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The classical, meta-heuristic and EM algorithms for the estimation of the KM-Weibull distribution under progressively first-failure censoring

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Abstract

In life testing, collecting data is difficult for high-technology products since they are too expensive and have long lifetimes. From this perspective, some censoring schemes are introduced, and the most popular one is the progressive censoring scheme. Recently, this scheme has also been modified and called a progressively-first-failure censoring scheme. This scheme allows lifetimes in short time intervals, even if the products are highly reliable. On the other hand, the maximum likelihood estimation cannot be obtained explicitly in many cases when the data are censored. Therefore, the estimates are obtained via numerical methods. One of the most important problems is determining the initial values for these numerical searching methods. Nowadays, meta-heuristic algorithms become popular for the optimization of a function. In this regard, these algorithms are being used to maximize the likelihood functions in the statistical theory. In this study, we considered the estimation problem for the KM-Weibull distribution under a progressively-first-failure censoring scheme. The fixed point iteration and Expectation-Maximization (EM) algorithms are also derived. Meta-heuristic algorithms and classical methods like Nelder-Mead, BFGS, etc. are employed to maximize the likelihood estimation of unknown parameters. The performances of all methods are compared with a simulation study. The methodology is exemplified through the inclusion of a numerical illustration.

Subject Classification: *62N02, 65K99, 68W50.*

Keywords: *EM algorithm, First-failure censoring sampling plan, KM-Weibull distribution, Maximum likelihood estimator, Meta-heuristic algorithms, Optimization.*

1. Introduction

Advancements in high technology have led to longer lifetimes for produced parts. However, it is difficult to draw statistical conclusions about the lifetime distributions of these parts due to the expected test time for lifetime experiments with high-technology parts is increasing. To overcome this problem, several censoring schemes have been developed such as Type-I, Type-II and progressive censoring which is the most widely used scheme. The progressively first-failure censoring scheme is developed by Wu and Kuş [1] combining the progressive censoring scheme with the first-failure censoring scheme. Since then, the progressively first-failure censoring scheme (PFFCS) is a topic with rapidly developing literature. Before $n \times k$ units are subjected to the life test, n groups of k units are separated. That is, there are n groups with size k before the test. Furthermore, a pre-specified censorship scheme $\mathbf{R} = (R_1,\ R_2,\ \ldots,\ R_m)$ is established. Upon initiating the experiment, all units in the group where the first failure (denoted as $X^{\mathbf{R}}_{1:m:n:k}$) occurs, and the R_1 randomly selected groups are eliminated from the test. Subsequently, all units in the group where the second failure (denoted as $X^{\mathbf{R}}_{2:m:n:k}$) occurs, and the R_2 randomly selected groups are removed from the test. Finally, upon observing the $m(\leq n)$th failure (denoted as $X^{\mathbf{R}}_{m:m:n:k}$), all remaining units are excluded from the test. This censorship scheme is referred to as PFFCS, and $X^{\mathbf{R}}_{1:m:n:k}$ < $X^{\mathbf{R}}_{2:m:n:k}$ $< \cdots <$ $X^{\mathbf{R}}_{m:m:n:k}$ designates a progressive first failure censored sample. The PFFCS is generalized to the many censoring schemes. PFFCS reduces to ordinary order statistics for $k=1$ and $\mathbf{R}$ = (0, …, 0), reduce to progressively censoring scheme for $k=1$ and reduces to Type-II right censoring scheme for $k=1$ and $\mathbf{R}=(0, \ldots, n-m)$. It is noticed that $X^{\mathbf{R}}_{1:m:n:k}$, $X^{\mathbf{R}}_{2:m:n:k}$, …, $X^{\mathbf{R}}_{m:m:n:k}$ can be thought as a progressive censored sample from $1-(1-F(x))^k$.

Parameter estimation problem for several distributions under PFFCS is addressed with the studies conducted by [2-7]. Recently, Saini et al. [8] is studied on the stress-strength reliability function for a generalized Maxwell failure distribution using maximum likelihood estimation (MLE) and Bayesian estimation under PFFCS. Chaturvedi et al. [9] dealt with the estimation and testing procedures for the reliability characteristics of Kumaraswamy-G distributions under the PFFCS.

When the data are censored, it is not always possible to obtain the MLE explicitly due to its complex structure of the likelihood functions. In these cases, some numerical methods such as " Nelder-Mead" , " Quasi-Newton" and " BFGS" are used to compute the estimates. Although these methods can easily be applied via functions such as " optim", " maxLik", " nlm" or " constrOptim" in R, they require an initial value selection for the computation. Since it is not always possible to determine the initial value for the real-life problems properly, the estimates obtained from these functions may lose their reliability. Thus, a need has arisen to use the meta-heuristic algorithms for maximizing the likelihood functions. These algorithms are not affected by the initial search space selection in comparison with the classical numerical methods. These advantageous properties of the meta-heuristic algorithms increased their preferability of them in different areas. Especially in recent years, the studies which aim to develop new algorithms and modify the existing meta-heuristic algorithms to tackle different problems are rapidly contributing to the literature. Some of them are given as follows. Pitombeira-Neto and Prata [10] proposed a new metaheuristic algorithm based on the " fix and optimize" strategy hybridized with random local search. Panwar and Deep [11] modified Grey Wolf Optimization (GWO) algorithm to solve permutation-coded problems such as the traveling salesman problem. Machado-Dominguez et al. [12] developed a metaheuristic algorithm for solving the multimode resource-constrained project scheduling problem with discounted cash flows. Nadimi-Shahraki et al. [13] also modified GWO algorithm to improve the capability of solving engineering problems. Recently, Etminaniesfahani et al. [14] developed a hybrid algorithm based on Artificial Bee Collony (ABC) and the Fibonacci indicator algorithm to improve the local search capacity of ABC.

To the best of our knowledge, there are not many studies conducted on parameter estimation for censored samples using metaheuristic algorithms. Karakoca and Pekgör [15] proposed the usage of the genetic algorithm (GA) instead of the Newton method to maximize the likelihood function for the Weibull distribution parameter under the progressively Type-II censored scheme due to the Newton method's limitations. Recently, doubly Type-II censoring is considered by Yalçınkaya et al. [16] to estimate the scale, location, and skewness parameters of the skew-normal distribution using GA. These studies only considered GA for the parameter estimation problem under different censoring schemes on different distributions. Therefore, considering both several classical and meta-heuristic algorithms for the above-mentioned problem demonstrates the importance of this study.

The Weibull distribution and some modifications of the Weibull distributions are considered in the following papers: [17-21]. Recently, a new lifetime distribution called KM-Weibull is introduced by Kavya and Manoharan [22]. The probability density function (pdf) and cumulative distribution function of KM-Weibull distribution are given, respectively, by

$$f(x;\theta)=\frac{\alpha\beta^{\alpha}}{e-1}x^{\alpha-1}\exp\{-(\beta x)^{\alpha}\}\exp\{e^{-(\beta x)^{\alpha}}\},x>0,\alpha,\beta>0$$

and

$$F(x;\theta)=\frac{e}{e-1}[1-\exp\{-(1-e^{-(\beta x)^{\alpha}})\}],$$

where $\theta=(\alpha,\beta)$ is a parameter vector, $\beta>0$ is a scale parameter and $\alpha>0$ is a shape parameter.

The KM-Weibull distribution finds broad applications in reliability theory and survival analysis. In the study conducted by Kavya and Manoharan [22], remission times (in months) of a random sample of 128 bladder cancer patients are used to indicate the modelling ability of this distribution. The statistical inference for the KM-Weibull distribution under PFFCS has not been examined yet since it is a newly proposed lifetime distribution, and this leads us to discuss this distribution in the present study.

In this study, several meta-heuristic algorithms such as Particle Swarm Optimization (PSO), Ant Lion Optimizer (ALO), GWO, Dragonfly (DA), Firefly (FFA), GA, Grasshopper Optimization Algorithm (GOA), Harmony Search Algorithm (HS), Moth Flame Optimizer (MFO), Sine Cosine Algorithm (SCA), Whale Optimization Algorithm (WOA), Clonal Selection Algorithm (CLOLG), Cuckoo Search (CS), Bat Algorithm (BA), Gravitation Based Search Algorithm (GBS), Black Hole Based Optimization Algorithm (BHO) are used which are available in " metaheuristicOpt" library in R which is provided by Riza and Nugroho [23], and the Simulated Annealing (SANN) algorithm that is available in " optim" function. Besides, classical optimization algorithms such as fixed point iteration, quasi-Newton method (BFGS), conjugate gradients (CG), Nelder-Mead (NM), Newton-Raphson (NR) and Berndt-Hall-Hall-Hausman (BHHH) methods are also considered in this study as well as the EM algorithms. An extensive simulation study is undertaken to assess the performance of these meta-heuristic algorithms based on criteria such as mean squared errors (MSE), bias, computation time, and detection rate, considering selected initial values. It is observed that computation time represents the duration needed to execute a computational process for any search methods.

Paper's structure is as follows: Section 2 introduces the MLEs for KM-Weibull distribution parameters under PFFCs. The derivation of EM and fixed-point iteration algorithms is provided, as these methods have not been explored for this problem before. Section 4 reports the simulation study results. In Section 4, a numerical example is presented to illustrate the performance of the algorithms mentioned earlier. Finally, Section 5 concludes and outlines potential future research.

