2109,0,1,2,6	0.9973,1.1353,0,0,1,3	0.9973,1.1353,0,0,1,3	0.9970,1.0046,0,0,0,1
0	0.9987,1.2109,0,1,2,6	0.9978,1.1353,0,0,1,3	0.9978,1.1353,0,0,1,3	0.9975,1.0046,0,0,0,1
0.4	0.9983,1.1328,0,0,1,3	0.9983,1.1328,0,0,1,3	0.9983,1.1328,0,0,1,3	0.9980,1.0059,0,0,0,1
0.8	0.9990,1.1328,0,0,1,3	0.9990,1.1328,0,0,1,3	0.9989,1.0034,0,0,0,1	0.9989,1.0034,0,0,0,1

Table 2

Optimal $(K, d_2, \{S_1, S_2, S_3, S_4\})$ combination of the 4 regions VSI RSR scheme in minimizing the ATS_1 for τ, where n = 5, d_1 = 0.1, $(\eta_X, \eta_Y) \in$ (0.2, 0.2), $\rho \in$ {−0.8, −0.4, 0, 0.4, 0.8}, $G \in$ {2, 3} and ATS_0 = 200

ρ	G = 2			
	$\tau = 0.9$	$\tau = 0.95$	$\tau = 0.98$	$\tau = 0.99$
−0.8	0.9994,1.0681,1,2,4,13	0.9994,1.0681,1,2,4,13	0.9994,1.0681,1,2,4,13	0.9994,1.0681,1,2,4,13
−0.4	0.9994,1.0681,1,2,4,13	0.9994,1.0681,1,2,4,13	0.9994,1.0681,1,2,4,13	0.9994,1.0681,1,2,4,13
0	0.9997,1.0669,1,2,4,13	0.9997,1.0669,1,2,4,13	0.9997,1.0669,1,2,4,13	0.9997,1.0669,1,2,4,13
0.4	0.9997,1.0681,1,2,4,13	0.9997,1.0681,1,2,4,13	0.9997,1.0681,1,2,4,13	0.9997,1.0681,1,2,4,13
0.8	0.9952,1.0876,0,2,7,14	0.9999,1.0669,1,2,4,13	0.9999,1.0669,1,2,4,13	0.9999,1.0669,1,2,4,13
	$\tau = 1.01$	$\tau = 1.02$	$\tau = 1.05$	$\tau = 1.1$
−0.8	0.9539,1.0559,0,2,3,12	0.9539,1.0559,0,2,3,12	0.9655,1.0669,0,1,2,6	0.9655,1.0669,0,1,2,6
−0.4	0.9591,1.0571,0,2,3,12	0.9591,1.0571,0,2,3,12	0.9694,1.0693,0,1,2,6	0.9694,1.0693,0,1,2,6
0	0.9653,1.0547,0,2,3,12	0.9653,1.0547,0,2,3,12	0.9740,1.0693,0,1,2,6	0.9740,1.0693,0,1,2,6
0.4	0.9729,1.0571,0,2,3,12	0.9729,1.0571,0,2,3,12	0.9798,1.0669,0,1,2,6	0.9798,1.0669,0,1,2,6
0.8	0.9842,1.0571,0,2,3,12	0.9882,1.0669,0,1,2,6	0.9882,1.0669,0,1,2,6	0.9930,1.0864,0,1,5,10
	G = 3			
	$\tau = 0.9$	$\tau = 0.95$	$\tau = 0.98$	$\tau = 0.99$
−0.8	0.9994,1.1841,1,2,4,13	0.9994,1.1841,1,2,4,13	0.9994,1.1841,1,2,4,13	0.9994,1.1841,1,2,4,13
−0.4	0.9994,1.1841,1,2,4,13	0.9994,1.1841,1,2,4,13	0.9994,1.1841,1,2,4,13	0.9994,1.1841,1,2,4,13

Contd...

0	0.9997,1.1841,1,2,4,13	0.9997,1.1841,1,2,4,13	0.9997,1.1841,1,2,4,13	0.9997,1.1841,1,2,4,13
0.4	0.9977,1.2134,1,2,6,15	0.9997,1.1841,1,2,4,13	0.9997,1.1841,1,2,4,13	0.9997,1.1841,1,2,4,13
0.8	0.9967,1.1792,0,2,5,11	0.9999,1.18411,2,4,13	0.9999,1.1841,1,2,4,13	0.9999,1.1841,1,2,4,13
	$\tau = 1.01$	$\tau = 1.02$	$\tau = 1.05$	$\tau = 1.1$
–0.8	0.9539,1.1719,0,2,3,12	0.9539,1.1719,0,2,3,12	0.9539,1.1719,0,2,3,12	0.9655,1.2134,0,1,2,6
–0.4	0.9591,1.1719,0,2,3,12	0.9591,1.1719,0,2,3,12	0.9694,1.2134,0,1,2,6	0.9694,1.2134,0,1,2,6
0	0.9653,1.1707,0,2,3,12	0.9653,1.1707,0,2,3,12	0.9740,1.2134,0,1,2,6	0.9740,1.2134,0,1,2,6
0.4	0.9729,1.1719,0,2,3,12	0.9729,1.1719,0,2,3,12	0.9798,1.2134,0,1,2,6	0.9798,1.2134,0,1,2,6
0.8	0.9842,1.1719,0,2,3,12	0.9842,1.1719,0,2,3,12	0.9882,1.2134,0,1,2,6	0.9882,1.2134,0,1,2,6

5. Performance evaluation of the two-sided VSI RSR chart

In the current section comparison of the performance of the VSI RSR scheme alongside the SHR chart by [4], and RSR scheme by [1], using the ATS_1 criterion. The ATS_1s computed using MATLAB for $n = 5$, $\rho \in \{-0.8, -0.4, 0, 0.4, 0.8\}$, $G \in \{2, 3\}$, $d_1 \in \{0.1, 0.3\}$ and $(\eta_X, \eta_Y) \in \{(0.01, 0.01), (0.2, 0.2), (0.01, 0.2), (0.2, 0.01)\}$ are given in Tables 5 – 8. The ATS_1 values in Tables 5 – 8 disclose that the VSI RSR scheme is by and large superior to SHR and RSR schemes, for entire shift sizes τ, notwithstanding the values of n, ρ and (η_X, η_Y).

Table 3

Optimal $(K, d_2, \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\})$ combination of the 7 regions VSI RSR scheme in minimizing the ATS_1 for τ, where $n = 5$, $d_1 = 0.1$, $(\eta_X, \eta_Y) \in (0.01, 0.01)$, $\rho \in \{-0.8, -0.4, 0, 0.4, 0.8\}$, $G \in \{2, 3\}$ and $ATS_0 = 200$

ρ	$G = 2$			
	$\tau = 0.9$	$\tau = 0.95$	$\tau = 0.98$	$\tau = 0.99$
–0.8	0.9989,1.0303,0,0,0,1,1,1,3	1.0000,1.0836,0,1,1,1,7,10,14	0.9997,1.0870,0,0,1,3,7,9,14	0.9997,1.0711,0,1,2,3,5,8,14
–0.4	0.9990,1.0301,0,0,0,1,1,1,3	1.0000,1.0836,0,1,1,1,7,10,14	1.0000,1.0836,0,1,1,1,7,10,14	0.9999,1.0670,0,1,1,3,6,9,14
0	0.9992,1.0309,0,0,0,1,1,1,3	1.0000,1.0837,0,1,1,1,7,10,14	1.0000,1.0837,0,1,1,1,7,10,14	0.9998,1.0867,0,0,1,3,7,9,14
0.4	0.9993,1.0314,0,0,0,1,1,1,3	1.0000,1.0840,0,1,1,1,7,10,14	1.0000,1.0840,0,1,1,1,7,10,14	0.9998,1.0879,0,0,1,3,7,9,14
0.8	0.9996,1.0314,0,0,0,1,1,1,3	0.9996,1.0314,0,0,0,1,1,1,3	1.0000,1.0838,0,1,1,1,7,10,14	1.0000,1.0838,0,1,1,1,7,10,14
	$\tau = 1.01$	$\tau = 1.02$	$\tau = 1.05$	$\tau = 1.1$
–0.8	0.9984,1.0665,0,1,2,3,4,6,15	0.9989,1.0880,0,0,0,2,7,7,14	0.9969,1.0367,0,0,0,1,1,7,14	0.9989,1.0303,0,0,0,1,1,1,3
–0.4	0.9988,1.0660,0,0,1,3,5,6,12	0.9990,1.0885,0,0,0,1,7,8,14	0.9973,1.0367,0,0,0,1,1,7,14	0.9990,1.0301,0,0,0,1,1,1,3
0	0.9990,1.0659,0,0,1,3,5,6,12	0.9992,1.0882,0,0,0,1,6,6,12	0.9977,1.0376,0,0,0,1,1,7,14	0.9992,1.0309,0,0,0,1,1,1,3
0.4	0.9994,1.0886,0,0,0,1,3,3,6	0.9993,1.0870,0,0,0,1,7,7,14	0.9982,1.0365,0,0,0,1,1,6,12	0.9993,1.0314,0,0,0,1,1,1,3
0.8	0.9996,1.0871,0,0,0,1,6,6,12	0.9990,1.037,0,0,0,1,1,8,15	0.9996,1.0314,0,0,0,1,1,1,3	0.9996,1.0314,0,0,0,1,1,1,3

Contd...

