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On strongly Schur-convex stochastic process

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Abstract

The study of strongly Schur convex stochastic processes is a main objective of this research. Additionally, certain properties of strongly Schur convex and Schur convex stochastic processes are given.

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1. Introduction

In 1974 B. Nagy [8] studied additive stochastic processes and so far researchers have devoted their efforts to introduce and study various convexity classes of stochastic processes and other related results.

As an illustration, the authors in [2] studied the classes of stochastic processes called $(p;h)$-convex and by using certain convexity classes of stochastic processes, the authors in [1] obtained a few estimates concerning the Hermite-Hadamard type inequality's right-hand side.

Introducing and providing some characterizations of stochastic processes using the strong schur-convexity represents the goal of this study.

Additionally, these findings results in this work may provide a practical and effective tool for additional research on Schur convexity.

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Throughout this work we will consider the random variable $\mathcal{C}:\eth\to\mathbb{R}$ where $\mathcal{C}\geq 0$.

We take the interval $\mathcal{H}\subset\mathbb{R}$, the stochastic process is defined as function $\mathcal{Z}:\mathcal{H}\times\eth\to\mathbb{R}$ if for every $\varkappa\in\mathcal{H}$, the function $\mathcal{Z}(\varkappa,.)$ is a random variable ($(\eth,\mathcal{F},P)$ being a probability space) .

The stochastic process $\mathcal{Z}:\mathcal{H}\times\eth\to\mathbb{R}$ is :

i) Strongly Wright-convex with modulus $\mathcal{C}(\cdot)$ or that $\mathcal{Z}$ belongs to the class $S_w(\mathcal{C};\mathcal{H}\times\eth)$ if

$$\mathcal{Z}(\nu\eta+(1-\nu)\rho,\cdot)+\mathcal{Z}((1-\nu)\eta+\nu\rho,\cdot)\leqslant\mathcal{Z}(\eta,\cdot)+\mathcal{Z}(\rho,\cdot)-2\mathcal{C}(\cdot)\nu(1-\nu)(\eta-\rho)^2 \tag{1.1}$$

holds for all $\nu\in[0,1]$ and $\eta,\rho\in\mathcal{H}$.

ii) Strongly convex with modulus $\mathcal{C}(\cdot)$ or that $\mathcal{Z}$ belongs to the class $S_c(\mathcal{C};\mathcal{H}\times\eth)$ if all $\nu\in[0;1]$ and $\eta;\rho\in\mathcal{H}$ we have :

$$\mathcal{Z}(\nu\eta+(1-\nu)\rho,\cdot)\leqslant\nu\mathcal{Z}(\eta,\cdot)+(1-\nu)\mathcal{Z}(\rho,\cdot)-\mathcal{C}(\cdot)\nu(1-\nu)(\eta-\rho)^2 \tag{1.2}$$

If (1.2) is true exclusively for $\nu=\frac{1}{2}$, $\mathcal{Z}$ belongs to the class $S_J(\mathcal{C};\mathcal{H}\times\eth)$ namely $\mathcal{Z}$ is strongly Jensen-convex .

iii) Additive if : $\mathcal{Z}(\eta+\rho,\cdot)=\mathcal{Z}(\eta,\cdot)+\mathcal{Z}(\rho,\cdot)$ hold for all $\eta,\rho\in\mathcal{H}$

Note that :

- if $\mathcal{C}(.)\equiv 0$ in (1.1) then $\mathcal{Z}$ is Wright-convex.
- if $\mathcal{C}(.)\equiv 0$ in (1.2) then $\mathcal{Z}$ is convex .

For more details we refer to [4]; [5] and [9].

Also $\mathcal{Z}:\mathcal{H}\times\eth\to(0,\infty)$ is named log-convex if :

$$\mathcal{Z}(\ell\eta+(1-\ell)\rho,\cdot)\leq[\mathcal{Z}(\eta,\cdot)]^{\ell}[\mathcal{Z}(\rho,\cdot)]^{1-\ell}$$

where $\eta,\rho\in\mathcal{H}$ and $\ell\in(0,1)$ (See [11]).

