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Optimality and duality results for minimax optimization problems under E-invexity

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Abstract

In this paper, minimax optimization problems involving (not necessarily) differentiable E-invex functions are considered. The sufficient optimality conditions for E-differentiable minimax optimization problems solving E-invex functions are established. To demonstrate the aforementioned results, an example of an E-minimax optimization problem is presented. The parametric and nonparametric Mond-Weir and Wolfe dual problems are formulated for the E-differentiable minimax optimization context. Furthermore, several duality theorems, including weak, strong, converse, and strict converse, are established under the assumptions of E-invexity.

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1. Introduction

Consider a (not necessarily) differentiable minimax optimization problem:

$$\begin{aligned} &\varphi(x) := \min_{x\in R^n} \max_{1\le t\le s} \ \varphi_t(x) \\ &\text{s.t. } g_w(x) \le 0,\, w \in W = \{1,\dots,m\}, \end{aligned} \tag{P}$$

where $\varphi_t : R^n \to R$, $t \in T = \{1,\dots,s\}$, and $g_w : R^n \to R$, $w \in W = \{1,\dots,m\}$, are (not necessarily) differentiable functions.

Minimax extremum problems occur frequently in many important applied disciplines like minimum risk problems, financial planning, Chebyshev approximation, game theory, economics, engineering, goal programming, facility location (see, for example, [19], [24], [21], [22], [23], [39], and others).

Numerous researchers have explored optimality and duality theories for various types of non-convex programming problems (refer, for instance, [6], [20], [42], [43], and others). During the last few years, minimax programming has received a lot of interest from authors in the fields of mathematics and specifically in the field of optimization theory. This is a result of the fact that minimax programming is being applied in many scientific disciplines. Optimality conditions and duality theorems for various classes of minimax optimization problems have been studied extensively in the literature (see, for example, [4], [5], [13], [14], [15], [16], [17], [18], [21], [22], [25], [27], [28], [29], [32], [33], [34], [35], [36], [37], [40], [44], [45], [47], [48], [49] and others).

One of the concepts of generalized convexity defined to weaken the convexity assumptions is the concept of *E*-convexity introduced by Youness [46]. These results motivated the authors to do extensive additional studies in optimization theory (see for example, [2], [3], [9], [10], [11], [12], [26], [41], [30], [31], and others). Megahed et al. [38] defined the notion of E-differentiable function. Recently, Antczak and Abdulaleem [8] introduced a new class of *E*-minimax programming under *E*-convexity. Abdulaleem [1] introduced a novel concept of generalized convexity termed *E*-invexity and established optimality conditions for cases where the functions are *E*-differentiable.

This study introduces a novel class of minimax optimization problems. The sufficient optimality conditions are proved under *E*-differentiable *E*-invexity hypotheses. Further, the example of an *E*-minimax optimization problem is generated to illustrate the sufficient optimality conditions results. Furthermore, Mond-Weir and Wolfe duals are formulated for the E-differentiable minimax optimization problem. Then, a number of weak duality theorems, strong duality theorems, converse duality theorems, strict converse duality theorems, and restricted strict converse duality theorems are established under appropriate E-invexity assumptions. It became clear from the optimality and duality results for E-differentiable E-invex minimax programming problem is not applicable to the optimality and duality results found in the literature, and this means that the results presented in this paper are expanded and generalized to the results established earlier in the literature.

2. Preliminaries

Abdulaleem [1] introduced the concept of E-invexity. Now, we recall the definitions of E-invex set, E-invex function. Suppose $E: R^n \to R^n$.

Definition 2.1: [1] $S \subseteq R^n$ is called an E-invex set with respect to (w.r.t.) $\eta: S \times S \to R^n$, iff $x^0, y^0 \in S$, $0 \le \lambda \le 1$ we have

$$E(y^0) + \lambda\eta(E(x^0), E(y^0)) \in S.$$

Definition 2.2: [38] Let $\varphi: R^n \to R$ be a function that is not necessarily differentiable at y^0. The function φ is called E-differentiable at y^0 iff $\varphi \circ E$ is differentiable at y^0 and, that is,

$$\varphi(E(x^0)) = \varphi(E(y^0)) + \nabla(\varphi \circ E)(y^0)(x^0 - y^0) + \theta(y^0, x^0 - y^0) \| x^0 - y^0 \|, \qquad (1)$$

where $\theta(y^0, x^0 - y^0) \to 0$ as $x^0 \to y^0$.

Definition 2.3: [1] Assume that $E: R^n \to R^n$. The function φ is said to be (strictly E-invex) E-invex w.r.t. η at $y^0 \in S$ on S if, there exists $\eta: S \times S \to R^n$ such that, for each $x^0 \in S$,

$$\varphi(E(x^0)) - \varphi(E(y^0)) \geqq \nabla(\varphi \circ E)(y^0)\eta(E(x^0), E(y^0)). \quad (>) \qquad (2)$$

If condition (2) is satisfied for every instance $y^0 \in S$, then φ is (strictly E-invex) E-invex w.r.t. η on S.