	$G = 3$			
	$\tau = 0.9$	$\tau = 0.95$	$\tau = 0.98$	$\tau = 0.99$
–0.8	0.9989,1.2871,0,0,0,1,1,1,3	0.9998,1.2802,0,1,1,5,6,6,15	0.9997,1.2728,0,0,1,5,5,8,14	0.9998,1.2802,0,1,1,5,6,6,15
–0.4	0.9990,1.2869,0,0,0,1,1,1,3	0.9999,1.2789,0,1,1,5,6,6,15	0.9998,1.2721,0,0,1,5,5,8,14	0.9999,1.2789,0,1,1,5,6,6,15
0	0.9992,1.2879,0,0,0,1,1,1,3	1.0000,1.0938,0,1,1,1,7,10,14	0.9998,1.2731,0,0,1,4,4,7,12	0.9998,1.2731,0,0,1,4,4,7,12
0.4	0.9993,1.2887,0,0,0,1,1,1,3	1.0000,1.1357,0,0,1,4,5,8,13	0.9999,1.2725,0,0,1,5,5,8,14	0.9999,1.2739,0,0,1,4,4,7,12
0.8	0.9996,1.2887,0,0,0,1,1,1,3	0.9996,1.2887,0,0,0,1,1,1,3	0.9999,1.2788,0,1,1,5,6,6,15	0.9999,1.2733,0,0,1,4,4,7,12
	$\tau = 1.01$	$\tau = 1.02$	$\tau = 1.05$	$\tau = 1.1$
–0.8	0.9980,1.1826,0,0,2,3,5,5,15	0.9989,1.2852,0,0,0,5,6,8,15	0.9970,1.1352,0,0,0,1,5,5,15	0.9989,1.2871,0,0,0,1,1,1,3
–0.4	0.9982,1.1813,0,0,2,3,5,5,15	0.9991,1.2860,0,0,0,4,5,6,12	0.9973,1.1343,0,0,0,1,5,5,15	0.9990,1.2869,0,0,0,1,1,1,3
0	0.9992,1.2879,0,0,0,3,4,4,9	0.9992,1.2853,0,0,0,5,6,8,15	0.9978,1.1335,0,0,0,1,5,5,15	0.9992,1.2879,0,0,0,1,1,1,3
0.4	0.9994,1.2852,0,0,0,5,6,8,15	0.9983,1.1346,0,0,0,1,5,5,15	0.9981,1.0419,0,0,0,1,1,5,14	0.9993,1.2887,0,0,0,1,1,1,3
0.8	0.9996,1.2859,0,0,0,5,6,8,15	0.9990,1.1348,0,0,0,1,5,5,15	0.9996,1.2887,0,0,0,1,1,1,3	0.9996,1.2887,0,0,0,1,1,1,3

For example, Table 6, when G =2, $d_1 = 0.1$, (η_X, η_Y) = (0.2, 0.2), $n = 5$, ρ = –0.4 and $\tau = 0.95$, the ATS_1 values for the SHR, RSR ($k = 4$), RS ratio ($k = 7$), VSI RSR ($k = 4$) and VSI RSR ($k = 7$) schemes are 136.0, 82.7, 81.9, 76.6 and 76.5, respectively, wherein the proposed VSI RSR schemes retains smaller ATS_1 values compared to the SHR and RSR schemes, indicating that the VSI RSR charts are expeditious in uncovering shifts in the process ratio

Table 4

Optimal $(K, d_2, \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\})$ combination of the 7 regions VSI RSR scheme in minimizing the ATS_1 for τ, where n = 5, d_1= 0.1, $(\eta_X, \eta_Y) \in$ (0.2, 0.2), $\rho \in$ {–0.8, –0.4, 0, 0.4, 0.8}, $G \in$ {2, 3} and ATS_0 = 200

ρ	$G = 2$			
	$\tau = 0.9$	$\tau = 0.95$	$\tau = 0.98$	$\tau = 0.99$
–0.8	0.9977,1.0616,0,1,3,3,4,8,15	0.9994,1.0681,1,1,2,2,4,4,13	0.9994,1.0681,1,1,2,2,4,4,13	0.9994,1.0681,1,1,2,2,4,4,13
–0.4	0.9988,1.0652,1,1,2,3,5,6,15	0.9988,1.0652,1,1,2,3,5,6,15	0.9994,1.0681,1,1,2,2,4,4,13	0.9994,1.0681,1,1,2,2,4,4,13
0	0.9971,1.0610,0,1,2,4,6,6,15	0.9997,1.0680,1,1,2,2,4,4,13	0.9997,1.0680,1,1,2,2,4,4,13	0.9997,1.0680,1,1,2,2,4,4,13
0.4	0.9977,1.0613,0,1,2,4,6,6,15	0.9997,1.0680,1,1,2,2,4,4,13	0.9997,1.0680,1,1,2,2,4,4,13	0.9997,1.0680,1,1,2,2,4,4,13
0.8	0.9981,1.869,0,0,1,3,7,9,14	0.9994,1.0605,0,1,3,3,4,8,15	0.9999,1.0680,1,1,2,2,4,4,13	0.9999,1.0680,1,1,2,2,4,4,13
	$\tau = 1.01$	$\tau = 1.02$	$\tau = 1.05$	$\tau = 1.1$
–0.8	0.9565,1.0581,0,1,2,2,3,4,14	0.9629,1.0499,0,1,2,3,4,4,15	0.9646,1.0503,0,1,2,3,4,5,15	0.9596,1.0563,0,0,2,3,4,5,14
–0.4	0.9615,1.0570,0,1,2,2,3,4,14	0.9672,1.0495,0,1,2,3,4,4,15	0.9687,1.0505,0,1,2,3,4,5,15	0.9662,1.0628,0,0,1,2,2,3,8
0	0.9673,1.0569,0,1,2,2,3,4,14	0.9721,1.0507,0,1,2,3,4,4,15	0.9733,1.0519,0,1,2,3,4,5,15	0.9713,1.0617,0,0,1,2,2,3,8

Contd...

0.4	0.9782,1.0515,0,1,2,3,4,4,15	0.9782,1.0515,0,1,2,3,4,4,15	0.9763,1.0566,0,0,2,3,4,5,14	0.9763,1.0606,0,0,1,2,3,5,10
0.8	0.9874,1.0507,0,1,2,3,4,4,15	0.9880,1.0506,0,1,2,3,4,5,15	0.9870,1.0635,0,0,1,2,2,3,8	0.9854,1.0644,0,0,0,3,3,5,10
	$G = 3$			
	$\tau = 0.9$	$\tau = 0.95$	$\tau = 0.98$	$\tau = 0.99$
−0.8	0.9977,1.1821,0,1,3,3,4,8,15	0.9994,1.1843,1,1,2,2,4,4,13	0.9994,1.1843,1,1,2,2,4,4,13	0.9589,1.1695,0,1,2,3,3,4,15
−0.4	0.9988,1.2473,1,1,2,3,5,6,15	0.9988,1.2473,1,1,2,3,5,6,15	0.9994,1.1843,1,1,2,2,4,4,13	0.9994,1.1843,1,1,2,2,4,4,13
0	0.9971,1.2081,0,1,2,4,6,6,15	0.9988,1.2476,1,1,2,3,5,6,15	0.9997,1.1842,1,1,2,2,4,4,13	0.9997,1.1842,1,1,2,2,4,4,13
0.4	0.9977,1.2086,0,1,2,4,6,6,15	0.9988,1.2486,1,1,2,3,5,6,15	0.9997,1.1842,1,1,2,2,4,4,13	0.9997,1.1842,1,1,2,2,4,4,13
0.8	0.9982,1.2740,0,0,1,4,4,7,12	0.9985,1.2100,0,1,2,4,6,6,15	0.9999,1.1842,1,1,2,2,4,4,13	0.9999,1.1842,1,1,2,2,4,4,13
	$\tau = 1.01$	$\tau = 1.02$	$\tau = 1.05$	$\tau = 1.1$
−0.8	0.9977,1.1821,0,1,3,3,4,8,15	0.9994,1.1843,1,1,2,2,4,4,13	0.9994,1.1843,1,1,2,2,4,4,13	0.9589,1.1695,0,1,2,3,3,4,15
−0.4	0.9988,1.2473,1,1,2,3,5,6,15	0.9988,1.2473,1,1,2,3,5,6,15	0.9994,1.1843,1,1,2,2,4,4,13	0.9994,1.1843,1,1,2,2,4,4,13
0	0.9971,1.2081,0,1,2,4,6,6,15	0.9988,1.2476,1,1,2,3,5,6,15	0.9997,1.1842,1,1,2,2,4,4,13	0.9997,1.1842,1,1,2,2,4,4,13
0.4	0.9977,1.2086,0,1,2,4,6,6,15	0.9988,1.2486,1,1,2,3,5,6,15	0.9997,1.1842,1,1,2,2,4,4,13	0.9997,1.1842,1,1,2,2,4,4,13
0.8	0.9982,1.2740,0,0,1,4,4,7,12	0.9985,1.2100,0,1,2,4,6,6,15	0.9999,1.1842,1,1,2,2,4,4,13	0.9999,1.1842,1,1,2,2,4,4,13