Consider the integer $n\geq 2$ and $\rho=(\rho_1,\ldots,\rho_n),\vartheta=(\vartheta_1,\ldots,\vartheta_n)\in\mathcal{H}^n$.

If there exists a doubly stochastic $n\times n$ *matrix* M where $\rho=\vartheta\cdot M$, we say that ρ is majorized by ϑ and write $\rho\preceq\vartheta$.

Given a stochastic process $\Psi:\mathcal{H}^n\times\eth\to\mathbb{R}$

Ψ is said to be Schur-convex if : $\rho\preceq\vartheta\Rightarrow\Psi(\rho,\cdot)\leq\Psi(\vartheta,\cdot)$

for all $\rho, \vartheta \in \mathcal{H}^n$.

For more details see [3] and [7].

2. Main results

Definition 1: Considering the stochastic process $\varphi : \mathcal{H}^n \times \eth \to \mathbb{R}$.

This process is named strongly Schur convex with modulus $\mathcal{C}(\cdot)$, $(\mathcal{C}(\cdot) \geq 0)$ or belongs to the class $S_p(\mathcal{C}; \mathcal{H}^n \times \eth)$ if for any $\rho, \vartheta \in \mathcal{H}^n$:

$$\rho \preceq \vartheta \Rightarrow \varphi(\rho;.) \leq \varphi(\vartheta;.) - \mathcal{C}(\cdot)\left(\sum_{i=1}^{i=n}(\vartheta_i^2 - \rho_i^2)\right)$$

Lemma 1: *Consider the stochastic processes* $\varphi : \mathcal{H}^n \times \eth \to \mathbb{R}$ *and* $\Psi : \mathcal{H}^n \times \eth \to \mathbb{R}$ *defined by* $\Psi(\rho_1, \ldots, \rho_n, \cdot) = \varphi(\rho_1, \ldots, \rho_n, \cdot) - \mathcal{C}(\cdot)\ (\sum_{i=1}^{i=n} \rho_i^2)$ *we have:*

$\Psi(\rho_1, \ldots, \rho_n, \cdot)$ *is schur convex* $\Leftrightarrow \varphi(\rho_1, \ldots, \rho_n, \cdot) \in S_p(\mathcal{C}; \mathcal{H}^n \times \eth)$.

Proof: It follows from definition 1 .

Theorem 1: *Consider the stochastic processes* $\mathcal{Z} : \mathcal{H} \times \eth \to \mathbb{R}$ *and* $\varphi : \mathcal{H}^n \times \eth \to \mathbb{R}$ *given by :*

$\varphi(\rho_1, \ldots, \rho_n, \cdot) = \mathcal{Z}(\rho_1, \cdot) + \cdots + \mathcal{Z}(\rho_n, \cdot)$ we have :

if $\mathcal{Z} \in S_c(\mathcal{C}; \mathcal{H} \times \eth)$ then $\varphi \in S_p(\mathcal{C}; \mathcal{H}^n \times \eth)$

Proof: Let $\rho, \vartheta \in \mathcal{H}^n$ such that $\rho \preceq \vartheta$.

Given that $\mathcal{Z} \in S_c(\mathcal{C}; \mathcal{H} \times \eth)$ this implies the convexity of $\mathcal{Z}(t, \cdot) - \mathcal{C}(.)t^2 = \mathcal{Y}(t, \cdot)$

(See [5]; lemma 2).

So,

$S_{\mathcal{Y}}(\rho_1; \ldots; \rho_n, \cdot) = \mathcal{Y}(\rho_1, \cdot) + \cdots + \mathcal{Y}(\rho_n, \cdot)$ is schur convex
consequently

$$\mathcal{Y}(\rho_1, \cdot) + \ldots + \mathcal{Y}(\rho_n, \cdot) \leq \mathcal{Y}(\vartheta_1, \cdot) + \ldots + \mathcal{Y}(\vartheta_n, \cdot)$$