3. *E*-minimax programming and optimality conditions

Let $E: R^n \to R^n$. We introduce the corresponding differentiable E-minimax problem (P_E) related to (P):

$$\begin{aligned}&(\varphi\circ E)(x):=\min_{x\in R^n}\max_{1\le t\le s}(\varphi_t\circ E)(x)\\&\text{s.t.}(g_w\circ E)(x)\le 0,\, w\in W=\{1,...,m\},\end{aligned}\qquad (\mathrm{P}_E)$$

where an objective function $\varphi_t\circ E:R^n\to R,\ \ t=1,2,...,s,$ and constraint function $g_w\circ E:R^n\to R,\ \ w=1,2,...,m,$ are differentiable. Assume that D_E is the set of all feasible solutions (FS) of (P$_E$)

$$D_E:=\{x\in R^n:(g_w\circ E)(x)\le 0,\, w\in W\}.$$

Now, for the considered problem (P) (and, the problem (P$_E$)), let us define its parametric problem (EP$_E$):

minimize q

$$\text{s.t.}(\varphi_t\circ E)(x)\le q,\, t\in T=\{1,...,s\}\qquad (\mathrm{EP}_E)$$

$$(g_w\circ E)(x)\le 0,\, w\in W=\{1,...,m\},$$

$$(x,q)\in R^n\times R.$$

Assume that Q_E is the set of all FS of (EP$_E$)

$$Q_E:=\{(x,q)\in R^n\times R:(\varphi_t\circ E)(x)\le q,\, t\in T,(g_w\circ E)(x)\le 0,\, w\in W\}.$$

Lemma 3.1: [8] *If (x,q) represents a FS for (EP$_E$), then x also qualifies as a FS for (P$_E$). Conversely, if x is a FS for (P$_E$), there exists some $q\in R$ such that (x,q) serves as a FS for (EP$_E$).*

Theorem 3.2: *Let $x^0\in D_E$ with its associated optimal value denoted as q^0. Moreover, φ_t, $t\in T$, g_w, $w\in W$, is E-differentiable at x^0, and the ACQ [3] is fulfilled at x^0 for (P$_E$). Then there exist $\zeta^0\in R^s$, $\xi^0\in R^m$ such that (x^0,q^0,ζ^0,ξ^0) satisfies the following conditions*

$$\sum_{t=1}^{s}\zeta_t^0\nabla(\varphi_t\circ E)(x^0)+\sum_{w\in W}\xi_w^0\nabla(g_w\circ E)(x^0)=0, \tag{3}$$

$$\zeta_t^0((\varphi_t\circ E)(x^0)-q^0)=0,\, t=1,...,s, \tag{4}$$

$$(\varphi_t\circ E)(x^0)\le q^0,\, t=1,...,s, \tag{5}$$

$$\xi_w^0(g_w\circ E)(x^0)=0,\, w=1,...,m, \tag{6}$$

$$(g_w \circ E)(x^0) \leq 0, w = 1, \ldots, m, \tag{7}$$

$$\zeta_t^0 \geq 0, t \in T, \sum_{t=1}^{s} \zeta_t^0 = 1, \xi_w^0 \geq 0, w \in W. \tag{8}$$

Theorem 3.3: *Let* $(x^0, q^0, \zeta^0, \xi^0) \in D_E \times R \times R_+^s \times R_+^m$ *satisfying relations (3)-(8). Further, assume that*

(a) each φ_t, $t \in T$, is E-invex at x^0 on D_E w.r.t. η,

(b) each g_w, $w \in W$, is E-invex at x^0 on D_E w.r.t. η.

Therefore, x^0 represents an optimal solution (OS) to (P_E), with the corresponding optimal objective value being q^0.

Proof: By $(x^0, q^0, \zeta^0, \xi^0) \in D_E \times R \times R_+^s \times R_+^m$ satisfies relations (3)-(8). By Definition 2.3 and by a)-b), The inequalities that follow

$$(\varphi_t \circ E)(x) - (\varphi_t \circ E)(x^0) \geq \nabla(\varphi_t \circ E)(x^0)\eta(E(x), E(x^0)), t \in T, \tag{9}$$

$$(g_w \circ E)(x) - (g_w \circ E)(x^0) \geq \nabla(g_w \circ E)(x^0)\eta(E(x), E(x^0)), w \in W \tag{10}$$

hold for all $x \in D_E$. Multiplying (9) and (10) by associated Lagrange multipliers, we get

$$\zeta_t^0(\varphi_t \circ E)(x) - \zeta_t^0(\varphi_t \circ E)(x^0) \geq \zeta_t^0 \nabla(\varphi_t \circ E)(x^0)\eta(E(x), E(x^0)), t \in T, \tag{11}$$

$$\xi_w^0(g_w \circ E)(x) - \xi_w^0 g_w(E(x^0)) \geq \xi_w^0 \nabla(g_w \circ E)(x^0)\eta(E(x), E(x^0)), w \in W. \tag{12}$$