2. Estimation based on the progressive first-failure censored sample

Let $X^{\mathbf{R}}_{j:m:n:k}$, $j = 1, 2, ..., m$ represent the ordered sample under the PFFCS from the KM-Weibull distribution with the censoring scheme **R**. In this case, the log-likelihood function is proportional to

$$\begin{aligned} l(\theta) \propto\ & m\log(\alpha) + m\alpha\log(\beta) + (\alpha-1)\sum_{j=1}^{m}\log(x_j) \\ & -\sum_{j=1}^{m}(\beta x_j)^{\alpha} + \sum_{j=1}^{m}\exp\{-(\beta x_j)^{\alpha}\} + \sum_{j=1}^{m}(k(R_j+1)-1) \\ & \times\log(\exp\{\exp\{-(\beta x_j)^{\alpha}\}\}-1), \end{aligned} \tag{1}$$

and thus, the associated likelihood equations are found to be

$$\frac{\partial \ell(\theta)}{\partial \alpha} = \frac{m}{\alpha} + m\log(\beta) + \sum_{j=1}^{m}\log(x_j) - \sum_{j=1}^{m} T^{(1)}_{\beta,x_j,\alpha} - \sum_{j=1}^{m} T^{(2)}_{\beta,x_j,\alpha}$$
$$-\sum_{j=1}^{m}\frac{(k(R_j+1)-1)T^{(2)}_{\beta,x_j,\alpha}T^{(3)}_{\beta,x_j,\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}, \tag{2}$$

and

$$\frac{\partial \ell(\theta)}{\partial \beta} = m\alpha - \sum_{j=1}^{m}\alpha(\beta x_j)^{\alpha} - \sum_{j=1}^{m}\frac{\alpha T^{(2)}_{\beta,x_j,\alpha}}{\log(\beta x_j)} - \sum_{j=1}^{m}\frac{(k(R_j+1)-1)\alpha T^{(2)}_{\beta,x_j,\alpha}T^{(3)}_{\beta,x_j,\alpha}}{\log(\beta x_j)[T^{(3)}_{\beta,x_j,\alpha}-1]}, \tag{3}$$

where $T^{(1)}_{a,b,c} = (ab)^{\alpha}\log(ab)$, $T^{(2)}_{a,b,c} = T^{(1)}_{a,b,c}\exp\{-(ab)^{c}\}$ and $T^{(3)}_{a,b,c} = \exp\{\exp\{-(ab)^{c}\}\}$.

The MLEs of parameters should be determined by solving both equations $\frac{\partial \ell(\theta)}{\partial \alpha} = 0$ and $\frac{\partial \ell(\theta)}{\partial \beta} = 0$, simultaneously. In the sequel, the MLEs of α and β are denoted by $\hat{\alpha}$, $\hat{\beta}$, respectively. Since these equations cannot be solved explicitly for α and β, numerical methods should be used. In the following sub-sections, the EM algorithm and fixed point iterations are discussed.

2.1 *EM Algorithm*

The data under PFFCS can be treated as incomplete data, and the EM algorithm can be employed to obtain the MLEs of the parameters. In the following, the notations from Ng et al. [24] are utilized to present the EM methodology. Let the observed and censored data be denoted by $\mathbf{X} = (X^{\mathbf{R}}_{1:m:k:n}, X^{\mathbf{R}}_{2:m:k:n}, \ldots, X^{\mathbf{R}}_{m:m:k:n})$ and $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_m)$, respectively, where Z_j is a $1\times(k(R_j+1)-1)$ vector with $\mathbf{Z}_j = (Z_{j1}, Z_{j2}, \ldots, Z_{j(k(R_j+1)-1)})$, for $j = 1,2,\ldots,m$. The censored data vector $\mathbf{Z}_j$ can be considered as the missing data. Combine $\mathbf{X}$ and $\mathbf{Z}$ to form $\mathbf{W}$, which constitutes the complete data set. The log-likelihood function $\lambda(\theta)$ based on complete data is proportional to

$$\lambda(\theta) \propto n_c\log(\alpha) + n_c\alpha\log(\beta) + (\alpha-1)\sum_{i=1}^{n_c}\log(w_i)$$
$$-\sum_{i=1}^{n_c}(\beta w_i)^{\alpha} + \sum_{i=1}^{n_c}\exp\{-(\beta w_i)^{\alpha}\}$$
$$= n\log(\alpha) + n\alpha\log(\beta) + (\alpha-1)\sum_{j=1}^{m}\log(x_j)$$
$$-\sum_{j=1}^{m}(\beta x_j)^{\alpha} + \sum_{j=1}^{m}\exp\{-(\beta x_j)^{\alpha}\}$$

$$+(\alpha-1)\sum_{j=1}^{m}\sum_{s=1}^{k(R_j+1)-1}\log(z_{js})-\sum_{j=1}^{m}\sum_{s=1}^{k(R_j+1)-1}(\beta z_{js})^{\alpha}$$
$$+\sum_{j=1}^{m}\sum_{s=1}^{k(R_j+1)-1}\exp\{-(\beta z_{js})^{\alpha})\},$$

where $n_c = m+\sum_{j=1}^{m}(k(R_j+1)-1)$, also x_j and z_{js} are realizations of $X^{\mathbf{R}}_{j:m:k:n}$ and Z_{js}, respectively. Then the E-step of the algorithm requires the computation of the conditional expectation

$$E(\lambda(\theta)\,|\,\mathbf{X}=\mathbf{x},\alpha^{(h)},\beta^{(h)}).$$

Hence, to execute the EM algorithm, it is necessary to have the conditional distribution of **Z** given **X**. Using Theorem 3 in Ng et al. [24], the conditional distribution of Z_{js} given $X^{\mathbf{R}}_{j:m:k:n}$ is a truncated KM-Weibull distribution with left truncation at $X^{\mathbf{R}}_{j:m:k:n}$ have pdf

$$f_{Z_{js}}(z_{js})=\frac{f(z_{js})}{1-F(x_j)},z_{js}>x_j,s=1,2,\cdots,k(R_j+1)-1$$
$$=\frac{\alpha}{z_{js}}\frac{T^{(2)}_{\beta,z_{js},\alpha}}{\log(\beta z_{js})}\frac{T^{(3)}_{\beta,z_{js},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}.$$

From complete data, the MLEs of parameters are obtained as simultaneous solutions to the equations

$$\hat{\alpha}=-n_c\left\{n_c\log(\hat{\beta})+\sum_{i=1}^{n_c}\log(w_i)-\sum_{i=1}^{n_c}T^{(1)}_{\hat{\beta},w_i,\hat{\alpha}}-\sum_{i=1}^{n_c}T^{(2)}_{\hat{\beta},w_i,\hat{\alpha}}\right\}^{-1},$$

and

$$\hat{\beta}=\left\{\frac{1}{n_c}\left(\sum_{i=1}^{n_c}w_i^{\hat{\alpha}}+\sum_{i=1}^{n_c}w_i^{\hat{\alpha}}\exp\{-(\hat{\beta}w_i)^{\hat{\alpha}})\}\right)\right\}^{-1/\hat{\alpha}},$$

simultaneously. Explicit expression of the required conditional expectations in the E-step are not necessary, and they can be obtained numerically. In our study, we use the Gauss-Legendre Quadrature approximation to perform the numerical integrals. Then the EM algorithm is derived by

$$\alpha^{(h+1)}=-n_c\left\{n_c\log(\beta^{(h)})+\sum_{j=1}^{m}\log(x_j)-\sum_{j=1}^{m}(\beta^{(h)}x_j)\alpha^{(h+1)}\log(\beta^{(h)}x_j)\right.$$
$$-\sum_{j=1}^{m}(\beta^{(h)}x_j)\alpha^{(h+1)}\log(\beta^{(h)}x_j)\exp\{-((\beta^{(h)}x_j)\alpha^{(h+1)})\}$$

$$+\sum_{j=1}^{m}(k(R_j+1)-1)E(\log Z_{j1}\,|\,x_j,\alpha^{(h)},\beta^{(h)})$$
$$-\sum_{j=1}^{m}(k(R_j+1)-1)E((\beta Z_{1j})^{\alpha}\log(\beta Z_{1j})\,|\,x_j,\alpha^{(h)},\beta^{(h)})$$
$$\left.-\sum_{j=1}^{m}(k(R_j+1)-1)E((\beta Z_{1j})^{\alpha}\log(\beta Z_{1j})\exp\{-(\beta Z_{1j})^{\alpha}\}\,|\,x_j,\alpha^{(h)},\beta^{(h)})\right\}^{-1},$$

and

$$\beta^{(h+1)}=n_c^{\frac{1}{\alpha^{(h+1)}}}\left\{\sum_{j=1}^{m}x_j^{\alpha^{(h+1)}}+\sum_{j=1}^{m}x_j^{\alpha^{(h+1)}}\exp\{-(\beta^{(h+1)}x_j)^{\alpha^{(h+1)}}\}\right.$$
$$+\sum_{j=1}^{m}(k(R_j+1)-1)E(Z_{j1}^{\alpha}\,|\,x_j,\alpha^{(h+1)},\beta^{(h)})$$
$$\left.+\sum_{j=1}^{m}(k(R_j+1)-1)E(Z_{j1}^{\alpha}\exp\{-(\beta Z_{1j})^{\alpha}\}\,|\,x_j,\alpha^{(h+1)},\beta^{(h)})\right\}^{-\frac{1}{\alpha^{(h+1)}}},$$

where

$$\begin{aligned}E(Z_{j1}^{\alpha}\big|x_j,\alpha,\beta)&=\int_0^{\infty}z_j^{\alpha}f_{Z_{j1}}(z_j)dz_j\\&=\int_0^{\infty}\frac{\alpha\beta^{\alpha}z_{j1}^{2\alpha-1}\exp\{-(\beta z_{j1})^{\alpha}\}T^{(3)}_{\beta,Z_{j1},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}dz_j\\&=\int_{-1}^{1}\frac{2(1+y)^{2\alpha-1}}{(1-y)^{2\alpha+1}}\frac{\alpha\beta^{\alpha}\exp\left\{-\left(\beta\frac{1+y}{1-y}\right)^{\alpha}\right\}T^{(3)}_{\beta,\frac{1+y}{1-y},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}dy\\&\approx\sum_{\ell=1}^{N}\overline{w_\ell}\,\frac{2(1+y_\ell)^{2\alpha-1}}{(1-y_\ell)^{2\alpha+1}}\times\frac{\alpha\beta^{\alpha}\exp\left\{-\left(\beta\frac{1+y_\ell}{1-y_\ell}\right)^{\alpha}\right\}T^{(3)}_{\beta,\frac{1+y_\ell}{1-y_\ell},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}.\end{aligned}\tag{4}$$

Here y_ℓ and $\overline{w_\ell}$ are the zeros and the corresponding Christoffel numbers of the Gauss-Legendre Quadrature formula on the interval $(-1,1)$, respectively (Canuto et al. [25]). The $\overline{w_\ell}$ is defined as

$$\overline{w_\ell}=\frac{2}{\left(1-y_\ell^2\right)\left\{\left.\frac{dL_N^{(y)}}{dy}\right|_{y=y_\ell}\right\}^2},\tag{5}$$

where, $L_N^{(.)}$ is the Legendre polynomial of degree N. It is noted that y_ℓ and $\overline{w_\ell}$ can be easily calculated by a R function **legendre.quadrature** in library the **gaussquad**. The R function is written to perform EM algorithm and it is available upon request from the authors.