Then

$$(\mathcal{Z}(\rho_1, \cdot) - \mathcal{C}(\cdot)\rho_1^2) + \ldots + (\mathcal{Z}(\rho_n, \cdot) - \mathcal{C}(\cdot)\rho_n^2) \leq (\mathcal{Z}(\vartheta_1, \cdot) - \mathcal{C}(\cdot)\vartheta_1^2) + \ldots + (\mathcal{Z}(\vartheta_n, \cdot) - \mathcal{C}(\cdot)\vartheta_n^2)$$

which gives

$$\mathcal{Z}(\rho_1;, \cdot) + \ldots + \mathcal{Z}(\rho_n;, \cdot) \leqslant \mathcal{Z}(\vartheta_1, \cdot) + \ldots + \mathcal{Z}(\vartheta_n, \cdot) - \mathcal{C}(\cdot)((\vartheta_1^2 - \rho_1^2) + \ldots + (\vartheta_n^2 - \rho_n^2))$$

Corollary 1: *Consider the stochastic processes* $\mathcal{Z};\mathcal{Y};\mathcal{T} : \mathcal{H}\times\eth\to\mathbb{R}$ *where* $\mathcal{Y}$ *is strictly positive then,* $\mathcal{Z}$ *is convex implies that the stochastic process :*

$$S_{\mathcal{Z}}(\rho_1,\ldots,\rho_n,\cdot)=\mathcal{T}(\textstyle\sum_{i=1}^n\rho_i,\cdot)+\mathcal{Y}(\sum_{i=1}^n\rho_i;.)(\sum_{i=1}^n\mathcal{Z}(\rho_i,\cdot))$$

is Schur convex.

Proof: In theorem 1 we take $\mathcal{C}(.)\equiv 0$ and we use the fact that if $\rho=(\rho_1,\ \ldots,\ \rho_n)\preceq\vartheta=(\vartheta_1,\ \ldots,\ \vartheta_n)$ then $\sum_{i=1}^n\rho_i=\sum_{i=1}^n\vartheta_i$

Theorem 2: *Consider the stochastic processes* $\mathcal{Z}:\mathcal{H}\times\eth\to\mathbb{R}$ *and* $\varphi:\mathcal{H}^n\times\eth\to\mathbb{R}$ *such that* $\varphi(\rho_1,\ \ldots,\ \rho_n,\cdot)=\mathcal{Z}(\rho_1,\cdot)+\cdots+\mathcal{Z}(\rho_n,\cdot)$

if $\varphi\in S_p(\mathcal{C};\mathcal{H}^n\times\eth)$ then $\mathcal{Z}\in S_J(\mathcal{C};\mathcal{H}\times\eth)$.

Proof: Let $\vartheta_1;\vartheta_2\in\mathcal{H}$

We take $\rho=(\rho_1;\rho_2;\vartheta_2;\vartheta_2\ldots;\vartheta_2),\vartheta=(\vartheta_1;\vartheta_2;\vartheta_2;\ldots;\vartheta_2)$ with $\rho_1=\rho_2=\frac{\vartheta_1+\vartheta_2}{2}$

So ,

$$\rho=M\times\vartheta\quad\text{with }M=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0&\cdots&0\\ \frac{1}{2}&\frac{1}{2}&0&\cdots&0\\ 0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&1\end{bmatrix}$$

consequently $\rho\preceq\vartheta$

Then by using the definition of strongly schur convexity of φ we obtain the desired result

Theorem 3: *We consider the stochastic processes* $\mathcal{Z}:\mathcal{H}\times\eth\to\mathbb{R}$ *and* $\varphi:\mathcal{H}^n\times\eth\to\mathbb{R}$ *given by :*

$$\varphi(\rho_1,\ \ldots,\ \rho_n,\cdot)=\mathcal{Z}(\rho_1,\cdot)+\ldots+\mathcal{Z}(\rho_n,\cdot).$$

the next assertions are equivalents :

i) $\varphi\in S_p(\mathcal{C};\mathcal{H}^n\times\eth)$, $(\forall n\geq 2)$

ii) $\varphi\in S_p(\mathcal{C};\mathcal{H}^n\times\eth)$, *for some* $n\geq 2$

iii) $\mathcal{Z}\in S_w(\mathcal{C};\mathcal{H}\times\eth)$

iv) *There exist the processes* $\mathcal{A}:\mathcal{H}\times\eth\to\mathbb{R}$ *and* $\mathcal{Y}\in S_c(\mathcal{C};\mathcal{H}\times\eth)$ *with* $\mathcal{Z}=\mathcal{Y}+\mathcal{A}$

where $\mathcal{A}$ *is additive.*

Proof: The implication $i) \Rightarrow ii)$ is clair.