Therefore, for every $x \in D_E$, we derive the following

$$\sum_{t \in T} \zeta_t^0(\varphi_t \circ E)(x) - \sum_{t \in T} \zeta_t^0(\varphi_t \circ E)(x^0) \geq \sum_{t \in T} \zeta_t^0 \nabla(\varphi_t \circ E)(x^0)\eta(E(x), E(x^0)), \tag{13}$$

$$\sum_{w \in W} \xi_w^0(g_w \circ E)(x) - \sum_{w \in W} \xi_w^0 g_w(E(x^0)) \geq \sum_{w \in W} \xi_w^0 \nabla(g_w \circ E)(x^0)\eta(E(x), E(x^0)). \tag{14}$$

By summing the two sides of inequalities (13) and (14), we get the following inequalities:

$$\sum_{t \in T} \zeta_t^0(\varphi_t \circ E)(x) - \sum_{t \in T} \zeta_t^0(\varphi_t \circ E)(x^0) + \sum_{w \in W} \xi_w^0(g_w \circ E)(x) - \sum_{w \in W} \xi_w^0 g_w(E(x^0)) \geq$$

$$[\sum_{t \in T} \zeta_t^0 \nabla(\varphi_t \circ E)(x^0) + \sum_{w \in W} \xi_w^0 \nabla(g_w \circ E)(x^0)]\eta(E(x), E(x^0)) \tag{15}$$

holds for each $x \in D_E$. Hence, the relation (3) implies that, for all $x \in D_E$,

$$\sum_{t\in T}\zeta_t^0(\varphi_t\circ E)(x)-\sum_{t\in T}\zeta_t^0(\varphi_t\circ E)(x^0)+\sum_{w\in W}\xi_w^0(g_w\circ E)(x)-\sum_{w\in W}\xi_w^0 g_w(E(x^0))\geq 0. \quad (16)$$

By relations (6) and (7), the inequalities

$$\sum_{t\in T}\zeta_t^0(\varphi_t\circ E)(x)\geq\sum_{t\in T}\zeta_t^0(\varphi_t\circ E)(x^0) \quad (17)$$

hold for each $x\in D_E$. If $x\in D_E$, then we have $q\in R$ such that $(x,q)\in Q_E$. Assume (x,q) is a FS of (EP$_E$) and $\zeta^0\in R_+^s$. Then, the first constraint of (EP$_E$) together with (4) yield

$$\sum_{t\in T}\zeta_t^0(q-q^0)\geq 0. \quad (18)$$

Therefore, the combination of (18) and (8) leads to the conclusion that

$$q\geq q^0$$

holds $\forall(x,q)\in Q_E$. This implies (x^0,q^0) is optimal solution (OS) of (EP$_E$). Therefore, by Lemma 3.1, conclusion that x^0 is optimal of (P$_E$) with the objective value equal to q^0.

Example 3.4: Consider a minimax optimization problem where the functions are nondifferentiable.

$$\begin{aligned}&\min_{x\in R^2}\max_{2\leq i\leq 4}\left(\sqrt[3]{x_1^2}+(i-3)\sqrt[3]{x_2}\right)\\ &\text{s.t. } g_1(x)=\sqrt[3]{x_1^2}-\sqrt[3]{x_2}\leq 0,\\ &\quad g_2(x)=\sqrt[3]{x_2^2}-\sqrt[3]{x_2}\leq 0.\end{aligned} \quad \text{(P1)}$$

Note $D=\{(x_1,x_2)\in R^2 : \sqrt[3]{x_1^2}-\sqrt[3]{x_2}\leq 0 \wedge \sqrt[3]{x_2^2}-\sqrt[3]{x_2}\leq 0\}$. Let $E:R^2\to R^2$ defined as follows $E(x_1,x_2)=(x_1^3,x_2^3)$ and η defined as follows $\eta(E(x), E(x^0)) = x_1^2+x_2$. Now, for (P1), we formulate its associated (P1$_E$) as it follows

$$\begin{aligned}&\min_{x\in R^2}\max_{2\leq i\leq 4}\left(x_1^2+(i-3)x_2\right)\\ &\text{s.t. } g_1(E(x))=x_1^2-x_2\leq 0,\\ &\quad g_2(E(x))=x_2^2-x_2\leq 0.\end{aligned} \quad (\text{P1}_E)$$

Note that $D_E=\{(x_1,x_2)\in R^2 : x_1^2-x_2\leq 0 \wedge x_2^2-x_2\leq 0\}$ is the set of all FSs in (P1$_E$), and $x^0=(0,0)$ is feasible in the problem (P1$_E$). Further, Lagrange multipliers $\zeta_1^0=0$, $\zeta_2^0=2$, $\xi_1^0=\xi_2^0=1$ satisfy the parametric Karush-Kuhn-Tucker necessary optimality conditions at x^0. Furthermore, each functions involved in (P1) is E-invex at x^0 on D_E w.r.t. η defined above. Then, by Theorem 3.3, $x^0 = (0,0)$ is optimal in (P1$_E$).

Remark 1: It should be noted that in Example 3.4, it is not possible to use the optimality results for finding KKT for minimax programming problems under E-differentiable E-convexity [8]. This is due to the fact that the functions in (P1) considered in Example 3.4 are not E-differentiable E-convex at x^0.