than the contesting schemes. The boldfaced values in Tables 5 – 8 manifest the cases where the VSI RSR schemes outperform the SHR and RSR schemes. With regards to the value of the short sampling interval (d_1) considered, it is found that $d_1 = 0.1$ gives the best performance. For instance, for $G = 2$, (η_X, η_Y) = (0.01, 0.2), $n = 5$, $\rho = 0$, $\tau = 1.01$ and $d_1 = 0.1$, the ATS_1 values for the VSI RSR (k = 4) and VSI RSR (k = 7) charts are 128.5 and 127.2, respectively, while for the same input parameter combination with $d_1 = 0.3$, the ATS_1 values for the VSI RSR (k = 4) and VSI RSR (k = 7) charts are 129.3 and 127.9, respectively. Additionally, when the non-negative integer that controls the rate of swapping between the short (d_1) and long (d_2) sampling intervals is increased from $G = 2$ to $G = 3$, the time required by the VSI RSR scheme in exposing shifts in the process R is reduced. For example, for $G = 2$, (η_X, η_Y) = (0.2, 0.01), $n = 5$, $\rho = 0.8$, $\tau = 1.01$ and $d_1 = 0.1$, the ATS_1 values for the VSI RSR (k = 4) and VSI RSR (k = 7) schemes are 124.7 and 123.3, respectively, but for the same parameter combination with $G = 3$, the respective ATS_1 values for the VSI RSR schemes with $k = 4$ and 7 are 122.9 and 121.4 (see Table 8). Due to space constraint, the ATS_1 values for $d_1 = 0.5$ are unavailable but these values can be sought from the 1st author.

Table 5

ATS_1 values of the SHR, optimal RSR and optimal VSI RSR schemes, for $(\eta_X, \eta_Y) \in (0.01, 0.01)$, $d_1 \in \{0.1, 0.3\}$, $\rho \in \{-0.8, -0.4, 0, 0.4, 0.8\}$, $n = 5$, $G = \{2, 3\}$ and $ATS_0 = 200$

τ	$G = 2$							$G = 3$						
				$d_1 = 0.1$		$d_1 = 0.3$					$d_1 = 0.1$		$d_1 = 0.3$	
	SH	RS ($k = 4$)	RS ($k = 7$)	VSI RS ($k = 4$)	VSI RS ($k = 7$)	VSI RS ($k = 4$)	VSI RS ($k = 7$)	SH	RS ($k = 4$)	RS ($k = 7$)	VSI RS ($k = 4$)	VSI RS ($k = 7$)	VSI RS ($k = 4$)	VSI RS ($k = 7$)
	$\rho = -0.8$													
0.9	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.95	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.98	3.0	1.6	1.4	**0.8**	**0.7**	**1.0**	**0.9**	3.0	1.6	1.4	**0.7**	**0.5**	**0.9**	**0.7**
0.99	19.1	7.8	6.9	**5.9**	**5.2**	**6.3**	**5.7**	19.1	7.8	6.9	**4.9**	**4.3**	**5.6**	**5.0**
1.01	19.6	6.2	6.0	**4.1**	**4.0**	**4.6**	**4.4**	19.6	6.2	6.0	**3.2**	**3.2**	**3.9**	**3.8**
1.02	3.1	1.3	1.2	**0.6**	**0.6**	**0.8**	**0.8**	3.1	1.3	1.2	**0.5**	**0.4**	**0.7**	**0.6**
1.05	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.1	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\rho = -0.4$													
0.9	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.95	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.98	2.2	1.1	0.9	**0.5**	**0.4**	**0.6**	**0.5**	2.2	1.1	0.9	**0.4**	**0.3**	**0.6**	**0.5**
0.99	14.0	5.9	5.3	**4.3**	**3.7**	**4.7**	**4.1**	14.0	5.9	5.3	**3.4**	**3.0**	**4.0**	**3.6**
1.01	14.3	4.8	4.6	**3.1**	**3.0**	**3.5**	**3.4**	14.3	4.8	4.6	**2.4**	**2.3**	**2.9**	**2.9**
1.02	2.3	0.9	0.8	**0.4**	**0.4**	**0.5**	**0.5**	2.3	0.9	0.8	**0.3**	**0.2**	**0.4**	**0.4**
1.05	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.1	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\rho = 0$													
0.9	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.95	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.98	1.5	0.6	0.5	**0.2**	**0.2**	**0.3**	**0.3**	1.5	0.6	0.5	**0.2**	**0.1**	**0.3**	**0.2**
0.99	9.0	4.1	3.7	**2.8**	**2.4**	**3.1**	**2.7**	9.0	4.1	3.7	**2.1**	**1.8**	**2.6**	**2.3**
1.01	9.2	3.4	3.3	**2.1**	**2.0**	**2.4**	**2.3**	9.2	3.4	3.3	**1.6**	**1.4**	**2.0**	**1.9**
1.02	1.6	0.4	0.4	**0.2**	**0.2**	**0.2**	**0.2**	1.6	0.4	0.4	**0.1**	**0.1**	**0.2**	**0.2**
1.05	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0

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1.1	1.0	0.0	0.0	0.0	00	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\rho = 0.4$													
0.9	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.95	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.98	1.1	0.2	0.1	**0.0**	**0.0**	**0.1**	0.1	1.1	0.2	0.1	**0.0**	**0.0**	**0.1**	**0.0**
0.99	4.4	2.4	2.1	**1.3**	**1.1**	**1.5**	**1.4**	4.4	2.4	2.1	**1.0**	**0.8**	**1.3**	**1.1**
1.01	4.6	1.9	1.8	**1.0**	**1.0**	**1.2**	**1.2**	4.6	1.9	1.8	**0.8**	**0.6**	**1.0**	**1.0**
1.02	1.1	0.1	0.1	**0.0**	**0.0**	0.1	0.1	1.1	0.1	0.1	**0.0**	**0.0**	**0.0**	**0.0**
1.05	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.1	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\rho = 0.8$													
0.9	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.95	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.98	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.99	1.3	0.4	0.3	**0.1**	**0.1**	**0.2**	**0.1**	1.3	0.4	0.3	**0.1**	**0.1**	**0.2**	**0.1**
1.01	1.3	0.2	0.2	**0.1**	**0.1**	**0.1**	**0.1**	1.3	0.2	0.2	**0.1**	**0.1**	**0.1**	**0.1**
1.02	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.05	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.1	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 6

ATS_1 values of the SHR, optimal RSR and optimal VSI RSR schemes, for $(\eta_X, \eta_Y) \in$ (0.2, 0.2), $d_1 \in$ {0.1, 0.3}, $\rho \in$ {–0.8, –0.4, 0, 0.4, 0.8}, n = 5, G = {2, 3} and ATS_0 = 200

τ	$G = 2$							$G = 3$						
				$d_1 = 0.1$		$d_1 = 0.3$					$d_1 = 0.1$		$d_1 = 0.3$	
	SH	RS (k = 4)	RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)	SH	RS (k = 4)	RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)
	$\rho = -0.8$													
0.9	73.5	31.2	28.9	**26.5**	**26.3**	**27.3**	**27.0**	73.5	31.2	28.9	**25.2**	**24.6**	**26.3**	**25.8**
0.95	146.4	98.5	95.2	**89.9**	**89.9**	**90.6**	**90.5**	146.4	98.5	95.2	**88.6**	**88.7**	**89.6**	**89.6**
0.98	189.5	177.4	174.8	**171.2**	**171.2**	**171.4**	**171.4**	189.5	177.4	174.8	**170.8**	**170.8**	**171.1**	**171.1**
0.99	197.3	195.6	193.9	**192.6**	**192.6**	**192.7**	**192.6**	197.3	195.6	193.9	**192.5**	**192.5**	**192.6**	**192.5**
1.01	197.4	162.5	161.5	**160.7**	**159.8**	**161.1**	**160.2**	197.4	162.5	161.5	**159.4**	**158.5**	**160.2**	**159.2**