For $ii) \Rightarrow iii)$: Let $\vartheta_1;\vartheta_2 \in \mathcal{H}$ and $t \in [0,1]$ we consider $\rho = (\rho_1, \ldots, \rho_n)$; $\vartheta = (\vartheta_1, \ldots, \vartheta_n) \in \mathcal{H}^n$ with $\rho_1 = t\vartheta_1 + (1-t)\vartheta_2$; $\rho_2 = (1-t)\vartheta_1 + t\vartheta_2$ and $\rho_i = \vartheta_i = \vartheta_2$ for $i \in \{3;\ldots;n\}$
i.e.,

$$\rho = (t\vartheta_1 + (1-t)\vartheta_2, (1-t)\vartheta_1 + t\vartheta_2; \vartheta_2; \vartheta_2; \vartheta_2; \ldots, \vartheta_2) \text{ and}$$
$$\vartheta = (\vartheta_1, \vartheta_2; \vartheta_2; \vartheta_2; \vartheta_2; \ldots, \vartheta_2)$$

Hence, we get $\rho \preceq \vartheta$

Since $\varphi(\rho;.) \in S_p(\mathcal{C};\mathcal{H}^n \times \eth)$

then $\varphi(\rho;.) - \mathcal{C}(.)(\rho_1^2 + \ldots + \rho_n^2)$ is schur convex consequently,

$$\mathcal{Z}(t\vartheta_1 + (1-t)\vartheta_2;.) + \mathcal{Z}((1-t)\vartheta_1 + t\vartheta_2;.) \le \mathcal{Z}(\vartheta_1;.) + \mathcal{Z}(\vartheta_2;.) - \mathcal{C}(.)(\textstyle\sum_{i=1}^{i=n}(\vartheta_i^2 - \rho_i^2))$$

with $$\textstyle\sum_{i=1}^{i=n}(\vartheta_i^2 - \rho_i^2) = \vartheta_2^2 + \vartheta_1^2 - (t\vartheta_1 + (1-t)\vartheta_2)^2 - ((1-t)\vartheta_1 + t\vartheta_2)^2$$
$$= 2t(1-t)(\vartheta_2 - \vartheta_1)^2$$

then we get the desired result .

$iii) \Rightarrow iv)$ See theorem 2 in [4]

$iv) \Rightarrow i)$ we have $\mathcal{Z} = \mathcal{Y} + \mathcal{A}$ then $\mathcal{Z} - \mathcal{C}(.)t^2 = \mathcal{Y} - \mathcal{C}(.)t^2 + \mathcal{A}$ with $\mathcal{Y} - \mathcal{C}(.)t^2$ is convex then,

The process $\psi(\rho_1, \ldots, \rho_n;.) = \mathcal{Z}(\rho_1,\cdot) + \mathcal{Z}(\rho_2,\cdot) + \ldots + \mathcal{Z}(\rho_n,\cdot) - \mathcal{C}(.)(\sum_{i=1}^{i=n}\rho_i^2)$ is schur convex

i.e., $\psi(\rho_1,\ldots,\rho_n;.) = \varphi(\rho_1,\ldots,\rho_n;.) - \mathcal{C}(.)(\sum_{i=1}^{i=n}\rho_i^2)$ is schur convex

So , $\varphi \in S_p(\mathcal{C};\mathcal{H}^n \times \eth)$.

Definition 2: If the binary operation $\oplus$ on $[\alpha,\beta]$ is continuous, associative, commutative, increasing and $\mathbf{0}$ its neutral element then $\oplus$ is called a pseudo-addition.

Definition 3: If the binary operation $\odot$ on $[\alpha,\beta]$ is positive, associative, commutative, increasing with $\odot$ is distributive over $\oplus$ and $\mathbf{1} \in [\alpha,\beta]$ its neutral element then $\odot$ is called a pseudo-multiplication.

For more details we refer to [6].

Remark 1: For continuous operations, there are three basic kinds of semirings $([\alpha,\beta],\oplus,\odot)$.