By similar arguments, the other required expectations are found to be

$$E(Z_{j1}^{\alpha}\exp(-(\beta Z_{j1})^{\alpha})\,|\,x_j,\alpha,\beta) \approx \sum_{\ell=1}^{N}\overline{w}_{\ell}\frac{2(1+y_{\ell})^{2\alpha-1}}{(1-y_{\ell})^{2\alpha+1}}\exp\left\{-\left(\beta\left(\frac{1+y_{\ell}}{1-y_{\ell}}\right)\right)^{\alpha}\right\} \times\frac{\alpha\beta^{\alpha}\exp\left\{-\left(\beta\frac{1+y_{\ell}}{1-y_{\ell}}\right)^{\alpha}\right\}T^{(3)}_{\beta,\frac{1+y_{\ell}}{1-y_{\ell}},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}, \tag{6}$$

$$E(\log(Z_{j1})\,|\,x_j,\alpha,\beta) \approx \sum_{\ell=1}^{N}\overline{w}_{\ell}\frac{2(1+y_{\ell})^{\alpha-1}}{(1-y_{\ell})^{\alpha+1}}\log\left(\frac{1+y_{\ell}}{1-y_{\ell}}\right)\times\frac{\alpha\beta^{\alpha}\exp\left\{-\left(\beta\frac{1+y_{\ell}}{1-y_{\ell}}\right)^{\alpha}\right\}T^{(3)}_{\beta,\frac{1+y_{\ell}}{1-y_{\ell}},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}, \tag{7}$$

$$E(T^{(2)}_{\beta,Z_{j1},\alpha}\,|\,x_j,\alpha,\beta) \approx \sum_{\ell=1}^{N}\overline{w}_{\ell}\frac{2(1+y_{\ell})^{2\alpha-1}}{(1-y_{\ell})^{2\alpha+1}}\log\left(\beta\frac{1+y_{\ell}}{1-y_{\ell}}\right)\times\exp\left\{-\left(\beta\left(\frac{1+y_{\ell}}{1-y_{\ell}}\right)\right)^{\alpha}\right\} \times\frac{\alpha\beta^{2\alpha}\exp\left\{-\left(\beta\frac{1+y_{\ell}}{1-y_{\ell}}\right)^{\alpha}\right\}T^{(3)}_{\beta,\frac{1+y_{\ell}}{1-y_{\ell}},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}, \tag{8}$$

and

$$E(T^{(1)}_{\beta,Z_{j1},\alpha}\,|\,x_j,\alpha,\beta) \approx \sum_{\ell=1}^{N}\overline{w}_{\ell}\frac{2(1+y_{\ell})^{2\alpha-1}}{(1-y_{\ell})^{2\alpha+1}}\log\left(\beta\frac{1+y_{\ell}}{1-y_{\ell}}\right) \times\frac{\alpha\beta^{2\alpha}\exp\left\{-\left(\beta\frac{1+y_{\ell}}{1-y_{\ell}}\right)^{\alpha}\right\}T^{(3)}_{\beta,\frac{1+y_{\ell}}{1-y_{\ell}},\alpha}}{T^{(3)}_{\beta,x_j,\alpha}-1}. \tag{9}$$

For the exact expectations given in Eq. (4), (6)-(9), there is no need the large values of N. For example, the expectation given in Eq. (4) is calculated as 2.2734 for $x_j = 1$, $\alpha = 1$ and $\beta = 2$. The approximation of the expectation in Eq. (4) is presented in Figure 1 for the same values of x_j, α and β.

In the EM algorithm, one-dimensional search method is needed in every step. However, this will cause fewer problems than optimization in two dimensions.

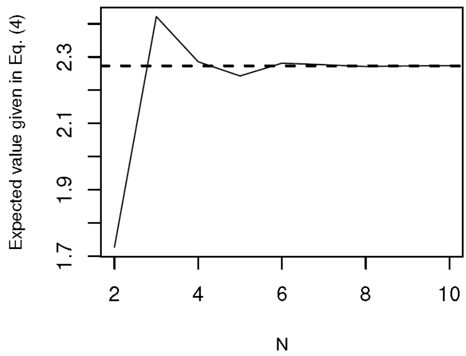

Figure 1

The relationship between the expected value given in (4) and the number of iterations N for $x_j = 1$, $\beta = 2$, , and $\alpha = 1$

2.2 *Fixed Point Iteration*

Fixed point iteration is one of the well-known numerical methods which can be implemented easily. Based on the likelihood gradients given in (2)-(3), the fixed point iterations can be written by

$$\alpha^{(h+1)} = \frac{m}{g(\alpha^{(h)}, \beta^{(h)})},$$

and

$$\beta^{(h+1)} = \left(\frac{m}{h(\alpha^{(h+1)}, \beta^{(h)})} \right)^{1/\alpha^{(h+1)}},$$

where

$$g(\alpha, \beta) = -\left\{ m\log(\beta) + \sum_{j=1}^{m} \log(x_j) - \sum_{j=1}^{m} T^{(1)}_{\beta, x_j, \alpha} - \sum_{j=1}^{m} T^{(2)}_{\beta, x_j, \alpha} \right.$$
$$\left. - \sum_{j=1}^{m} \frac{(k(R_j+1)-1) T^{(2)}_{\beta, x_j, \alpha} T^{(3)}_{\beta, x_j, \alpha}}{T^{(3)}_{\beta, x_j, \alpha} - 1} \right\},$$

and

$$h(\alpha, \beta) = \sum_{j=1}^{m} x_j^{\alpha} + \sum_{j=1}^{m} x_j^{\alpha} \exp\{-(\beta x_j)^{\alpha})\} + \sum_{j=1}^{m} \frac{(k(R_j+1)-1) T^{(2)}_{\beta, x_j, \alpha} T^{(3)}_{\beta, x_j, \alpha}}{\beta^{\alpha} \log(\beta x_j)[T^{(3)}_{\beta, x_j, \alpha} - 1]}$$

It is noticed that the fixed point iterations do not require any one-dimensional searching as in the EM algorithm. However, we will see that the EM algorithm is much stronger in terms of convergence than the fixed point iteration in the next section.

3. Simulation Study

In this section, an extensive simulation study is conducted to compare the performances of the above-mentioned meta-heuristic algorithms and classical methods with different cases when the true parameter is fixed as $\theta = (0.5, 1)$ and (2, 3). Different cases of k and censoring scheme with $m = 50$ as shown in Table 1. The MSE, bias, computation times and detection rates (if the algorithm is converged, then detection is done) are obtained with 1000 replications. The tolerance value is determined as $2.220446e-16,$ and the maximum number of iterations is determined as 500 for the classical, EM and fixed point algorithms. Only the maximum number of iterations is determined as 100, and other parameters are chosen as default for the meta-heuristic algorithms. Results for $m = 50$ are given in Figures 2-13, and the results for $m = 25$ and $m = 100$ are available upon request from the authors. Larger values on the vertical axis are omitted and not shown in figures in order to simplify the comparison among algorithms.

Table 1

The censoring schemes used in the simulation study

Case	(α, β)	m	k	**R**
1	(0.5, 1)	50	2	$(10, 0^{49})$
2	(0.5, 1)	50	2	$(0^{49}, 10)$
3	(0.5, 1)	50	2	$(0^{24}, 10, 0^{25})$
4	(0.5, 1)	50	5	$(10, 0^{49})$
5	(0.5, 1)	50	5	$(0^{49}, 10)$
6	(0.5, 1)	50	5	$(0^{24}, 10, 0^{25})$
7	(2,3)	50	2	$(10, 0^{49})$
8	(2,3)	50	2	$(0^{49}, 10)$
9	(2,3)	50	2	$(0^{24}, 10, 0^{25})$
10	(2,3)	50	5	$(10, 0^{49})$
11	(2,3)	50	5	$(0^{49}, 10)$
12	(2,3)	50	5	$(0^{24}, 10, 0^{25})$