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1.02	189.9	130.2	128.5	**127.1**	**125.8**	**127.9**	**126.4**	189.9	130.2	128.5	**125.0**	**123.5**	**126.2**	**124.6**
1.05	150.3	65.9	64.2	**61.2**	**60.0**	**62.2**	**60.9**	150.3	65.9	64.2	**58.6**	**57.1**	**60.3**	**58.7**
1.1	84.3	24.9	24.5	**20.7**	**20.4**	**21.6**	**21.3**	84.3	24.9	24.5	**18.7**	**18.4**	**20.1**	**19.8**
	$\rho = -0.4$													
0.9	61.2	23.9	22.4	**20.3**	**19.6**	**21.0**	**20.3**	61.2	23.9	22.4	**19.0**	**18.0**	**20.0**	**19.0**
0.95	136.0	82.7	81.9	**76.6**	**76.5**	**77.4**	**77.2**	136.0	82.7	81.9	**75.3**	**74.8**	**76.3**	**75.9**
0.98	186.9	168.5	168.5	**164.4**	**164.4**	**164.7**	**164.6**	186.9	168.5	168.5	**163.9**	**163.9**	**164.3**	**164.3**
0.99	196.6	192.5	192.3	**190.6**	**190.6**	**190.7**	**190.6**	196.6	192.5	192.3	**190.4**	**190.4**	**190.6**	**190.5**
1.01	196.7	157.9	156.7	**156.0**	**155.0**	**156.3**	**155.4**	196.7	157.9	156.7	**154.4**	**153.4**	**155.2**	**154.2**
1.02	187.4	122.4	120.7	**119.3**	**117.7**	**119.9**	**118.4**	187.4	122.4	120.7	**116.8**	**115.3**	**118.1**	**116.5**
1.05	140.5	56.8	55.3	**52.0**	**51.1**	**53.0**	**52.0**	140.5	56.8	55.3	**49.4**	**48.2**	**51.0**	**49.8**
1.1	71.4	19.9	19.6	**16.0**	**15.9**	**16.9**	**16.7**	71.4	19.9	19.6	**14.3**	**14.0**	**15.5**	**15.3**
	$\rho = 0$													
0.9	46.0	17.0	15.6	**13.8**	**13.1**	**14.5**	**13.7**	46.0	17.0	15.6	**12.7**	**11.7**	**13.6**	**12.7**
0.95	120.3	65.7	63.5	**59.2**	**59.2**	**60.0**	**60.0**	120.3	65.7	63.5	**57.8**	**57.7**	**58.9**	**58.9**
0.98	182.2	158.4	156.5	**152.0**	**152.1**	**152.5**	**152.4**	182.2	158.4	156.5	**151.5**	**151.5**	**152.0**	**152.0**
0.99	195.3	189.6	188.1	**186.1**	**186.3**	**186.4**	**186.3**	195.3	189.6	188.1	**186.1**	**186.1**	**186.2**	**186.2**
1.01	195.4	150.9	149.7	**148.5**	**147.5**	**149.2**	**148.0**	195.4	150.9	149.7	**146.8**	**145.8**	**147.6**	**146.6**
1.02	182.9	111.2	109.6	**107.6**	**106.1**	**108.7**	**106.8**	182.9	111.2	109.6	**105.0**	**103.5**	**106.4**	**104.9**
1.05	125.3	45.4	44.2	**40.6**	**40.0**	**41.7**	**41.0**	125.3	45.4	44.2	**38.2**	**37.3**	**39.7**	**38.8**
1.1	55.1	14.6	14.4	**11.2**	**11.1**	**12.0**	**11.8**	55.1	14.6	14.4	**9.7**	**9.5**	**10.8**	**10.6**
	$\rho = 0.4$													
0.9	27.4	10.0	9.0	**7.7**	**6.9**	**8.2**	**7.4**	27.4	10.0	9.0	**6.8**	**5.9**	**7.6**	**6.6**
0.95	93.6	43.3	40.5	**37.8**	**37.8**	**38.6**	**38.6**	93.6	43.3	40.5	**36.3**	**36.0**	**37.5**	**37.2**
0.98	172.2	138.6	136.1	**130.5**	**130.5**	**131.0**	**130.9**	172.2	138.6	136.1	**129.6**	**129.6**	**130.3**	**130.2**
0.99	192.4	183.0	180.9	**178.1**	**178.1**	**178.3**	**178.2**	192.4	183.0	180.9	**177.8**	**177.8**	**178.0**	**178.0**
1.01	192.5	138.2	136.7	**135.6**	**134.3**	**136.1**	**134.8**	192.5	138.2	136.7	**133.4**	**132.2**	**134.5**	**133.2**
1.02	173.1	92.7	90.9	**88.9**	**87.1**	**89.8**	**88.0**	173.1	92.7	90.9	**85.8**	**84.3**	**87.5**	**85.8**
1.05	99.1	30.7	30.1	**26.2**	**25.9**	**27.2**	**26.8**	99.1	30.7	30.1	**24.1**	**23.7**	**25.5**	**25.1**
1.1	34.1	9.1	8.8	**6.4**	**6.3**	**7.0**	**6.9**	34.1	9.1	8.8	**5.3**	**5.2**	**6.1**	**6.0**
	$\rho = 0.8$													
0.9	6.6	3.1	2.7	**1.8**	**1.6**	**2.2**	**1.9**	6.6	3.1	2.7	**1.4**	**1.1**	**1.8**	**1.5**

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0.95	39.8	14.3	12.9	**11.4**	**10.7**	**12.1**	**11.3**	39.8	14.3	12.9	**10.4**	**9.5**	**11.3**	**10.4**
0.98	133.8	79.6	76.2	**72.2**	**72.3**	**73.1**	**73.0**	133.8	79.6	76.2	**70.9**	**70.9**	**72.0**	**72.0**
0.99	178.9	151.0	148.9	**143.8**	**143.9**	**144.4**	**144.3**	178.9	151.0	148.9	**143.2**	**143.2**	**143.8**	**143.8**
1.01	179.2	103.1	101.3	**99.4**	**97.7**	**100.2**	**98.5**	179.2	103.1	101.3	**96.5**	**95.0**	**98.1**	**96.4**
1.02	135.6	52.0	50.6	**47.2**	**46.4**	**48.3**	**47.3**	135.6	52.0	50.6	**44.7**	**43.6**	**46.3**	**45.1**
1.05	43.9	11.4	11.1	**8.3**	**8.3**	**9.0**	**8.9**	43.9	11.4	11.1	**7.1**	**6.9**	**8.0**	**7.9**
1.1	8.8	3.1	3.0	**1.8**	**1.8**	**2.1**	**2.1**	8.8	3.1	3.0	**1.4**	**1.2**	**1.8**	**1.7**

Table 7

ATS_1 values of the SHR, optimal RSR and optimal VSI RSR schemes, for $(\eta_X, \eta_Y) \in$ (0.01, 0.2), $d_1 \in$ {0.1, 0.3}, $\rho \in$ {−0.8, −0.4, 0, 0.4, 0.8}, n = 5, G = {2, 3} and ATS_0 = 200

τ	$G = 2$							$G = 3$						
				$d_1 = 0.1$		$d_1 = 0.3$					$d_1 = 0.1$		$d_1 = 0.3$	
	SH	RS (k = 4)	RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)	SH	RS (k = 4)	RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)
	$\rho = -0.8$													
0.9	14.1	8.1	6.8	**6.3**	**5.1**	**6.7**	**5.5**	14.1	8.1	6.8	**5.4**	**4.4**	**6.0**	**5.1**
0.95	59.4	36.1	32.8	**34.4**	**28.4**	**35.1**	**29.1**	59.4	36.1	32.8	**33.1**	**26.9**	**34.1**	**28.0**
0.98	142.4	129.1	121.2	132.6	**117.2**	133.2	**117.7**	142.4	129.1	121.2	131.9	**115.9**	132.5	**116.7**
0.99	175.9	179.8	172.5	184.0	**171.5**	184.3	**171.7**	175.9	179.8	172.5	183.9	**171.1**	184.1	**171.4**
1.01	206.2	134.4	132.7	**130.9**	**129.5**	**131.6**	**130.3**	206.2	134.4	132.7	**128.6**	**127.4**	**129.9**	**128.6**
1.02	194.2	88.2	86.3	**83.2**	**81.7**	**84.2**	**82.8**	194.2	88.2	86.3	**80.0**	**78.6**	**81.8**	**80.3**
1.05	118.6	28.4	27.6	**23.6**	**23.3**	**24.7**	**24.3**	118.6	28.4	27.6	**21.2**	**20.9**	**22.8**	**22.5**
1.1	43.5	8.6	8.4	**5.7**	**5.6**	**6.3**	**6.3**	43.5	8.6	8.4	**4.6**	**4.5**	5.5	5.4
	$\rho = -0.4$													
0.9	13.4	7.7	6.5	**6.0**	**4.8**	**6.4**	**5.2**	13.4	7.7	6.5	**5.1**	**4.2**	**5.7**	**4.7**
0.95	57.4	34.8	31.5	**33.0**	**27.2**	**33.8**	**27.9**	57.4	34.8	31.5	**31.8**	**25.8**	**32.8**	**26.8**
0.98	140.7	127.3	119.7	130.8	**115.2**	131.4	**115.7**	140.7	127.3	119.7	130.1	**113.8**	130.7	**114.6**
0.99	175.1	179.1	171.6	183.4	**170.5**	183.7	**170.7**	175.1	179.1	171.6	183.3	**170.1**	183.5	**170.4**

Contd...