Theorem 3.5: *Assume that $x^0 \in D_E$ is an optimal point of (PE). Further, (ACQ) [3] is fulfilled at x^0. Then there are $\zeta^0 \in R^s$, $\xi^0 \in R^m$ such that*

$$\sum_{t=1}^{s}\zeta_t^0 \nabla(\varphi_t \circ E)(x^0) + \sum_{w=1}^{m}\xi_w^0 \nabla(g_w \circ E)(x^0) = 0, \tag{19}$$

$$\xi_w^0 (g_w \circ E)(x^0) = 0,\, w = 1,...,m, \tag{20}$$

$$(g_w \circ E)(x^0) \le 0,\, w = 1,...,m, \tag{21}$$

$$\zeta_t^0 \ge 0,\, t \in T,\, \sum_{t=1}^{s}\zeta_t^0 = 1,\, \xi_w^0 \ge 0,\, w \in W. \tag{22}$$

Proof: By replacing q^0 with $\max_{1\le t\le s}\varphi_t(E(x^0))$ in Theorem 3.2, this theorem is directly proved.

4. Mond-Weir *E*-duality

Next, we derive the Mond-Weir dual for the problem (EP$_E$).

$$\lambda \to max$$

subject to $$\sum_{t=1}^{s}\zeta_t \nabla(\varphi_t \circ E)(y) + \sum_{w\in W(E(y))}\xi_w \nabla(g_w \circ E)(y) = 0, \tag{23}$$

$$\zeta_t((\varphi_t \circ E)(y) - \lambda) \ge 0,\, t = 1,...,s, \quad (\text{MWD}_E) \tag{24}$$

$$\xi_w g_w(E(y)) \ge 0,\, w = 1,...,m, \tag{25}$$

$$\sum_{t=1}^{s}\zeta_t = 1, \tag{26}$$

$$\zeta \in R_{+}^{s}, y \in R^{n}, \xi \in R_{+}^{m}, \lambda \in R. \tag{27}$$

Let Y_E represent the projection of the set all FSs Γ_E for (MWD$_E$), which is, $Y_E = pr_{R^n}\Gamma_E = \{y \in R^n : (y,\lambda,\zeta,\xi) \in \Gamma_E\}$.

Lemma 4.1: *Assume that* $(y,\lambda,\zeta,\xi) \in \Gamma_E$. *Moreover,* g_w, $w \in W$, *is E-invex at y on* $D_E \cup Y_E$ *w.r.t.* η. *Then, the inequality*

$$\sum_{w \in W(E(y))} \xi_w \nabla (g_w \circ E)(y)\eta(E(x),E(y)) \leq 0 \tag{28}$$

is true for all $x \in D_E$.

Proof: Definition 2.3 provides direct proof of this lemma.

Now, we prove the duality theorems relating to (EP$_E$) and (MWD$_E$) under *E*-invexity hypotheses.

Theorem 4.2: *(Weak duality). Let* (x,q) *be any FS for the problem (EP$_E$), and let* (y,λ,ζ,ξ) *be any FS for (MWD$_E$). Furthermore, suppose that*

(a) function φ_t, $t \in T$, is *E*-invex at *y* on $pr_{R^n}Q_E \cup Y_E$ w.r.t. η,

(b) each inequality constraint g_w, $w \in W$, is *E*-invex at *y* on $pr_{R^n}Q_E \cup Y_E$ w.r.t. η.

Then

$$q \geq \lambda. \tag{29}$$

Proof: Since $(x,q) \in Q_E$, by $x \in D_E$ and Lemma 3.1. By assumption *b*), each function g_w, $w \in W$, is *E*-invex at *y* on $pr_{R^n}Q_E \cup Y_E$ w.r.t. η. Therefore, by Lemma 4.1, that

$$\sum_{w \in W(E(y))} \xi_w \nabla (g_w \circ E)(y)\eta(E(x),E(y)) \leq 0 \tag{30}$$

holds. Further, by *a*), φ_t, $t \in T$, is *E*-invex at *y* on $pr_{R^n}Q_E \cup Y_E$ w.r.t. η. Then, by Definition 2.3, that

$$(\varphi_t \circ E)(u) - (\varphi_t \circ E)(y) \geq \nabla(\varphi_t \circ E)(y)\eta(E(u),E(y)), t \in T \tag{31}$$

hold for all $u \in pr_{R^n}Q_E \cup Y_E$. As a result, $u = x \in D_E$ is also satisfied. Multiplying (31) by associated Lagrange multipliers, that

$$\zeta_t(\varphi_t \circ E)(x) - \zeta_t(\varphi_t \circ E)(y) \geq \zeta_t \nabla(\varphi_t \circ E)(y)\eta(E(x),E(y)), t \in T \tag{32}$$

hold. The result of summing the two sides of the inequality (32) and after adding the resulting inequality to (30), we obtain

$$\sum_{t\in T}\zeta_t(\varphi_t\circ E)(x)-\sum_{t\in T}\zeta_t(\varphi_t\circ E)(y)\geq$$
$$[\sum_{t\in T}\zeta_t\nabla(\varphi_t\circ E)(y)+\sum_{w\in W(E(y))}\xi_w\nabla(g_w\circ E)(y)]\eta(E(x),E(y)). \tag{33}$$

Thus, by (23) yields

$$\sum_{t\in T}(\varphi_t\circ E)(x)-\sum_{t\in T}(\varphi_t\circ E)(y)\geq 0. \tag{34}$$

By (23) and the first constraint of (EP_E), we have

$$\sum_{t\in T}\zeta_t(q-\lambda)\geq 0. \tag{35}$$

Hence, (26) and (35) imply (29). The demonstration of this theorem has been concluded.