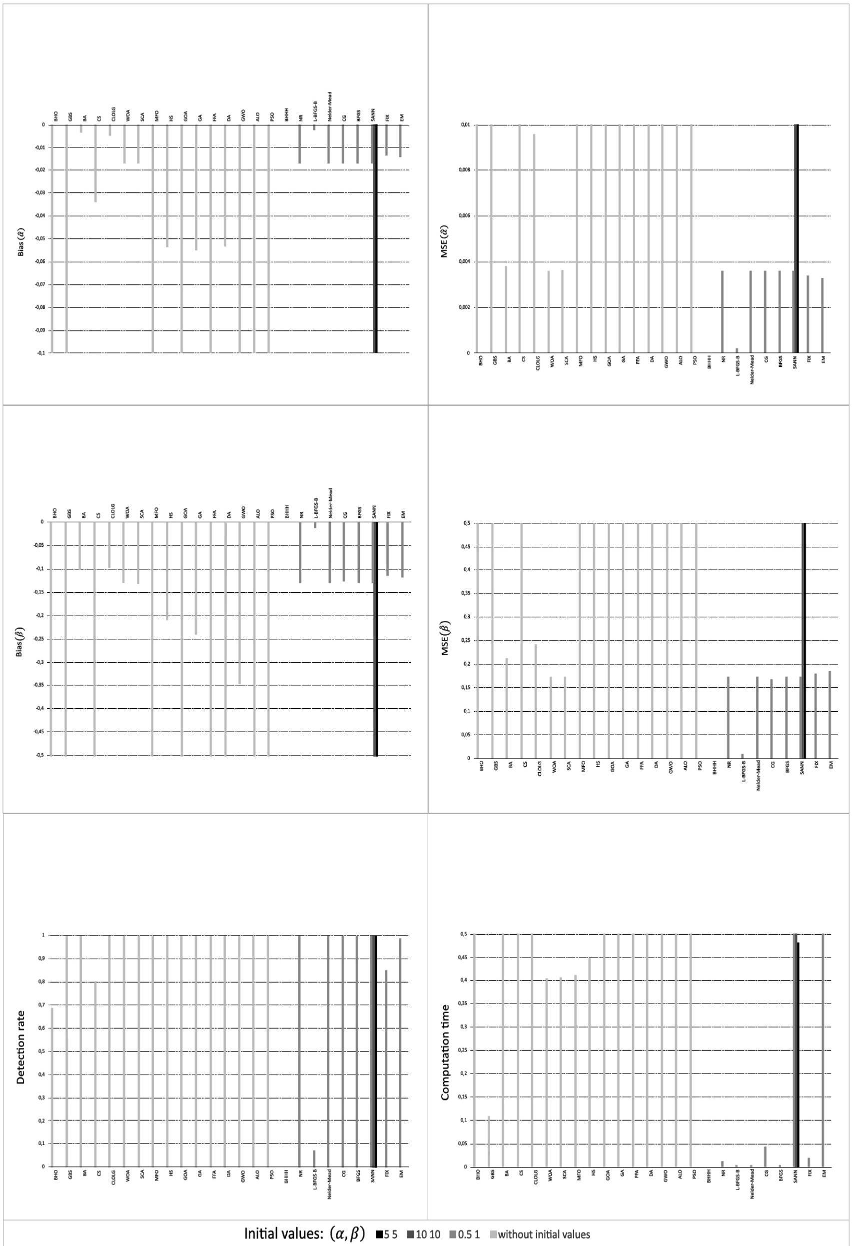

Figure 2

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 1

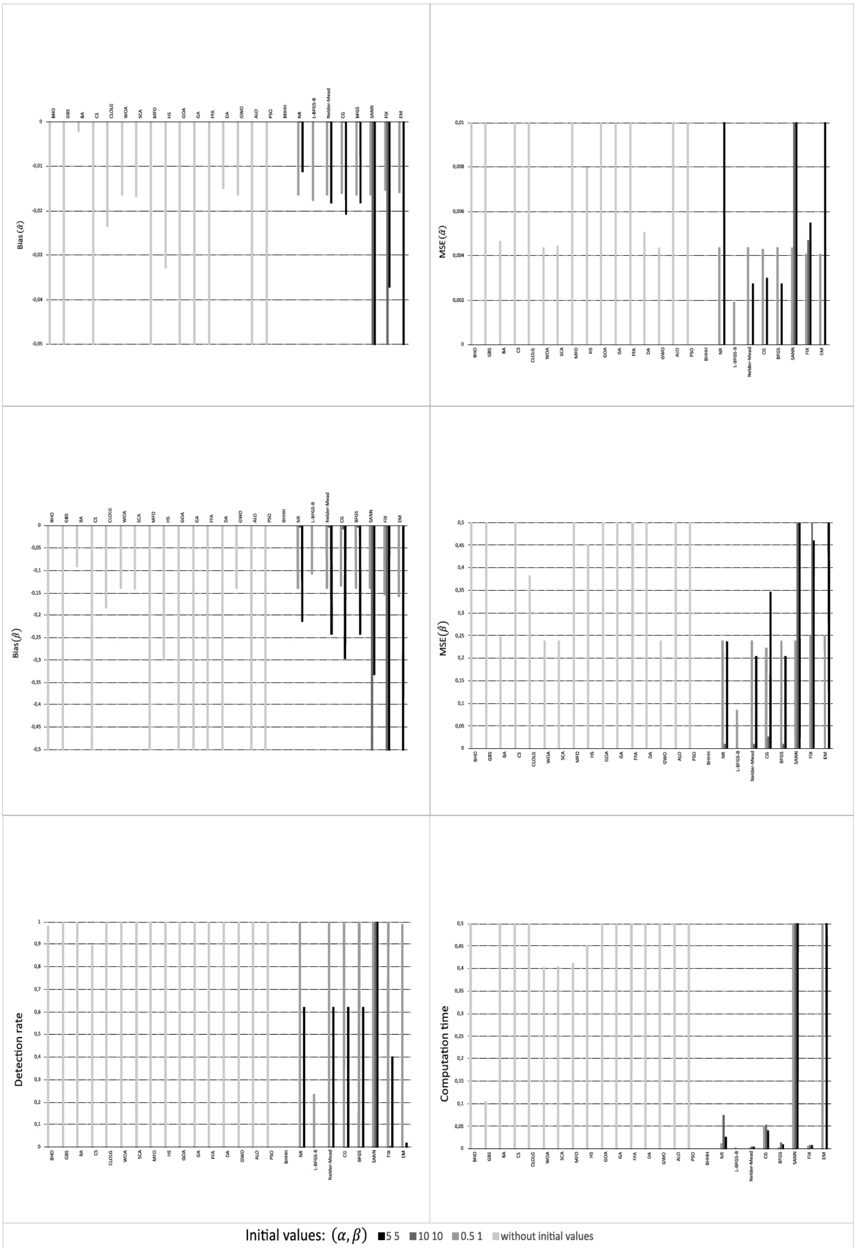

Figure 3

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 2

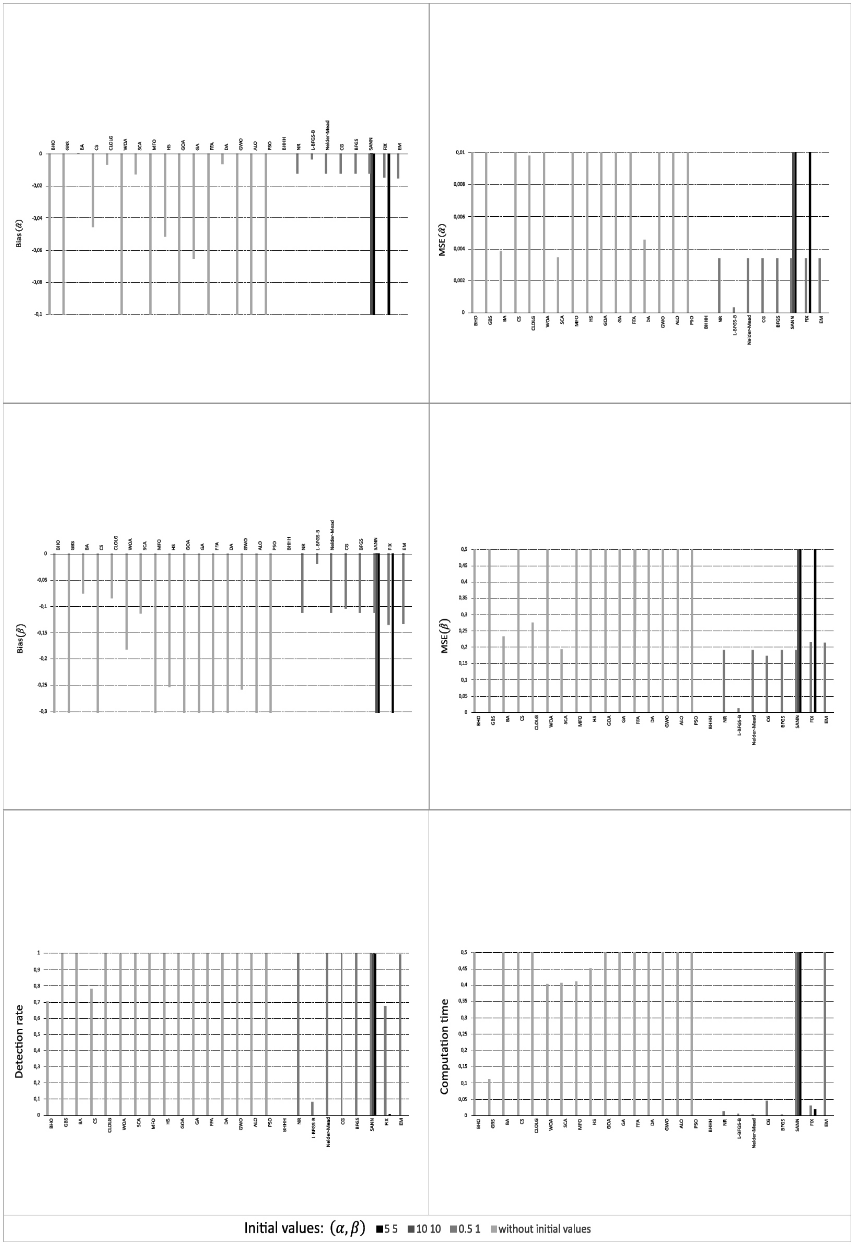

Figure 4

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 3

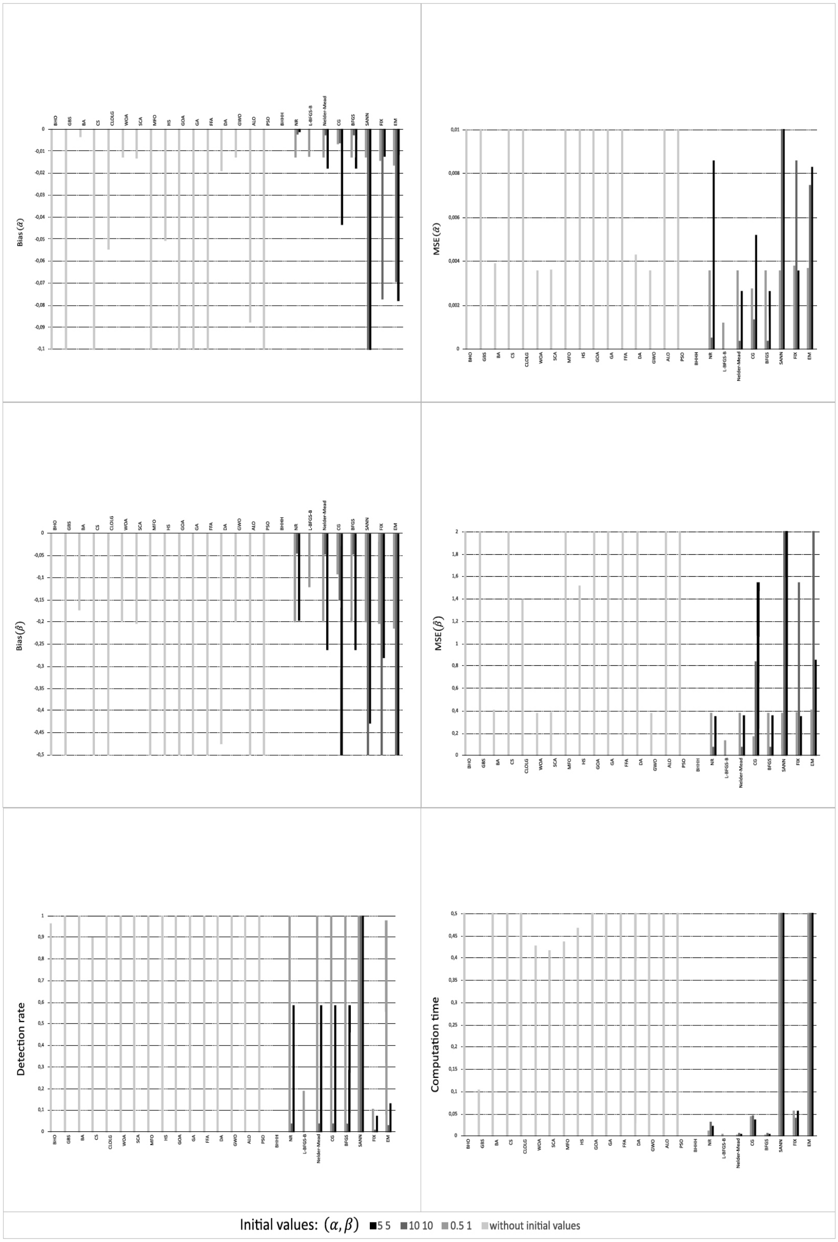

Figure 5

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 4

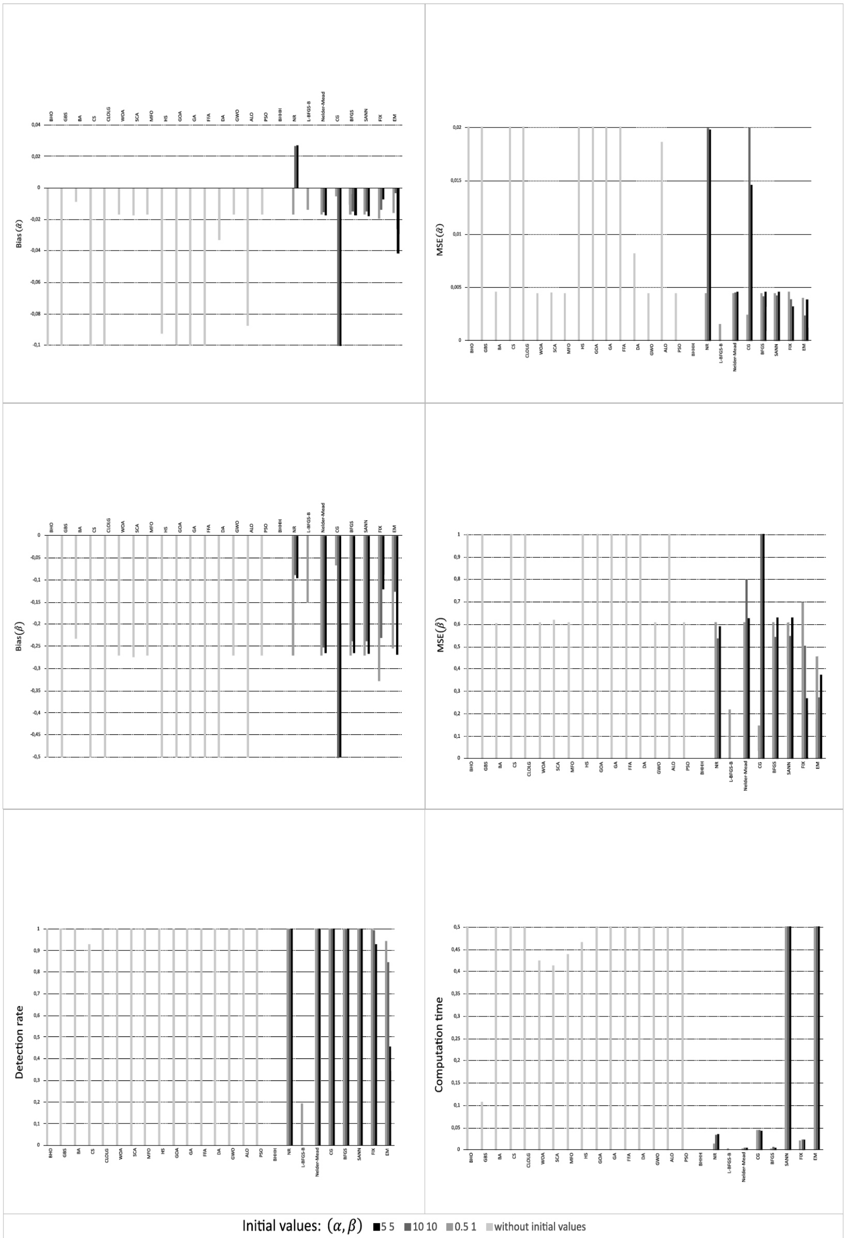

Figure 6

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 5

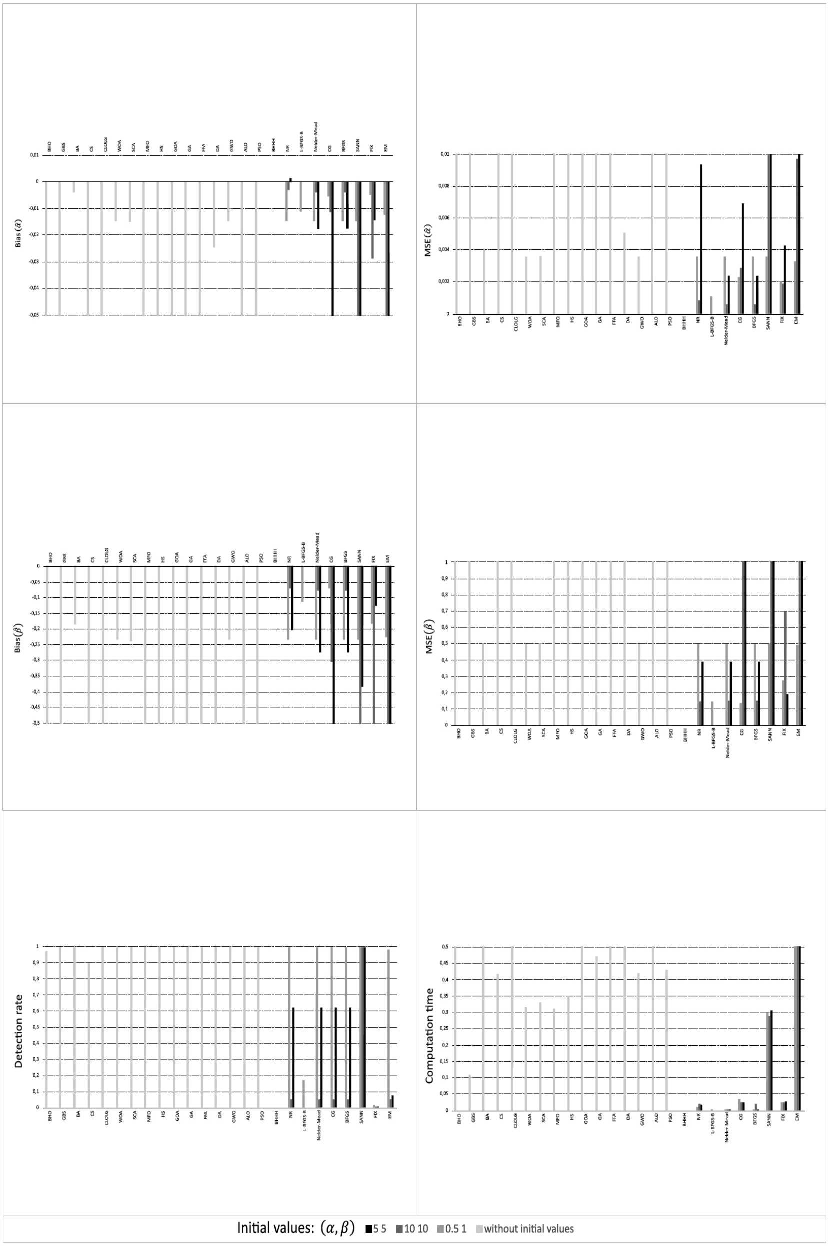

Figure 7

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 6

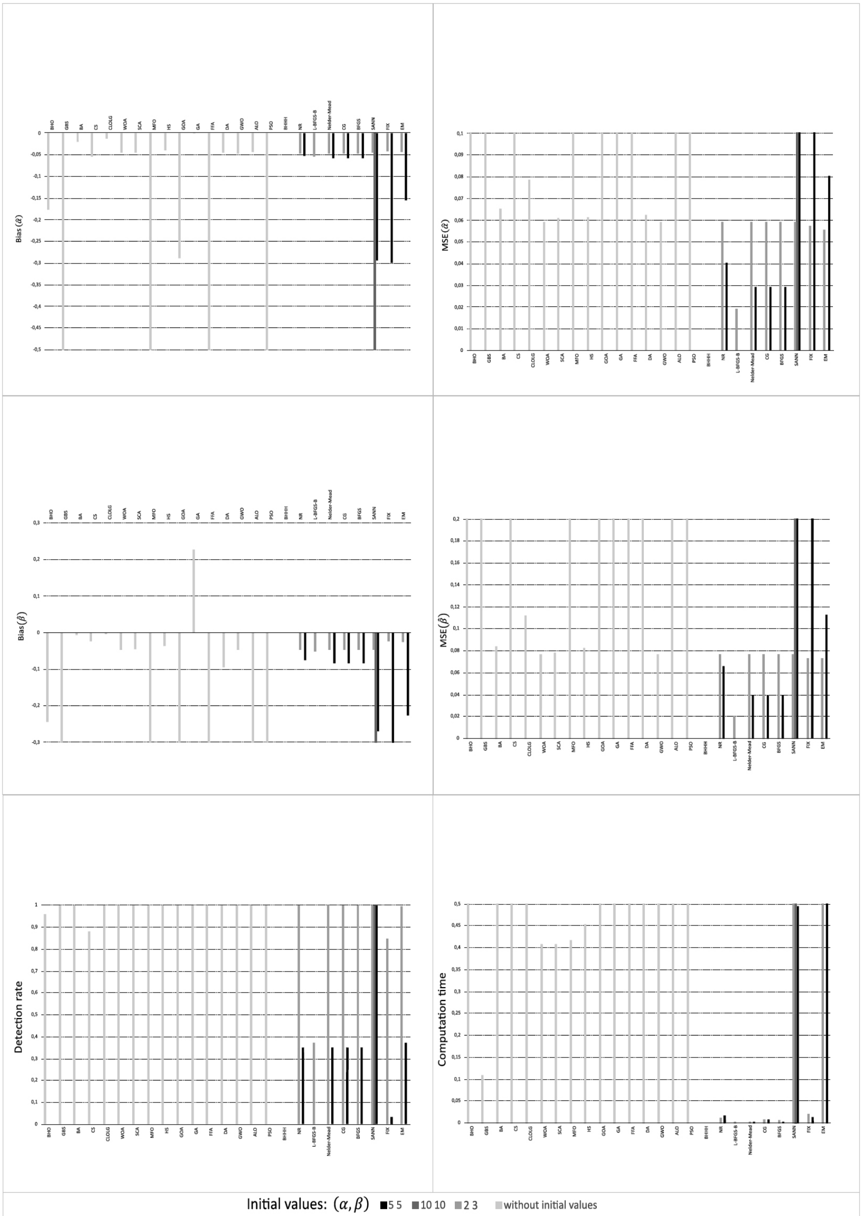

Figure 8