1.01	206.5	133.4	131.8	**129.7**	**128.4**	**130.5**	**129.1**	206.5	133.4	131.8	127.5	126.2	128.8	127.4
1.02	193.9	86.8	84.9	**81.6**	**80.2**	**82.8**	**81.3**	193.9	86.8	84.9	**78.5**	**77.1**	**80.4**	**78.8**
1.05	116.9	27.5	26.8	**22.8**	**22.5**	**23.9**	**23.5**	116.9	27.5	26.8	**20.4**	**20.1**	**22.0**	**21.7**
1.1	42.3	8.3	8.1	**5.5**	**5.4**	**6.1**	**6.0**	42.3	8.3	8.1	**4.4**	**4.3**	**5.2**	**5.2**
	$\rho = 0$													
0.9	12.6	7.4	6.2	**5.7**	**4.6**	**6.1**	**5.0**	12.6	7.4	6.2	**4.8**	**4.0**	**5.4**	**4.5**
0.95	55.3	33.5	30.2	**31.7**	**26.1**	**32.5**	**26.8**	55.3	33.5	30.2	**30.5**	**24.6**	**31.5**	**25.7**
0.98	138.8	125.3	117.7	128.9	**113.1**	129.4	**113.6**	138.8	125.3	117.7	128.2	**111.6**	128.8	**112.5**
0.99	174.1	178.4	170.7	182.8	**169.8**	183.1	**169.9**	174.1	178.4	170.7	182.7	**169.1**	182.9	**169.4**
1.01	206.7	132.0	130.5	**128.5**	**127.2**	**129.3**	**127.9**	206.7	132.0	130.5	**126.2**	**125.0**	**127.6**	**126.2**
1.02	193.7	85.2	83.4	**80.1**	**78.7**	**81.2**	**79.8**	193.7	85.2	83.4	**77.0**	**75.6**	**78.8**	**77.3**
1.05	115.1	26.6	25.9	**21.9**	**21.6**	**23.0**	**22.6**	115.1	26.6	25.9	**19.6**	**19.3**	**21.2**	**20.9**
1.1	40.9	8.0	7.8	**5.2**	**5.2**	**5.8**	**5.8**	40.9	8.0	7.8	**4.2**	**4.1**	5.0	5.0
	$\rho = 0.4$													
0.9	11.9	7.0	6.2	**5.4**	**4.4**	**5.8**	**4.7**	11.9	7.0	6.2	**4.5**	**3.8**	**5.1**	**4.2**
0.95	53.1	32.2	30.2	**30.4**	**24.9**	**31.1**	**25.7**	53.1	32.2	30.2	**29.2**	**23.5**	**30.1**	**24.5**
0.98	136.8	123.2	121.3	126.9	**110.8**	127.4	**111.3**	136.8	123.2	121.3	126.1	**109.4**	126.7	**110.2**
0.99	173.1	177.6	173.3	182.1	**168.6**	182.5	**168.8**	173.1	177.6	173.3	182.1	**167.9**	182.2	**168.3**
1.01	207.0	131.0	129.3	**127.4**	**125.9**	**128.0**	**126.6**	207.0	131.0	129.3	124.9	123.6	126.2	124.9
1.02	193.4	83.7	81.8	**78.6**	**77.1**	**79.6**	**78.2**	193.4	83.7	81.8	**75.3**	**74.0**	**77.2**	**75.7**
1.05	113.3	25.7	25.0	**21.1**	**20.8**	**22.1**	**21.8**	113.3	25.7	25.0	**18.7**	**18.5**	**20.3**	**20.1**
1.1	39.6	7.7	7.6	**5.0**	**5.0**	**5.6**	**5.6**	39.6	7.7	7.6	**4.0**	**3.9**	**4.8**	**4.8**
	$\rho = 0.8$													
0.9	11.2	6.7	5.9	**5.1**	**4.1**	**5.5**	**4.4**	11.2	6.7	5.9	**4.3**	**3.5**	**4.8**	**4.0**
0.95	50.9	30.9	28.9	**29.0**	**24.4**	**29.8**	**25.1**	50.9	30.9	28.9	**27.9**	**22.8**	**28.8**	**23.8**
0.98	134.7	121.1	119.1	124.7	**109.4**	125.3	**109.9**	134.7	121.1	119.1	123.9	**107.9**	124.6	**108.7**
0.99	172.1	176.8	172.3	181.4	**167.5**	181.7	**167.6**	172.1	176.8	172.3	181.3	**166.8**	181.5	**167.1**
1.01	207.2	129.6	128.0	**125.9**	**124.5**	**126.7**	**125.3**	207.2	129.6	128.0	**123.5**	**122.2**	**124.9**	**123.5**
1.02	193.1	82.1	80.2	**76.8**	**75.4**	**78.0**	**76.6**	193.1	82.1	80.2	**73.6**	**72.3**	**75.5**	**74.1**
1.05	111.4	24.8	24.2	**20.2**	**19.9**	**21.3**	**20.9**	111.4	24.8	24.2	**17.9**	**17.6**	**19.5**	**19.2**
1.1	38.3	7.5	7.3	**4.8**	**4.8**	**5.4**	**5.3**	38.3	7.5	7.3	**3.8**	**3.7**	4.6	4.5

Table 8

ATS_1 values of the SHR, optimal RSR and optimal VSI RSR schemes, for $(\eta_X, \eta_Y) \in$ (0.2, 0.01), $d_1 \in$ {0.1, 0.3}, $\rho \in$ {–0.8, –0.4, 0, 0.4, 0.8}, n = 5, G = {2, 3} and ATS_0 = 200

τ	$G = 2$							$G = 3$						
	SH	RS (k = 4)	RS (k = 7)	$d_1 = 0.1$		$d_1 = 0.3$		SH	RS (k = 4)	RS (k = 7)	$d_1 = 0.1$		$d_1 = 0.3$	
				VSI RS (k = 4)	VSI RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)				VSI RS (k = 4)	VSI RS (k = 7)	VSI RS (k = 4)	VSI RS (k = 7)
	$\rho = -0.8$													
0.9	35.3	9.7	8.7	**7.1**	**6.3**	**7.7**	**6.9**	35.3	9.7	8.7	**6.1**	**5.3**	**6.9**	**6.1**
0.95	112.5	42.5	40.0	**36.5**	**36.4**	**37.4**	**37.2**	112.5	42.5	40.0	**35.1**	**34.6**	**36.3**	**35.8**
0.98	193.4	142.3	139.7	**134.7**	**134.7**	**135.2**	**135.2**	193.4	142.3	139.7	**133.9**	**134.0**	**134.6**	**134.6**
0.99	206.2	188.7	186.8	**184.7**	**184.6**	**184.8**	**184.8**	206.2	188.7	186.8	**184.5**	**184.5**	**184.7**	**184.7**
1.01	176.2	132.1	130.8	**129.8**	**128.6**	**130.4**	**129.1**	176.2	132.1	130.8	**128.4**	**126.8**	**129.1**	**127.7**
1.02	143.8	84.7	83.2	**80.8**	**79.9**	**81.7**	**80.6**	143.8	84.7	83.2	**78.6**	**77.3**	**79.9**	**78.6**
1.05	64.0	25.7	25.2	**22.0**	**21.8**	**22.8**	**22.6**	64.0	25.7	25.2	**20.1**	**19.8**	**21.3**	**21.0**
1.1	17.9	7.2	7.0	**5.2**	**5.0**	**5.7**	**5.5**	17.9	7.2	7.0	**4.3**	**4.2**	**4.9**	**4.8**
	$\rho = -0.4$													
0.9	34.2	9.3	8.3	**6.8**	**6.0**	**7.4**	**6.6**	34.2	9.3	8.3	**5.7**	**5.0**	**6.5**	**5.8**
0.95	110.8	40.8	38.6	**35.2**	**35.1**	**36.1**	**35.9**	110.8	40.8	38.6	**33.7**	**33.2**	**34.9**	**34.5**
0.98	193.1	140.0	137.9	**133.0**	**133.0**	**133.5**	**133.5**	193.1	140.0	137.9	**132.2**	**132.2**	**132.9**	**132.9**
0.99	206.4	187.8	186.3	**184.1**	**184.1**	**184.3**	**184.2**	206.4	187.8	186.3	**183.9**	**183.9**	**184.1**	**184.1**
1.01	175.4	131.1	129.7	**128.6**	**127.4**	**129.2**	**127.9**	175.4	131.1	129.7	**127.1**	**125.6**	**127.8**	**126.4**
1.02	142.1	83.1	81.7	**79.2**	**78.4**	**80.1**	**79.1**	142.1	83.1	81.7	**77.0**	**75.8**	**78.3**	**77.1**
1.05	61.9	24.8	24.3	**21.1**	**21.0**	**22.0**	**21.8**	61.9	24.8	24.3	**19.3**	**19.1**	**20.5**	**20.2**
1.1	17.0	6.9	6.7	**5.0**	**4.8**	**5.4**	**5.2**	17.0	6.9	6.7	**4.1**	**4.0**	**4.7**	4.6
	$\rho = 0$													
0.9	33.1	8.9	8.0	**6.5**	**5.8**	**7.1**	**6.3**	33.1	8.9	8.0	**5.4**	**4.8**	**6.2**	**5.5**
0.95	109.0	39.4	37.3	**33.9**	**33.7**	**34.7**	**34.6**	109.0	39.4	37.3	**32.4**	**31.9**	**33.6**	**33.2**
0.98	192.8	138.2	136.1	**131.1**	**131.1**	**131.7**	**131.6**	192.8	138.2	136.1	**130.3**	**130.3**	**131.0**	**131.0**
0.99	206.7	187.3	185.7	**183.5**	**183.5**	**183.7**	**183.6**	206.7	187.3	185.7	**183.3**	**183.3**	**183.5**	**183.5**
1.01	174.5	129.9	128.4	**127.4**	**126.1**	**127.9**	**126.6**	174.5	129.9	128.4	**125.6**	**124.2**	**126.6**	**125.1**
1.02	140.2	81.6	80.2	**77.7**	**76.8**	**78.5**	**77.5**	140.2	81.6	80.2	**75.3**	**74.2**	**76.7**	**75.5**

Contd...