Theorem 4.3: *(Strong duality). Suppose (x^0,q^0) is optimal of the problem (EP_E) and the (ACQ) is fulfilled at x^0. Then, there exist $\zeta^0\in R^s$, $\xi^0\in R^m$, in such a manner that the point (x^0,q^0,ζ^0,ξ^0) satisfies the feasibility conditions for (MWD_E), and at this point, the objective functions of (EP_E) and (MWD_E) attain identical values. Moreover, if the conditions of Theorem 4.2 are fully satisfied, then (x^0,q^0,ζ^0,ξ^0) represents an OS of maximum type in (MWD_E).*

Proof: Assuming that (x^0,q^0) is optimal of (EP_E) and the (ACQ) is satisfied at x^0. Then, by relations (3)-(8) for the problem (EP_E), we conclude that (x^0,q^0,ζ^0,ξ^0) is a FS of (MWD_E). At x^0, it is evident that the values of problems (EP_E) and (MWD_E) are identical, as both share the same objective functions. By Theorem 4.2, we have

$$q^0\geq\lambda$$

The statement applies to any valid point (y,λ,ζ,ξ) within (MWD_E), demonstrating that (x^0,q^0,ζ^0,ξ^0) represents an OS of a maximum type of (MWD_E). Thus, the proof is concluded.

Theorem 4.4: *(Converse duality). Let $(y^0,\lambda^0,\zeta^0,\xi^0)$ be an OS of (MWD_E) and the (ACQ) be fulfilled at y^0. Moreover, assume*

(a) each function φ_t, $t\in T$, is E-invex at y^0 on $pr_{R^n}Q_E\cup Y_E$ w.r.t. η,
(b) each inequality constraint g_w, $w\in W$, is E-invex at y^0 on $pr_{R^n}Q_E\cup Y_E$ w.r.t. η.

Then (y^0, λ^0) is a OS of the problem (EP$_E$).

Proof: Because $(y^0, \lambda^0, \zeta^0, \xi^0)$ is an OS of (MWD$_E$) and the Abadie constraint qualification (ACQ) is fulfilled at y^0, therefore, $(y^0, \lambda^0, \zeta^0, \xi^0)$ relations (3)-(8). This means that (y^0, λ^0) is feasible for (EP$_E$). By assumption, g_w, $w \in W$, are E-invex at y^0 on $pr_{R^n} Q_E \cup Y_E$ w.r.t. η. Then, by Lemma 4.1, that

$$\sum_{w \in W(E(y^0))} \xi^0{}_w \nabla g_w(E(y^0))\eta(E(x), E(y^0)) \leq 0 \tag{36}$$

is true for all $x \in D_E$. Since, φ_t, $t \in T$, is E-invex at y^0 on $pr_{R^n} Q_E \cup Y_E$ w.r.t. η, by Definition 2.3, it follows that the inequality

$$(\varphi_t \circ E)(u) - (\varphi_t \circ E)(y^0) \geq \nabla(\varphi_t \circ E)(y^0)\eta(E(u), E(y^0)), t \in T \tag{37}$$

hold for all $u \in pr_{R^n} Q_E \cup Y_E$. As a consequence, the conditions hold for every $u = x \in D_E$. By applying the corresponding Lagrange multipliers to equation (37) and summing up both sides, we obtain the following result

$$\sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(x) - \sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(y^0) \geq \sum_{t \in T} \zeta_t^0 \nabla(\varphi_t \circ E)(y^0)\eta(E(x), E(y^0)). \tag{38}$$

By combining (36) with (38), we get

$$\begin{aligned} &\sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(x) - \sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(y^0) \geq \\ &[\sum_{t \in T} \zeta_t^0 \nabla(\varphi_t \circ E)(y^0) + \sum_{w \in W(E(y))} \xi^0{}_w \nabla(g_w \circ E)(y^0)]\eta(E(x), E(y^0)). \end{aligned} \tag{39}$$

Hence, first constraint in (MWD$_E$) yields

$$\sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(x) - \sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(y^0) \geq 0. \tag{40}$$

By first constraint in (EP$_E$) and (24), we get

$$\sum_{t \in T} \zeta_t^0 (q - \lambda^0) \geq 0. \tag{41}$$

Hence, by $\sum_{t=1}^{s} \zeta_t^0 = 1$, (41) implies

$$q \geq \lambda^0.$$

Thus, (y^0, λ^0) is an OS of (EP$_E$). The demonstration of this theorem has been concluded.