One of these kinds that we will use in this work is the class whose the pseudo-operations are given by a continuous bijection and strictly monotone $g:[\alpha,\beta]\to[0,\infty]$, where :

$$s\oplus k=g^{-1}(g(s)+g(k)) \quad \text{and} \quad s\odot k=g^{-1}(g(s)g(k))$$

(See [10] for more details).

Theorem 4: *Assume that the function g is increasing and convex .*

Consider the stochastic process $\mathcal{Z}:\mathcal{H}\times\eth\to\mathbb{R}$.

Suppose that $\mathcal{Z}$ is convex

Let $\rho=(\rho_1;\rho_2;....;\rho_n)$; $\vartheta=(\vartheta_1;\vartheta_2;....;\vartheta_n)\in\mathcal{H}^n$.

if $\rho\preceq\vartheta$ then $\oplus_{i=1}^{n}\mathcal{Z}(\rho_i;.)\leqslant\oplus_{i=1}^{n}\mathcal{Z}(\vartheta_i;.)$

Proof: $\mathcal{Z}$ and g are convex then $g\circ\mathcal{Z}$ is convex .

We suppose that $\rho\preceq\vartheta$ by using theorem 1 with $\mathcal{C}(.)\equiv0$ we get

$$g\circ\mathcal{Z}(\rho_1,\cdot)+g\circ\mathcal{Z}(\rho_2,\cdot)+....+g\circ\mathcal{Z}(\rho_n,\cdot)\le g\circ\mathcal{Z}(\vartheta_1,\cdot)+g\circ\mathcal{Z}(\vartheta_2,\cdot) +....+g\circ\mathcal{Z}(\vartheta_n,\cdot)$$

Since the function g^{-1} is increasing

Then;

$$g^{-1}(g\circ\mathcal{Z}(\rho_1,\cdot)+g\circ\mathcal{Z}(\rho_2,\cdot)+....+g\circ\mathcal{Z}(\rho_n,\cdot)) \le g^{-1}(g\circ\mathcal{Z}(\vartheta_1,\cdot)+g\circ\mathcal{Z}(\vartheta_2,\cdot)+....+g\circ\mathcal{Z}(\vartheta_n,\cdot))$$

i.e.,

$$\mathcal{Z}(\rho_1,\cdot)\oplus\mathcal{Z}(\rho_2,\cdot)\oplus....\oplus\mathcal{Z}(\rho_n,\cdot)\le\mathcal{Z}(\vartheta_1,\cdot)\oplus\mathcal{Z}(\vartheta_2,\cdot)\oplus....\oplus\mathcal{Z}(\vartheta_n,\cdot)$$

If we take $g=id(ie\,g(x)=x)$ in theorem 4 we obtain the following corollary

Corollary 2: *(See [3]) We assume that the stochastic process* $\mathcal{Z}:\mathcal{H}\times\eth\to\mathbb{R}$ *is convex then the stochastic process* $\mathcal{T}:\mathcal{H}^n\times\eth\to\mathbb{R}$ *given by*

$$\mathcal{T}(\rho_1,\dots,\rho_n,\cdot)=\mathcal{Z}(\rho_1,\cdot)+\dots+\mathcal{Z}(\rho_n,\cdot)$$

is Schur-convex.(for every $n\geqslant2$)

Corollary 3: *Let* $\mathcal{Z};\mathcal{Y};\mathcal{T}:\mathcal{H}\times\eth\to\mathbb{R}$ *are stochastic processes, such that* $\mathcal{Z}$ *and* $\mathcal{Y}$ *are stricly positives .*

The stochastic process $\mathcal{Z}$ is logconvex implies that the stochastic process $\Theta(\rho_1, \ldots, \rho_n,\cdot) = \mathcal{T}(\rho_1 + \ldots + \rho_n,\cdot) + \mathcal{Y}(\rho_1 + \ldots + \rho_n,\cdot)$ $(\log\mathcal{Z}(\rho_1,\cdot) + \ldots + \log\mathcal{Z}(\rho_n,\cdot))$ is Schur convex.

Proof: Suppose that $g = id$ in theorem 4 and we use the fact that $\log\mathcal{Z}$ is convex if $\mathcal{Z}$ is log convex , then we get the desired result

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