1.05	59.7	23.9	23.4	**20.3**	**20.2**	**21.1**	**21.0**	59.7	23.9	23.4	**18.5**	**18.3**	**19.7**	**19.4**
1.1	16.1	6.6	6.4	**4.8**	**4.6**	**5.2**	**5.0**	16.1	6.6	6.4	3.9	3.8	4.5	4.4
	$\rho = 0.4$													
0.9	31.9	8.5	7.7	**6.2**	**5.5**	**6.8**	**6.0**	31.9	8.5	7.7	**5.1**	**4.5**	**5.9**	**5.3**
0.95	107.1	38.0	35.9	**32.5**	**32.1**	**33.4**	**33.0**	107.1	38.0	35.9	**31.1**	**30.3**	**32.2**	**31.6**
0.98	192.5	136.4	134.2	**129.2**	**129.2**	**129.7**	**129.7**	192.5	136.4	134.2	**128.3**	**128.4**	**129.1**	**129.0**
0.99	206.9	186.8	185.1	**182.8**	**182.8**	**183.0**	**183.0**	206.9	186.8	185.1	**182.6**	**182.6**	**182.9**	**182.8**
1.01	173.5	128.5	127.1	**126.1**	**124.7**	**126.5**	**125.2**	173.5	128.5	127.1	**124.3**	**122.8**	**125.3**	**123.8**
1.02	138.3	80.0	78.6	**76.1**	**75.1**	**76.8**	**75.9**	138.3	80.0	78.6	**73.7**	**72.6**	**75.1**	**73.9**
1.05	57.4	23.0	22.5	**19.5**	**19.4**	**20.2**	**20.1**	57.4	23.0	22.5	**17.7**	**17.5**	**18.9**	**18.6**
1.1	15.2	6.4	6.1	**4.5**	**4.3**	**4.9**	**4.7**	15.2	6.4	6.1	**3.7**	**3.6**	**4.3**	**4.2**
	$\rho = 0.8$													
0.9	30.7	8.1	7.4	**5.9**	**5.2**	**6.4**	**5.7**	30.7	8.1	7.4	**4.8**	**4.3**	**5.6**	**5.0**
0.95	105.2	36.6	34.5	**31.1**	**30.8**	**32.0**	**31.6**	105.2	36.6	34.5	**29.7**	**29.0**	**30.9**	**30.2**
0.98	192.2	134.5	132.2	**127.1**	**127.1**	**127.7**	**127.6**	192.2	134.5	132.2	**126.2**	**126.3**	**127.0**	**127.0**
0.99	207.2	186.2	184.5	**182.1**	**182.1**	**182.3**	**182.3**	207.2	186.2	184.5	**181.9**	**181.9**	**182.2**	**182.1**
1.01	172.4	127.1	125.7	**124.7**	**123.3**	**125.1**	**123.8**	172.4	127.1	125.7	**122.9**	**121.4**	**123.7**	**122.3**
1.02	136.2	78.1	76.8	**74.3**	**73.4**	**75.0**	**74.2**	136.2	78.1	76.8	**71.9**	**70.9**	**73.2**	**72.2**
1.05	55.1	22.1	21.6	**18.6**	**18.5**	**19.4**	**19.2**	55.1	22.1	21.6	**16.9**	**16.6**	**18.0**	**17.8**
1.1	14.3	6.1	5.8	**4.3**	**4.1**	**4.7**	**4.5**	14.3	6.1	5.8	**3.5**	**3.4**	**4.1**	**4.0**

6. Illustration with an application

The VSI RSR (k = 4) scheme in practice is presented in this section. The data used in this application are adopted from [4]. The data deal with the recipe of muesli brand derived by blending several ingredients. The recipe needs same weight of 'pumpkin seeds' and 'flaxseeds' for the taste of mixture to be preserved. In preserving the taste, the in-control R of blends may encounter undesirable shifts ascribable to setbacks occurring during device administration. Assume that, VSI RSR (k = 4) scheme is used to detect shifts in the R process in-control of $z_0 = \frac{\mu_{X,i}}{\mu_{Y,i}} = 1$, wherein the individual average weights of the mixture are $\mu_{X,i}$ and $\mu_{Y,i}$, and i denotes sample number. The Phase-II data adopted from [4] are given in Table 9. Samples i (= 1, 2, ..., 15), each containing n = 5 containers are realized every 30 minutes. Averages of 'pumpkin seeds' and 'flaxseeds' for samples i (= 1, 2, ..., 15) are computed as $\overline{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{i,j}$ and $\overline{Y}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{i,j}$, respectively. Then the ratios of these samples, i.e., $\hat{Z}_i = \frac{\overline{X}_i}{\overline{Y}_i}$, for i = 1, 2, …, 15, are computed and given in Table 9. By using established data, it is discovered that $X_{i,j}$ and $Y_{i,j}$ could be represented as normal variables with constant CVs of $\eta_X = 0.02$ and $\eta_Y = 0.01$, respectively, hence, $X_{i,j} \sim N(\mu_{X,i}, (0.02 \times \mu_{X,i})^2)$ and $Y_{i,j} \sim N(\mu_{Y,i}, (0.01 \times \mu_{Y,i})^2)$, while the coefficient of correlation between variables $X_{i,j}$ and $Y_{i,j}$ is $\rho = 0.8$ [4].

Table 9

Muesli brand recipe data (Phase-II) for the illustrative application

Observed sample, i	$\bar{X}_i$	$\bar{Y}_i$	$\hat{Z}_i = \frac{\bar{X}_i}{\bar{Y}_i}$	$S(\hat{Z}_i)$	(W_i, F_i)	Sampling interval (d_1 or d_2 hours)	Elapsed time (hours)
1	25.122	25.046	1.003	+0	(+0, 0)	0	0
2	24.956	24.954	1.000	+0	(+0, 0)	1.2085	1.2085
3	25.044	24.929	1.005	+1	(+1, 0)	1.2085	2.4170
4	24.950	24.974	0.999	+0	(+1, 0)	1.2085	3.6255
5	25.111	25.163	0.998	–0	(0, –0)	1.2085	4.8340
6	24.864	24.932	0.997	–0	(0, –0)	1.2085	6.0425
7	49.562	49.588	0.999	+0	(+0, 0)	1.2085	7.2510
8	49.205	49.720	0.990	–1	(0, –1)	1.2085	8.4595
9	49.454	49.781	0.993	–0	(0, –1)	1.2085	9.6680
10	50.192	50.102	1.002	+0	(+0, 0)	1.2085	10.8765
11	50.920	50.045	1.017	+6	(+6, 0)*	1.2085	12.0850
12	51.138	49.966	1.023	+6	(+12, 0)		
13	50.949	50.152	1.016	+2	(+14 0)		
14	50.101	49.712	1.008	+1	(+15, 0)		
15	24.870	24.977	0.996	–0	(0, –0)		

Remark. " * " denotes the first out-of-control signal

The VSI RSR (k = 4) scheme is designed for a quick detection of the shift size, $\tau = 1.01$, where $n = 5$, $\rho = 0.8$, $G = 3$ and $d_1 = 0.1$ hour are earmarked. Inevitably, the optimal parameter and scores of the scheme are computed as $(K, d_2, \{S_1, S_2, S_3, S_4\}) = (0.9988, 1.2085, \{0, 1, 2, 6\})$ using MATLAB. As $z_0 = 1$ and $\omega = z_0 \times \frac{\eta_X}{\eta_Y} = 1 \times \frac{0.02}{0.01} = 2$, then by using Equations (12) – (14), UCL_t, LCL_t (for t = 1, 2, 3, 4) and CL of the VSI RS ratio (k = 4) chart are computed as UCL_1 = 1.0048, UCL_2 = 1.0107, UCL_3 = 1.0167, UCL_4 = ∞, LCL_1 = 0.9928, LCL_2 = 0.9868, LCL_3 = 0.9807, LCL_4 = $-\infty$ and CL = 0.9988.

The preliminary scores (W_0, F_0) = (0, 0) prevailed. Cumulative sums, W_i and F_i of the VSI RSR (k = 4) scheme at sample i, are evaluated based upon vicinity $\hat{Z}_i$ is plotted on the scheme. If $\hat{Z}_i \in (UCL_{t-1}, UCL_t)$, for t = 1, 2, 3 and 4, the accumulated sum W_i is calculated as $W_i = W_{i-1} + S_t$ (see Equations (15) and (17)), while the accumulated sum F_i is altered as zero (see Equation (18)). concurrently, if $\hat{Z}_i \in (LCL_t, LCL_{t-1})$, for t = 1, 2, 3 and 4, the accumulated sum F_i is computed as $F_i = F_{i-1} - S_t$ (see Equations (16) and (18)) and W_i is altered as zero (see Equation (17)). If $W_i \geq S_4$ (= +6) or $F_i \leq -S_4$ (= –6) the VSI RSR (k = 4) scheme gives an out-of-control signal at sample i.