Theorem 4.5: *(Strict converse duality). Assume that* (x^0, q^0) *and* $(y^0, \lambda^0, \zeta^0, \xi^0)$ *are optimal solutions of (EP$_E$) and (MWD$_E$), respectively, and the Abadie constraint qualification (ACQ) is fulfilled at* x^0. *Moreover, suppose that*

(a) each function φ_t, $t \in T$, is strictly E-invex at y^0 on $pr_{R^n} Q_E \cup Y_E$ w.r.t. η,

(b) each inequality constraint g_w, $w \in W$, is E-invex at y^0 on $pr_{R^n} Q_E \cup Y_E$ w.r.t. η.

Then $(x^0, q^0) = (y^0, \lambda^0)$.

Proof: suppose that, contrary to the outcome, that $(x^0, q^0) \neq (y^0, \lambda^0)$. By assumption, (x^0, q^0) is an OS of (EP$_E$). Therefore, by Theorem 4.3, there exist $\zeta^0 \in R_+^s$, and $\xi^0 \in R_+^m$, such that $(x^0, q^0, \zeta^0, \xi^0)$ is an OS of a maximum type in (MWD$_E$). Thus, by $(y^0, \lambda^0, \zeta^0, \xi^0) \in \Gamma_E$, it follows that

$$q^0 = \lambda^0, \tag{42}$$

$$\sum_{t=1}^{s} \zeta_t^0 \nabla(\varphi_t \circ E)(y^0) + \sum_{w \in W} \xi_w^0 \nabla(g_w \circ E)(y^0) = 0, \tag{43}$$

$$\zeta_t^0 ((\varphi_t \circ E)(y^0) - \lambda^0) \geq 0, t = 1, \ldots, s, \tag{44}$$

$$\xi_w^0 (g_w \circ E)(y^0) \geq 0, w = 1, \ldots, m, \tag{45}$$

$$\sum_{t=1}^{s} \zeta_t^0 = 1. \tag{46}$$

Since $(y^0, \lambda^0, \zeta^0, \xi^0)$ is a FS of (MWD$_E$), similarly to the proof of Theorem 4.2, we obtain

$$\sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(x^0) - \sum_{t \in T} \zeta_t^0 (\varphi_t \circ E)(y^0) > 0.$$

By $(x^0, q^0) \in Q_E$ and $(y^0, \lambda^0, \zeta^0, \xi^0) \in \Gamma_E$, (4) and (44) yield

$$\sum_{t \in T} \zeta_t^0 (q^0 - \lambda^0) \geq 0.$$

Hence, (46) implies the inequality

$$q^0 \geq \lambda^0,$$

contradicting (42). The demonstration of this theorem has been concluded.

Theorem 4.6: *(Restricted strict converse duality). Suppose* $(y^0, \lambda^0, \zeta^0, \xi^0)$ *is a feasible point of (MWD$_E$) and also there is* $(x^0, q^0) \in Q_E$ *such that* $q^0 = \lambda^0$. *If* φ_t, $t \in T$, *is strictly E-invex functions at* y^0 *on* $pr_{R^n} Q_E \cup Y_E$ *w.r.t.* η, *the functions*

g_w, $w \in W$, is E-invex at y^0 on $pr_{R^n}Q_E \cup Y_E$ w.r.t. η, then (x^0, q^0) is an OS of (EP$_E$).

Proof: Let $(y^0, \lambda^0, \zeta^0, \xi^0)$ be a FS of (MWD$_E$). By g_w, $w \in W$, is *E*-invex at y^0 on $pr_{R^n}Q_E \cup Y_E$ w.r.t. η. Then, by Lemma 4.1, for each $x \in D_E$. As a result, the following inequality is derived

$$\sum_{w \in W(E(y^0))} \xi^0{}_w \nabla(g_w \circ E)(y^0)\eta(E(x), E(y^0)) \leq 0. \tag{47}$$

They are satisfied for $x = x^0$. Also by φ_t, $t \in T$, are strictly *E*-invex at y^0 on $pr_{R^n}Q_E \cup Y_E$ w.r.t. η. Then, by Definition 2.3, the inequalities

$$(\varphi_t \circ E)(x^0) - (\varphi_t \circ E)(y^0) > \nabla(\varphi_t \circ E)(y^0)\eta(E(x^0), E(y^0)), t \in T \tag{48}$$

hold. Multiplying each inequality (48) by ζ_t^0 and then adding both sides of the resulting inequalities, we get

$$\sum_{t \in T}\zeta_t^0(\varphi_t \circ E)(x^0) - \sum_{t \in T}\zeta_t^0(\varphi_t \circ E)(y^0) > \sum_{t \in T}\zeta_t^0\nabla(\varphi_t \circ E)(y^0)\eta(E(x^0), E(y^0)). \tag{49}$$