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 7

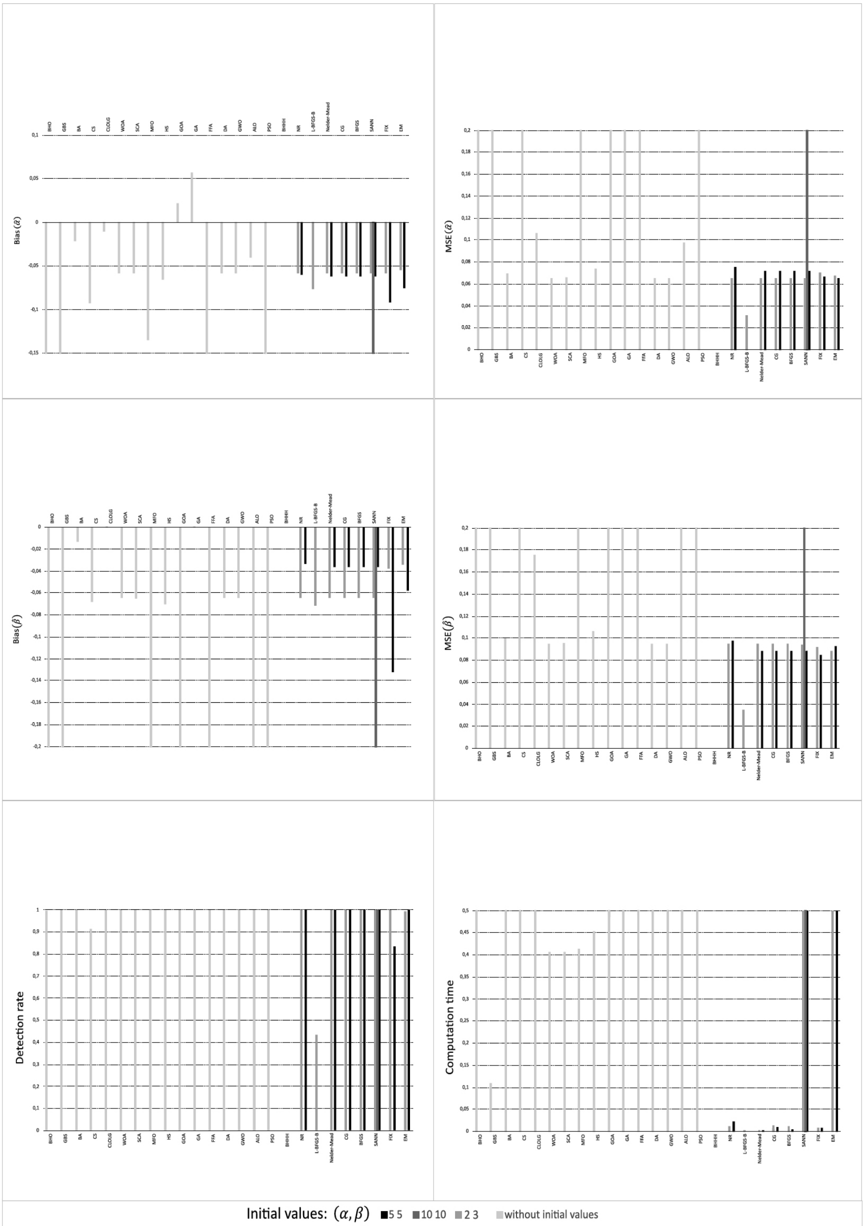

Figure 9

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 8

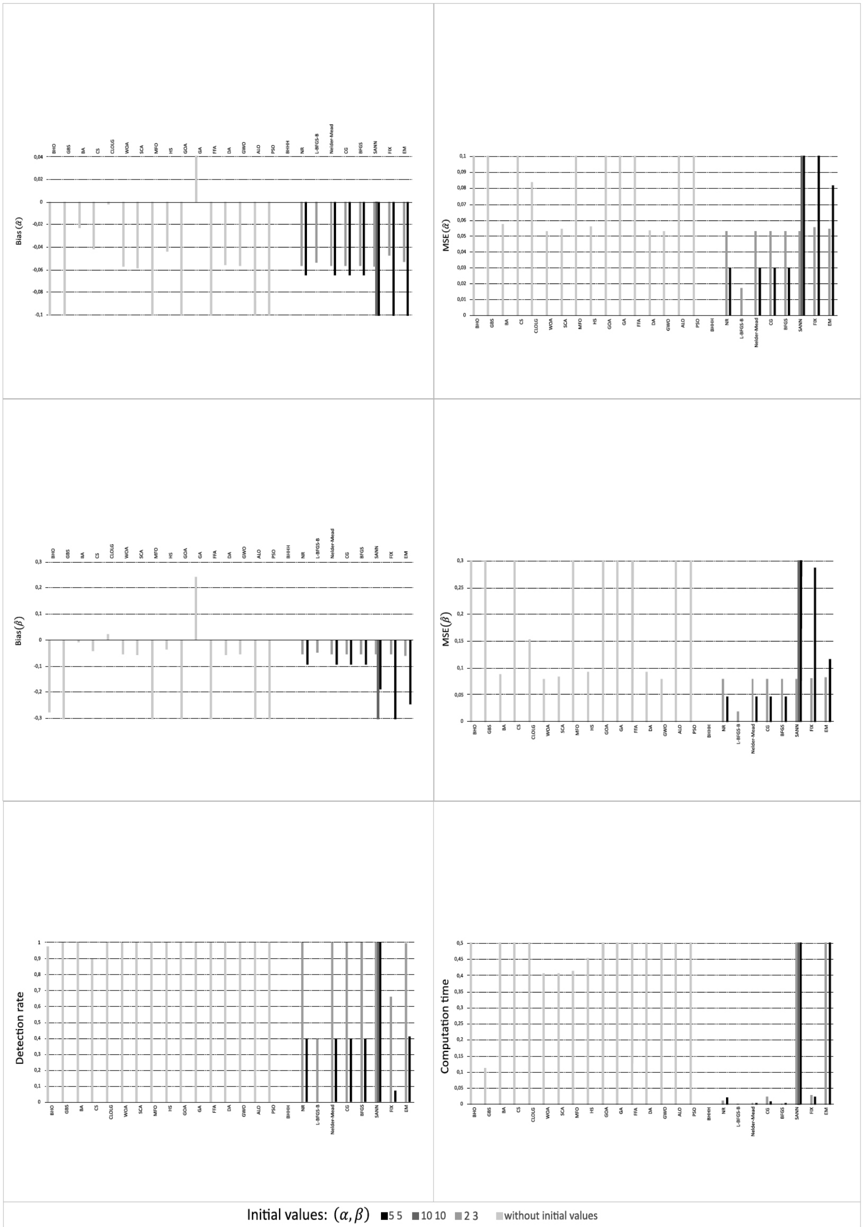

Figure 10

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 9

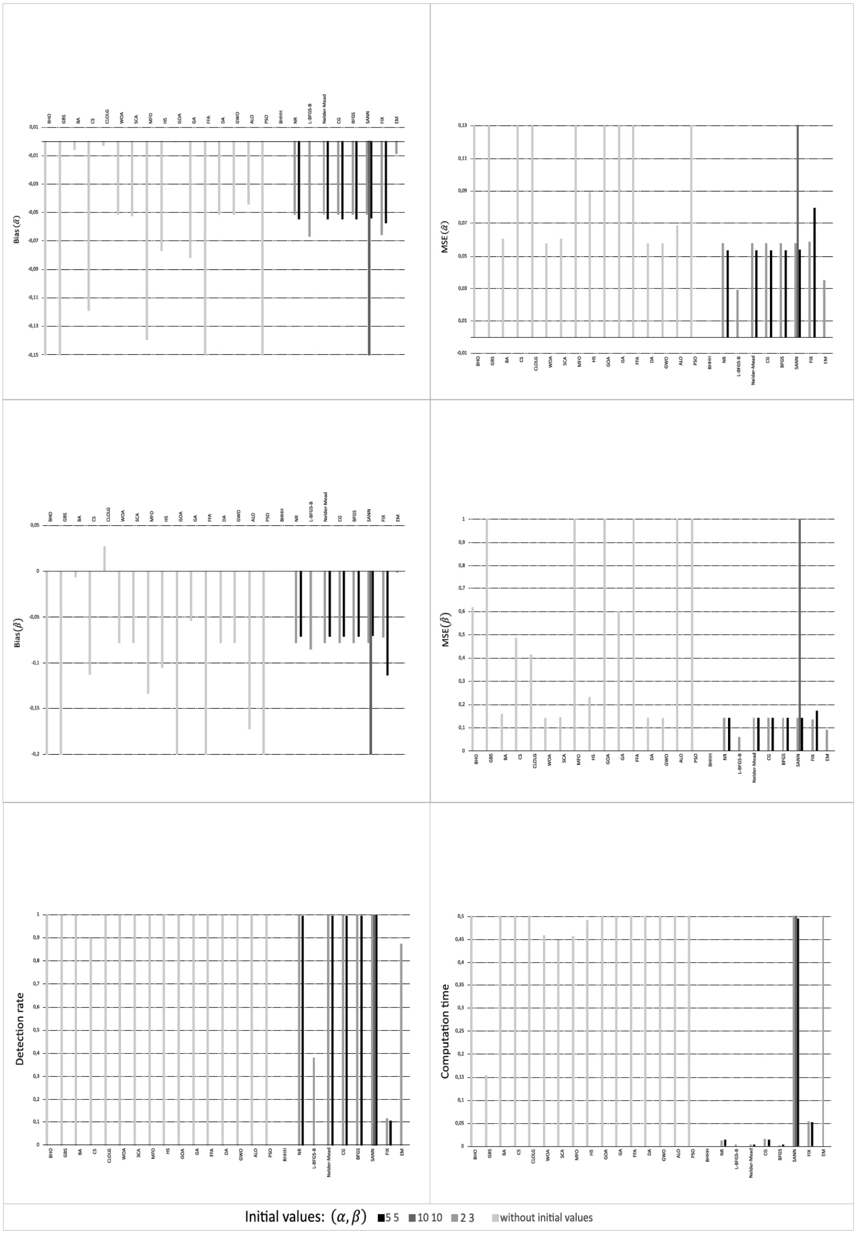

Figure 11

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 10

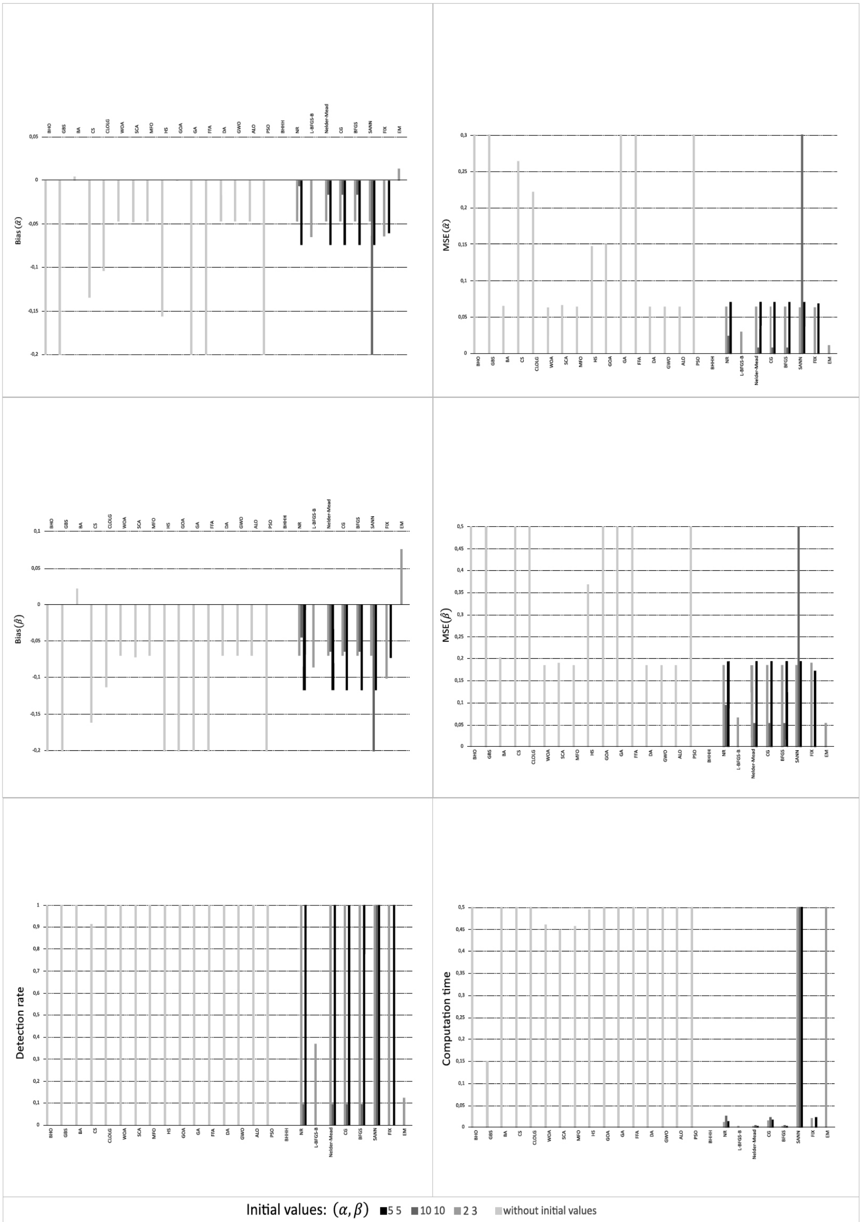

Figure 12

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 11

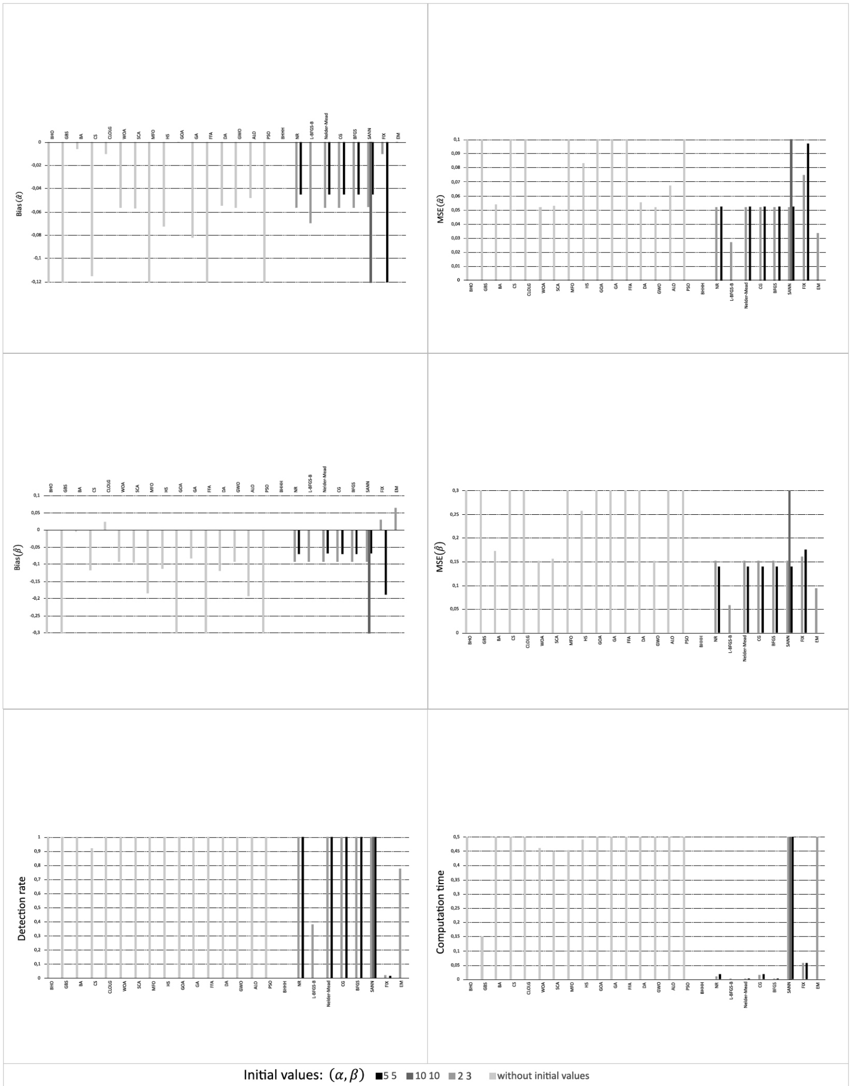

Figure 13

The bias, MSE, computation times and detection rates for the $(\hat{\alpha}, \hat{\beta})$ under Case 12

According to the Figures 2-13, several merit points are achieved. First of all, it should be noted that meta-heuristic algorithms are superior to classical methods because they do not require initial values. It should be noted that the percentage of meta-heuristic algorithms that achieve estimation values (detection rates) is one hundred percent in almost all cases. Dramatically, classical methods are heavily influenced by initial values. The EM algorithm, which is a stochastic method, is also negatively affected by the initial values. With classical methods, the detection rates decrease as the initial values deviate from the true parameters. Not surprisingly, as m increases, MSE and bias values decrease while computation time increases. A disadvantage of meta-heuristic algorithms is that they have much more computation time. In order of best, the meta-heuristic algorithms SCA, BA, WOA, GWO, DA are better than the other algorithms in terms of MSE and bias criteria. Their MSE and bias are almost identical to those of classical methods with true initial parameters. In general, the classical methods and EM algorithms can not reach to the solution without true initial parameters. However, they have good MSE and bias when the true initial parameters are used in the simulation. On the other hand, as the sample size increases, bias and MSE decrease in all cases. Moreover, in the experiment, censoring at the beginning is better than censoring in the middle and at the end in terms of lower bias and MSE.