Presumably, sample 1 (i = 1) is taken instantaneously when Phase-II process monitoring begins. Thus, in Table 9, the elapsed time corresponding to sample 1 is 0 hour. The VSI RSR (k = 4) scheme's statistic calibrated at sample 1 is $\hat{Z}_1$ = 1.003, which plots in the interval $[CL, UCL_1)$. Hence, the score $S(\hat{Z}_1)$ = $+S_1$ = +0 is added to W_0 (= 0) and F_1 is renew as zero, thus, conceding (W_1, F_1) = (+0, 0). By using Equation (21), the sampling interval embraced in taking sample 2 (or equivalently the interval length between samples 1 and 2) is d = 1.2085 hours, while the elapsed time until sample 2 is taken is (0+1.2085) hours = 1.2085 hours. Next, $\hat{Z}_2$ = 1.000 is computed at sample 2. As $\hat{Z}_2 \in [CL, UCL_1)$, the score $S(\hat{Z}_2)$ = +0 is added to W_1, while F_2 is reset as zero, which gives (W_2, F_2) = (+0, 0). Based on Equation (21), the lengthy sampling interval, i.e., d_2 = 1.2085 hours, is employed in taking sample 3, resulting in an elapsed time of (1.2085+1.2085) hours = 2.4170 hours until sample 3 is taken. For the third sample, $\hat{Z}_3$ = 1.005 $\in$ $[UCL_1, UCL_2)$, consequently, the score $S(\hat{Z}_3)$ = +1 is add on to W_2 and F_3 is renew to zero, hence, changing the cumulative scores to (W_3, F_3) = (+1, 0) and resulting in d_2 = 1.2085 hours being adopted in taking sample 4, as well as an elapsed time of (2.4170+1.2085) hours = 3.6255 hours until sample 4 is taken.

The approach of assigning scores, updating (W_i, F_i), deciding on the sampling interval for fetching the succeeding sample and computing the elapsed time continues in a similar way for the remaining samples. At sample 11, $\hat{Z}_{11}$ = 1.017 $\in$ $[UCL_3, UCL_4)$, where a score $S(\hat{Z}_{11})$ = +6 is assigned, producing the cumulative scores (W_{11}, F_{11}) = (+6, 0), wherein the VSI RSR (k = 4) scheme issues the 1st out-of-control signal at this sample. This is because W_{11} (= +6) ≥ $+S_4$ (= +6). The elapsed time until the 1st out-of-control signal is issued is 12.0850 hours (see Table 9). Following this

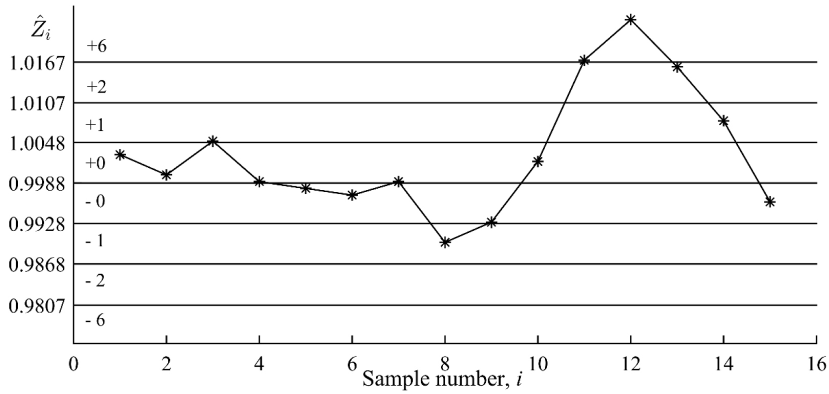

Figure 2

VSI RSR (k = 4) chart with $(K, d_2, \{S_1, S_2, S_3, S_4\})$ = (0.9988, 1.2085, {0, 1, 2, 6}), for the data on muesli brand recipe (Phase-II) in Table 9

signal, a search for the special cause should be carried out so that the necessary curative measures can be implemented. VSI RSR (k = 4) scheme for this example using the information in Table 9 is plotted in Figure 2.

7. Conclusion

Boosting wherewithal of the SHR scheme in uncovering shifts in the RTNVs, i.e., $Z = \frac{X}{Y}$, where X and Y are normally distributed, this study evolves the VSI RSR scheme. A process shift eventuates as the in-control R of Z, i.e., z_0 shifts to an out-of-control value $z_1 = \tau z_0$, where $\tau \neq 1$. The revelation as regards findings in this study is that VSI RSR scheme outshines the SHR and RSR schemes for nearly all shift sizes in the process ratio, with respect to ATS performance. Optimal design, performance assessment and demonstrated application for the new scheme are proffered in this paper. Tables of optimally designed parameters of the new VSI RSR scheme are made available to make the execution of the new scheme seamless so that practitioners will benefit from the findings in this research. Prior to this work, the VSI RSR scheme that monitors the RTNVs could not be found in the existing literature. Areas to be investigated in prospective research works include investigations of the two-sided VSI RSR schemes with measurement errors and in short runs production, as well as the EWMA and CUSUM R schemes for short runs production with and without measurement errors. In addition, an adaptive EWMA R chart can also be developed in further research. An adaptive EWMA scheme works by first estimating the mean shift using an estimator based on an EWMA statistic and then finds the smoothing constant of the scheme by using a continuous function.

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Appendix

The setting up of the tpm for VSI RSR scheme will be explained here. The VSI RSR (k = 4) scheme with scores $\{S_1, S_2, S_3, S_4\}$ = {0, 0, 1, 7} is considered. The procedure explained in [22] and [24] in coming up with the tpm is bring to bear. Suppose the initially assigned scores are (W_0, F_0) = (0, 0), based on the scores $\{S_1, S_2, S_3, S_4\}$ = {0, 0, 1, 7}, all the possible cumulative scores (W_i, F_i) at sample i for the in-control process are (0, 0), (0, –6), (0, –5), (0, –4), (0, –3), (0, –2), (0, –1), (+1, 0), (+2, 0), (+3, 0), (+4, 0), (+5, 0) and (+6, 0), which are designated by states 1, 2, …, 13, respectively (see Table 10). In addition, state 14 (last row and last column in Table 10) stands for the absorbing (Ab) state, where $W_i \geq +7$ or $F_i \leq -7$.

To ascertain how the entries depicted in Table 10 are actualized, consider finding the transition probability from state 1 (current state) to state 8. In state 1, the accumulative scores are (W_i, F_i) = (0, 0), while in state 8, the accumulative scores are (W_{i+1}, F_{i+1}) = (+1, 0). As $W_{i+1} = W_i + S(\hat{Z}_{i+1})$ (see Equation (17)), then $S(\hat{Z}_{i+1}) = W_{i+1} - W_i = +1 - 0 = +1$. But $S(\hat{Z}_{i+1}) = +1$ corresponds to $\hat{Z}_{i+1} \in C_{+3}$. It follows that $\Pr(\hat{Z}_{i+1} \in C_{+3}) = \Pr(\hat{Z}_{i+1} \in [UCL_2, UCL_3)) = \pi_{+3}$, where π_{+3} is obtained using Equation (19). Table 10 details the complete transition probabilities for the tpm of the VSI RSR (k = 4) chart with scores $\{S_1, S_2, S_3, S_4\}$ = {0, 0, 1, 7}. Note that the tpm acquired by getting rid of the last row and last column (Abs state) in Table 10 represents the tpm corresponding to the transient states, $\boldsymbol{Q}$, of the VSI RSR (k = 4) scheme with scores $\{S_1, S_2, S_3, S_4\}$ = {0, 0, 1, 7}.