Combining (47) and (49), we obtain

$$\sum_{t \in T}\zeta_t^0(\varphi_t \circ E)(x^0) - \sum_{t \in T}\zeta_t^0(\varphi_t \circ E)(y^0) >$$
$$[\sum_{t \in T}\zeta_t^0\nabla(\varphi_t \circ E)(y^0) + \sum_{w \in W}\xi^0{}_w\nabla(g_w \circ E)(y^0)]\eta(E(x^0), E(y^0)).$$

Since $(y^0, \lambda^0, \zeta^0, \xi^0)$ is feasible in (MWD$_E$), the first constraint of (MWD$_E$) yields

$$\sum_{t \in T}\zeta_t^0(\varphi_t \circ E)(x^0) - \sum_{t \in T}\zeta_t^0(\varphi_t \circ E)E(y^0) > 0.$$

By (4) and (44), it follows that

$$\sum_{t \in T}\zeta_t^0(q^0 - \lambda^0) > 0. \tag{50}$$

Hence, (50) implies that

$$q^0 > \lambda^0$$

holds, which contradicts the assumption $q^0 = \lambda^0$. Then, according to Lemma 3.1, (x^0, q^0) is an OS of (EP$_E$). Thus, this theorem's proof is complete.

5. Non-parametric Wolfe *E*-duality

Lemma 5.1: [7] *For all* $z \in R^n$, *we have*

$$\varphi(z) = \max_{1\le t\le s} \varphi_t(z) = \max_{\zeta\in\Lambda} \sum_{t=1}^{s} \zeta_t \varphi_t(z), \tag{51}$$

where $\Lambda = \{\zeta = (\zeta_1, ..., \zeta_s) \in R_+^s : \sum_{t=1}^{s} \zeta_t = 1\}$.

Now, for (P_E), we define its Wolfe *E*-dual (WD_E) as it follows

$$\sum_{t=1}^{s}\zeta_t(\varphi_t \circ E)(y) + \sum_{w=1}^{m}\xi_w(g_w \circ E)(y) \to \max$$

$$\text{subject to } \sum_{t=1}^{s}\zeta_t\nabla(\varphi_t \circ E)(y) + \sum_{w=1}^{m}\xi_w\nabla(g_w \circ E)(y) = 0, \quad (\mathrm{WD}_E)$$

$$y \in R^n, \zeta \in R_+^s, \sum_{t=1}^{s}\zeta_t = 1, \xi \in R_+^m.$$

Let χ_E be the set all FSs of (WD_E) and $Y_E^W = pr_{R^n}\chi_E = \{y \in R^n : (y,\zeta,\xi) \in \chi_E\}$.

Theorem 5.2: *(Wolfe weak duality). Let* $x \in D_E$ *and* $(y,\zeta,\xi) \in Y_E^W$. *Moreover, suppose that*

(a) each function φ_t, $t \in T$, is *E*-invex at *y* on $D_E \cup Y_E^W$ w.r.t. η,

(b) each constraint function g_w, $w \in W$, is *E*-invex at *y* on $D_E \cup Y_E^W$ w.r.t. η. Then

$$\max_{1\le t\le s}(\varphi_t \circ E)(x) \ge \sum_{t=1}^{s}\zeta_t(\varphi_t \circ E)(y) + \sum_{w=1}^{m}\xi_w(g_w \circ E)(y). \tag{52}$$

Proof: Let $x \in D_E$ and $(y,\zeta,\xi) \in Y_E^W$. By φ_t, $t \in T$, and g_w, $w \in W$, are *E*-invex at *y* on $D_E \cup Y_E^W$ w.r.t. η. Then, by Definition 2.3, the inequalities

$$(\varphi_t \circ E)(x) - (\varphi_t \circ E)(y) \ge \nabla(\varphi_t \circ E)(y)\eta(E(x),E(y)), t \in T \tag{53}$$

$$(g_w \circ E)(x) - g_w(E(y)) \ge \nabla(g_w \circ E)(y)\eta(E(x),E(y)), w \in W \tag{54}$$

hold. Multiplying (53) by ζ_t and (54) by ξ_w and then adding both sides, we get

$$\sum_{t=1}^{s}\zeta_t(\varphi_t\circ E)(x)-\sum_{t=1}^{s}\zeta_t(\varphi_t\circ E)(y)+\sum_{w=1}^{m}\xi_w(g_w\circ E)(x)-\sum_{w=1}^{m}\xi_w(g_w\circ E)(y)\geq \left[\sum_{t=1}^{s}\zeta_t\nabla(\varphi_t\circ E)(y)+\sum_{w=1}^{m}\xi_w\nabla(g_w\circ E)(y)\right]\eta(E(x),E(y)).$$

Hence, the first constraint of (WD$_E$) gives

$$\sum_{t=1}^{s}\zeta_t(\varphi_t\circ E)(x)+\sum_{w=1}^{m}\xi_w(g_w\circ E)(x)\geq\sum_{t=1}^{s}\zeta_t(\varphi_t\circ E)(y)+\sum_{w=1}^{m}\xi_w(g_w\circ E)(y).$$