4. A Numerical Example

In this section, a data set concerning aircraft windshield failure times in Murthy et al. [26] is used for illustration purposes. For the complete data, the parameter estimation of the $\hat{\alpha}$ and $\hat{\beta}$ are obtained using maximum likelihood methodology. The MLEs and also some of the goodness of fit values such as Akaike Information Criterion (AIC), corrected Akaike's Information Criterion (CAIC), Bayesian Information Criterion (BIC), Anderson-Darling (A) and Kolmogorov-Smirnov (KS) and corresponding p-values (in parentheses) are given in Table 2. Since the both p-values are greater than 0.05, it can be concluded that the KM-Weibull distribution is adequate for modeling this data.

Table 2

Some results for the windshield failure times data for complete data

$\hat{\alpha}$	$\hat{\beta}$	$\lambda(\hat{\theta})$	*AIC*	*CAIC*	*BIC*	*A*	*KS*
2.5769	0.3101	-131.0329	266.0659	266.2141	270.9275	0.6124	0.0582
						(0.6358)	(0.9382)

Let us censoring the complete data with the censoring schemes $\mathbf{R}=(3,3,0^{20})$ and $\mathbf{R}=(0^{20},3,3)$ under PFFCS when $k=3$. Then, the censored data are produced as given in Table 3. In both scenarios, $\hat{\alpha}$ and $\hat{\beta}$ are obtained through the classical, meta-heuristic and EM algorithms in which are held in this study. Different initial values for α and β are pre-determined such as (1,1), (5,5) and (10,10) to observe the performances of the meta-heuristic, EM and classical algorithms. It should be noted that the tolerance value considered as 10^{-6} for the EM and fixed point methods to the ease of convergence. Also, set.seed(1) is considered in our R code. Results are given for Scenario 1 and Scenario 2 in Tables 4-5, respectively.

Table 3

Censored data set

Scenario 1:	$k=3$	$\mathbf{R}=(3,3,0^{20})$	data:	0.040	0.301	0.309	0.557,
				0.943	1.070	1.281	1.432,
				1.505	1.506	1.568	1.619,
				1.757	1.911	1.912	2.038,
				2.089	2.224	2.324	2.481,
				2.902	2.962.		
Scenario 2:	$k=3$	$\mathbf{R}=(0^{20},3,3)$	data:	0.040	0.301	0.309	0.557,
				0.943	1.070	1.248	1.281,
				1.281	1.432	1.505	1.506,
				1.568	1.619	1.757	1.911,
				1.912	2.038	2.085	2.089,
				2.154	2.224.		

Table 4

Results for the numerical data under Scenario 1

Methods	Initial Parameters	$\hat{\alpha}$	$\hat{\beta}$	$\ell(\hat{\alpha},\hat{\beta})$
EM	(1,1)	1.9504	0.2623	-52.1079
	(5,5)	*	*	*
	(10,10)	*	*	*
Fixed Point	(1,1)	*	*	*

Contd...

	(5,5)	*	*	*
	(10,10)	*	*	*
SANN	(1,1)	1.9443	0.2614	-52.1081
	(5,5)	1.9640	0.2630	-52.1087
	(10,10)	0.0922	13.8049	-185.5988
BFGS	(1,1)	1.9494	0.2622	-52.1079
	(5,5)	*	*	*
	(10,10)	*	*	*
CG	(1,1)	1.9540	0.2626	-52.1079
	(5,5)	*	*	*
	(10,10)	*	*	*
Nelder-Mead	(1,1)	1.9502	0.2622	-52.1079
	(5,5)	*	*	*
	(10,10)	*	*	*
L-BFGS-B	(1,1)	*	*	*
	(5,5)	*	*	*
	(10,10)	*	*	*
NR	(1,1)	1.9504	0.2623	-52.1079
	(5,5)	*	*	*
	(10,10)	*	*	*
BHHH	(1,1)	*	*	*
	(5,5)	*	*	*
	(10,10)	*	*	*
PSO		1.9496	0.2622	-52.1079
ALO		0.1035	10.1581	-182.5616
GWO		1.9509	0.2623	-52.1079
DA		1.9504	0.2623	-52.1079
FFA		0.0088	4.2664	-218.6301
GA		2.0522	0.2732	-52.1517
GOA		4.4265	0.3589	-66.9463
HS		1.7020	0.2446	-52.4008
MFO		1.9504	0.2623	-52.1079
SCA		2.0217	0.2622	-52.1491
WOA		1.9508	0.2623	-52.1079
CLOLG		2.1951	0.2924	-52.4009
CS		0.6626	0.1819	-71.2277
BA		2.0388	0.2519	-52.3215
GBS		5.2545	15.2169	-Inf
BHO		1.9499	0.2623	-52.1079

*: The method does not provide a solution.

Table 5

Results for the numerical data under Scenario 2

Methods	Initial Parameters	$\hat{\alpha}$	$\hat{\beta}$	$\ell(\hat{\alpha},\hat{\beta})$
EM	(1,1)	1.9132	0.2355	-57.4139
	(5,5)	*	*	*
	(10,10)	*	*	*
Fixed Point	(1,1)	*	*	*
	(5,5)	*	*	*
	(10,10)	*	*	*
SANN	(1,1)	1.9182	0.2356	-57.4141
	(5,5)	1.9330	0.2370	-57.4155
	(10,10)	0.0845	9.1852	-185.7619
BFGS	(1,1)	1.9132	0.2355	-57.4139
	(5,5)	*	*	*
	(10,10)	*	*	*
CG	(1,1)	1.8834	0.2324	-57.4171
	(5,5)	*	*	*
	(10,10)	*	*	*
Nelder-Mead	(1,1)	1.9136	0.2356	-57.4139
	(5,5)	*	*	*
	(10,10)	*	*	*
L-BFGS-B	(1,1)	*	*	*
	(5,5)	*	*	*
	(10,10)	*	*	*
NR	(1,1)	1.9132	0.2355	-57.4139
	(5,5)	*	*	*
	(10,10)	*	*	*
BHHH	(1,1)	*	*	*
	(5,5)	*	*	*
	(10,10)	*	*	*
PSO		24.6573	35.1258	-Inf
ALO		1.9132	0.2355	-57.4139
GWO		1.9127	0.2355	-57.4139
DA		1.9139	0.2356	-57.4139
FFA		1.9595	0.0910	-79.8147
GA		0.8777	0.0200	-75.1760

GOA		1.7220	0.2155	-57.5554
HS		1.9134	0.2356	-57.4139
MFO		13.1914	27.9145	-Inf
SCA		1.8744	0.2329	-57.4206
WOA		1.9133	0.2355	-57.4139
CLOLG		1.2679	0.1243	-60.0004
CS		2.0091	0.2441	-57.4467
BA		0.9062	0.0499	-65.6372
GBS		47.2866	26.5966	-Inf
BHO		2.8328	0.3534	-61.6819
*: The method does not provide a solution.				

From the Table 4, it is concluded that the fixed point algorithm failed to reach the $\hat{\alpha}$ and $\hat{\beta}$ for all initial parameters. SANN, BFGS, CG, Nelder-Mead and NR methods can be considered successful algorithms only when the initial value is chosen as $(1,1)$ among all classical algorithms. Surprisingly, the SANN algorithm which is also available in " optim" package is performed well for all initial parameter values. Considering meta-heuristic algorithms, PSO, GWO, DA, GA, MFO, SCA, WOA, CLOLG, BA and BHO algorithms are achieved to estimate $\hat{\alpha}$ and $\hat{\beta}$.

In Scenario 2, from Table 5 indicate that only BFGS, SANN, Nelder-Mead and NR algorithms are presented reasonable results when the initial value is (1,1) among all classical algorithms. Besides, SANN algorithm also performed well when the initial value is (5,5). ALO, GWO, DA, FFA, GOA, HS, SCA, CLOLG and CS meta-heuristic algorithms are considered to be successful in Scenario 2. Considering successful algorithms under Scenario 1 and Scenario 2, one should mention that SCA, WOA, GWO and DA algorithms presented similar estimates and likelihood function values, with only slight differences.

In light of these results, considering both scenarios, it can be concluded that the SCA algorithm performed best in comparison with all algorithms. These results coincide with the results of the simulation study.

5. Conclusion

This paper deals with the parameter estimation problem for the KM-Weibull distribution under PFFCS. A comprehensive simulation study is conducted to compare the performances of classical, EM, and several meta-heuristic algorithms using different criteria. Also, a numerical example is performed with the same algorithms under different scenarios. According to the results, considering the bias, MSE, computation time, and detection rate, despite having the longest computation time, the SCA algorithm is suggested to use for the KM-Weibull distribution under PFFCS.

Lifetime distributions have a range field of applications not only in statistics but also in actuarial science, medicine, engineering, etc. The requirement for saving time and money for the parameter estimation problems of these distributions leads practitioners to use censoring schemes. Since this study deals with the estimation problem of a new lifetime distribution under one of the most advantageous censoring schemes considering several optimization algorithms, and we believe that these results will provide convenience for the practitioners. For future study, the meta-heuristic algorithms can be compared in terms of control parameters such as the number of population, number of maximum iterations, etc., under different censoring schemes. Furthermore, different censoring schemes can also be examined to see if the meta-heuristic algorithms used in this study perform well under other censoring types.

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