Table 10

Current state to next state transition probabilities for $\{S_1, S_2, S_3, S_4\}$ = {0, 0, 1, 7}

Current state (sample i)	Cumulative scores (W_i, F_i) at sample i	Next state (sample i + 1)													
		1	2	3	4	5	6	7	8	9	10	11	12	13	Abs
1	(0, 0)	$\pi_{-2}+\pi_{-1}+\pi_{+1}+\pi_{+2}$	0	0	0	0	0	π_{-3}	π_{+3}	0	0	0	0	0	$\pi_{-4}+\pi_{+4}$
2	(0, −6)	$\pi_{+1}+\pi_{+2}$	$\pi_{-2}+\pi_{-1}$	0	0	0	0	0	π_{+3}	0	0	0	0	0	$\pi_{-4}+\pi_{-3}+\pi_{+4}$
3	(0, −5)	$\pi_{+1}+\pi_{+2}$	π_{-3}	$\pi_{-2}+\pi_{-1}$	0	0	0	0	π_{+3}	0	0	0	0	0	$\pi_{-4}+\pi_{+4}$
4	(0, −4)	$\pi_{+1}+\pi_{+2}$	0	π_{-3}	$\pi_{-2}+\pi_{-1}$	0	0	0	π_{+3}	0	0	0	0	0	$\pi_{-4}+\pi_{+4}$
5	(0, −3)	$\pi_{+1}+\pi_{+2}$	0	0	π_{-3}	$\pi_{-2}+\pi_{-1}$	0	0	π_{+3}	0	0	0	0	0	$\pi_{-4}+\pi_{+4}$
6	(0, −2)	$\pi_{+1}+\pi_{+2}$	0	0	0	π_{-3}	$\pi_{-2}+\pi_{-1}$	0	π_{+3}	0	0	0	0	0	$\pi_{-4}+\pi_{+4}$
7	(0, −1)	$\pi_{+1}+\pi_{+2}$	0	0	0	0	π_{-3}	$\pi_{-2}+\pi_{-1}$	π_{+3}	0	0	0	0	0	$\pi_{-4}+\pi_{+4}$
8	(+1, 0)	$\pi_{-2}+\pi_{-1}$	0	0	0	0	0	π_{-3}	$\pi_{+1}+\pi_{+2}$	π_{+3}	0	0	0	0	$\pi_{-4}+\pi_{+4}$
9	(+2, 0)	$\pi_{-2}+\pi_{-1}$	0	0	0	0	0	π_{-3}	0	$\pi_{+1}+\pi_{+2}$	π_{+3}	0	0	0	$\pi_{-4}+\pi_{+4}$
10	(+3, 0)	$\pi_{-2}+\pi_{-1}$	0	0	0	0	0	π_{-3}	0	0	$\pi_{+1}+\pi_{+2}$	π_{+3}	0	0	$\pi_{-4}+\pi_{+4}$
11	(+4, 0)	$\pi_{-2}+\pi_{-1}$	0	0	0	0	0	π_{-3}	0	0	0	$\pi_{+1}+\pi_{+2}$	π_{+3}	0	$\pi_{-4}+\pi_{+4}$
12	(+5, 0)	$\pi_{-2}+\pi_{-1}$	0	0	0	0	0	π_{-3}	0	0	0	0	$\pi_{+1}+\pi_{+2}$	π_{+3}	$\pi_{-4}+\pi_{+4}$
13	(+6, 0)	$\pi_{-2}+\pi_{-1}$	0	0	0	0	0	π_{-3}	0	0	0	0	0	$\pi_{+1}+\pi_{+2}$	$\pi_{-4}+\pi_{+3}+\pi_{+4}$
Abs	$W_i \geq +7$ or $F_i \leq -7$	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Remark. Abs denotes the absorbing state

References

[1] S. S. Abubakar, M. B. C. Khoo, S. Saha, and W. L. Teoh, "RS control chart for monitoring the ratio of population means of a bivariate normal distribution," *Communications in Statistics - Theory and Methods,* vol. 51, no. 13, pp. 4559–4588 (2022).

[2] S. S. Abubakar, M. B. C. Khoo, S. Saha, A. A. Umar, and W. C. Yeong, "RS ratio chart when measurement errors are present," *Quality and Reliability Engineering International,* vol. 38, no. 8, pp. 4049–4072 (2022).

[3] A. Cedilnik, K. Kosmelj, and A. Blejec, "The distribution of the ratio of jointly normal variables," *Metodoloski Zvezki,* vol. 1, no. 1, pp. 99–108 (2004).

[4] G. Celano and P. Castagliola, "Design of phase II control chart for monitoring the ratio of two normal variables," *Quality and Reliability Engineering International,* vol. 32, no. 1, pp. 291–308 (2016).

[5] G. Celano and P. Castagliola, "A synthetic control chart for monitoring the ratio of two normal variables," *Quality and Reliability Engineering International,* vol. 32, no. 1, pp. 681–696 (2016).

[6] G. Celano, P. Castagliola, A. Faraz, and S. Fichera, "Statistical performance of a control chart for individual observations monitoring the ratio of two normal variables," *Quality and Reliability Engineering International,* vol. 30, no. 8, pp. 1361–1377 (2014).

[7] C. W. Champ and S. E. Rigdon, "An analysis of the run sum control chart," *Journal of Quality Technology,* vol. 29, no. 4, pp. 407–417 (1997).

[8] C. H. Chen and C. Y. Chou, "Optimum profit model under the single variable sampling plan with consumer's quality protection," *Journal of Information and Optimization Sciences*, vol. 42, no. 4, pp. 835–844 (2021).

[9] X. Y. Chew, M. B. C. Khoo, S. Y. Teh, and P. Castagliola, "The variable sampling interval run sum control chart," *Computers & Industrial Engineering*, vol. 90, pp. 25–38 (2015).

[10] A. F. B. Costa, "Charts with variable sample size," *Journal of Quality Technology,* vol. 26, no. 3, pp. 155–163 (1994).

[11] A. F. B. Costa, "Charts with variable sample size and sampling intervals," *Journal of Quality Technology,* vol. 29, no. 2, pp. 197–294 (1997).

[12] A. F. B. Costa, “Charts with variable parameters,” *Journal of Quality Technology*, vol. 31, no. 4, pp. 408–416 (1999).

[13] J. Hayya, D. Armstrong, and N. Gressis, “A note on the ratio of two normally distributed variables,” *Management Science*, vol. 21, no. 11, pp. 1338–1341 (1975).

[14] S. B. Mahadik, “Charts with variable control and warning limits,” *Economic Quality Control*, vol. 28, no. 2, pp. 117–124 (2013).

[15] S. B. Mahadik and D. T. Shirke, “Economic design of a modified variable sample size and sampling interval chart,” *Economic Quality Control*, vol. 22, no. 2, pp. 273–293 (2007).

[16] H. D. Nguyen and K. P. Tran, “Effect of the measurement errors on two one-sided Shewhart control charts for monitoring the ratio of two normal variables,” *Quality and Reliability Engineering International*, vol. 36, no. 5, pp. 1731–1750 (2020).

[17] H. D. Nguyen, K. P. Tran, and C. Heuchenne, “Monitoring the ratio of two normal variables using variable sampling interval exponentially weighted moving average control charts,” *Quality and Reliability Engineering International*, vol. 35, no. 1, pp. 439–460 (2019).

[18] H. D. Nguyen, K. P. Tran, and H. L. Heuchenne, “CUSUM control charts with variable sampling interval for monitoring the ratio of two normal variables,” *Quality and Reliability Engineering International*, vol. 36, no. 2, pp. 474–497 (2020).

[19] H. D. Nguyen, K. P. Tran, and K. D. Tran, “The effect of measurement errors on the performance of the exponentially weighted moving average control charts for the ratio of two normally distributed variables,” *European Journal of Operational Research*, vol. 293, no. 1, pp. 203–218 (2021).

[20] D. Öksoy, E. Boulos, and L. D. Pye, “Statistical process control by the quotient of two correlated normal variables,” *Quality Engineering*, vol. 6, no. 2, pp. 179–194 (1993).

[21] M. R. Reynolds Jr., R. W. Amin, J. C. Arnold, and J. A. Nachlas, “Charts with variable sampling intervals,” *Technometrics*, vol. 30, no. 2, pp. 181–192 (1988).

[22] S. Saha, M. B. C. Khoo, P. S. Ng, and Z. L. Chong, “Variable sampling interval run sum median charts with known and estimated process parameters,” *Computers & Industrial Engineering*, vol. 127, pp. 571–587 (2019).

[23] A. W. Spisak, “A control chart for ratios,” *Journal of Quality Technology*, vol. 22, no. 1, pp. 34–37 (1990).

[24] W. L. Teoh, M. B. C. Khoo, P. Castagliola, W. C. Yeong, and S. Y. Teh, “Run-sum control charts for monitoring the coefficient of variation,” *European Journal of Operational Research*, vol. 257, no. 1, pp. 144–158 (2017).

[25] K. P. Tran, P. Castagliola, and G. Celano, “Monitoring the ratio of two normal variables using run rules type control charts,” *International Journal of Production Research*, vol. 54, no. 6, pp. 1670–1688 (2016).

[26] K. P. Tran, P. Castagliola, and G. Celano, “Monitoring the ratio of two normal variables using EWMA type control charts,” *Quality and Reliability Engineering International*, vol. 32, no. 5, pp. 1853–1869 (2016).

[27] K. P. Tran, P. Castagliola, and G. Celano, “The performance of the Shewhart-RZ control chart in the presence of measurement error,” *International Journal of Production Research*, vol. 54, no. 24, pp. 7504–7522 (2016).

[28] K. P. Tran, P. Castagliola, and G. Celano, “Monitoring the ratio of population means of a bivariate normal distribution using CUSUM type control charts,” *Statistical Papers*, vol. 59, no. 1, pp. 387–413 (2018).

[29] K. D. Tran, Q. U. A. Khaliq, A. A. Nadi, T. H. Nguyen, and K. P. Tran, “One-sided Shewhart control charts for monitoring the ratio of two normal variables in short production runs,” *Journal of Manufacturing Processes*, vol. 69, pp. 273–289 (2021).

[30] K. P. Tran and S. Knoth, “Steady-state ARL analysis of ARL-unbiased EWMA-RZ control chart monitoring the ratio of two normal variables,” *Quality and Reliability Engineering International*, vol. 34, no. 3, pp. 377–390 (2018).

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