Given the feasibility of x in (P$_E$) and the feasibility of (y,ζ,ξ) in (WD$_E$), we can deduce that

$$\sum_{t=1}^{s}\zeta_t(\varphi_t\circ E)(y)\geq\sum_{t=1}^{s}\zeta_t(\varphi_t\circ E)(y)+\sum_{w=1}^{m}\xi_w(g_w\circ E)(y).$$

Thus,

$$\max_{\hat{\zeta}\in\Lambda}\sum_{t=1}^{s}\hat{\zeta}_t(\varphi_t\circ E)(x)\geq\sum_{t=1}^{s}\zeta_t(\varphi_t\circ E)(y)+\sum_{w=1}^{m}\xi_w(g_w\circ E)(y). \tag{55}$$

Then, by Lemma 5.1, (55) implies (52). Thus, weak duality between (P$_E$) and (WD$_E$) holds.

Theorem 5.3: *(Wolfe strong duality). Assume that $x^0\in D_E$ serves as an OS to (P$_E$) and that the (ACQ) condition holds at x^0. In this case, there exist vectors $\zeta^0\in R^s$, $\xi^0\in R^m$ such that (x^0,ζ^0,ξ^0) becomes a FS for (WD$_E$), with the objective values of (P$_E$) and (WD$_E$) being equivalent. Moreover, if the assumptions of Theorem 5.2 are met, then (x^0,ζ^0,ξ^0) is not just feasible but also an OS of a maximization type for (WD$_E$).*

Proof: Since $x^0\in D_E$ and the (ACQ) is satisfied at x^0, by Theorem 3.5, there are $\zeta^0\in R^s$, $\xi^0\in R^m$ such (x^0,ζ^0,ξ^0) is a FS of (WD$_E$). By Lemma 5.1 (that is, (51)) lead to the conclusion that the associated objective values of (P$_E$) and (WD$_E$) are equal. As it follows from conditions (3)-(8), (x^0,ζ^0,ξ^0) is a FS in (WD$_E$). Given that (x^0,ζ^0,ξ^0) is optimal for (WD$_E$) based on the principle of weak duality, the corresponding optimal value is expressed as $\sum_{t=1}^{s}\zeta_t^0\varphi_t(x^0)$. Thus, the extremum problems (P$_E$) and (WD$_E$) share the same optimal value as indicated by (51). Moreover, (x^0,ζ^0,ξ^0) serves as an OS of a maximization nature in the dual problem (WD$_E$) following Wolfe's sense, provided that all conditions required for weak duality (as stated in Theorem 5.2) are fulfilled.

Theorem 5.4: *(Wolfe strict converse duality). Suppose* x^0 *and* (y^0, ζ^0, ξ^0) *are optimal solutions for* (P_E) *and (WDE), respectively. Additionally, supposing that the aforementioned assumptions are true:*

(a) φ_t, $t \in T$, are strictly E-invex functions at y^0 on $D_E \cup Y_E^W$ w.r.t. η,

(b) g_w, $w \in W$, are E-invex functions at y^0 on $D_E \cup Y_E^W$ w.r.t. η.

Then $x^0 = y^0$, that is, y^0 is optimal of (P_E).

Proof: Suppose the opposite of the outcome, that $x^0 \neq y^0$. According to Theorem 5.3, we know that

$$\max_{1\leq t\leq s}(\varphi_t \circ E)(x^0) = \sum_{t=1}^{s}\zeta_t^0\varphi_t(E(y^0)) + \sum_{w=1}^{m}\xi_w^0 g_w(E(y^0)).$$

Hence, by Lemma 5.1, we get

$$\max_{\zeta\in\zeta}\sum_{t=1}^{s}\zeta_t(\varphi_t \circ E)(x^0) = \sum_{t=1}^{s}\zeta_t^0(\varphi_t \circ E)(y^0) + \sum_{w=1}^{m}\xi_w^0(g_w \circ E)(y^0). \tag{56}$$

Thus, (56) gives

$$\sum_{t=1}^{s}\zeta_t^0(\varphi_t \circ E)(x^0) \leq \sum_{t=1}^{s}\zeta_t^0(\varphi_t \circ E)(y^0) + \sum_{w=1}^{m}\xi_w^0(g_w \circ E)(y^0). \tag{57}$$

Given that the conditions of Theorem 5.2 are satisfied, we can follow the approach outlined in Theorem 5.2 to derive the inequality

$$\sum_{t=1}^{s}\zeta_t^0(\varphi_t \circ E)(x^0) > \sum_{t=1}^{s}\zeta_t^0(\varphi_t \circ E)(y^0) + \sum_{w=1}^{m}\xi_w^0(g_w \circ E)(y^0),$$

contradicting (57). The proof has been completed.

6. Conclusion

This study focuses on a specific class of minimax optimization problems. In particular, a novel class of such problems, which incorporates E-differentiable E-invex functions, has been introduced. Further, sufficient conditions for optimality, as well as duality results in the contexts of Mond-Weir and Wolfe, have been derived. For a broader range of nondifferentiable minimax optimization problems, the results on optimality and duality previously found in the literature have been extended. An illustrative example of an E-minimax optimization problem is also provided.

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