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Mixture unit Weibull distribution : Properties and applications

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Abstract

In this study, two component mixture unit-Weibull (TCMUW) model is proposed for fitting data on (0, 1) interval by mixing two unit-Weibull models. The density function plots give the impression that the new model can exhibit negative or right skewness, unimodal or bimodal shapes. The hazard function displayed various desirable shapes. The relevance of the TCMUW model are exemplified utilizing datasets. The outcome of the findings intimates that the TCMUW model proffer reasonable fit with minimal loss of information than the other models compared with.

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1. Introduction

Countless researchers in the statistical jurisdiction have proffered alternative distributions for data modeling. The intent behind these developments is to obtain distributions that are more flexible or improve on the performances of the existing distributions. The Weibull distribution

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is cardinal in data modeling, especially in the sphere of product reliability or survival analysis of lifetime data. The mixture of Weibull distributions is highly helpful in statistical analysis, due to their numerous applications. As a result, many researchers have proposed mixture distributions for modeling datasets with varied traits. These distributions include: Mixture of two inverse Weibull distribution [25], mixture Weibull-generalized gamma distribution [12], log-Lindley [11], generalized mixtures of Weibull components distribution [10], bivariate mixture of inverse Weibull [2], unit-improved second-degree Lindley [3], serial Weibull Rayleigh distribution [20], two improved mixture Weibull distribution [21], heterogeneous mixture distribution [22], finite mixture of two Weibull distribution [30], Fréchet-Weibull mixture distribution [26], two-component mixture of transmuted Weibull distribution [29], two-component mixture of inverse Weibull distributions [23], weighted inverted Weibull distribution [9], generalized logistic Weibull distribution [27] and so forth. We propose the two-component mixture unit-Weibull (TCMUW) model in this study as an appendage of the unit-Weibull distribution of Mazucheli et al. [15]. Our objective for this extension is to enhance the performance the unit-Weibull distribution.

The remnants are: Section 2 presents the TCMUW model, Section 3 introduces properties of the model, Section 4 gives the estimation method, Section 6 demonstrates the applicability of the model and Section 7 proffers the conclusions.

2. TCMUW Model

According to Mazucheli et al. [15, 17], a random variable X, follows the unit-Weibull model if its cumulative distribution function (CDF) is

$$F(x) = \exp[-\alpha(-\log x)^{\beta}], 0 < x < 1, \alpha > 0, \beta > 0. \tag{1}$$

The related probability density function (PDF) is

$$f(x) = \frac{1}{x}\alpha\beta(-\log x)^{\beta-1}\exp[-\alpha(-\log x)^{\beta}], 0 < x < 1. \tag{2}$$

Suppose X_1 is a random variable of the first unit-Weibull model, with CDF

$$F_{X_1}(x) = \exp[-\alpha(-\log x)^{\beta}], 0 < x < 1 \tag{3}$$

and PDF defined as

$$f_{X_1}(x) = \frac{1}{x}\alpha\beta(-\log x)^{\beta-1}\exp[-\alpha(-\log x)^{\beta}], 0 < x < 1. \tag{4}$$

Let X_2 be a random variable of the second unit-Weibull model, with CDF

$$F_{X_2}(x) = \exp[-\gamma(-\log x)^{\lambda}], 0 < x < 1 \tag{5}$$

and PDF given by

$$f_{X_2}(x) = \frac{1}{x}\gamma\lambda(-\log x)^{\lambda-1}\exp[-\gamma(-\log x)^{\lambda}], 0 < x < 1. \tag{6}$$

In order to obtain the TCMUW model, let

$$X = \begin{cases} X_1 \text{ with } & \text{probability } v_1 \\ X_2 \text{ with } & \text{probability } v_2 \end{cases},$$

where $0 \le v_i \le 1$, $i = 1,2$ and $v_1 + v_2 = 1$. Thus, the CDF of the mixture model is developed using X_1 and X_2, and CDF of X is

$$F(x) = v_1 F_{X_1}(x) + v_2 F_{X_2}(x), \tag{7}$$

where $v_1 = \frac{\theta}{\theta+1}$ and $v_2 = \frac{1}{\theta+1}$. Substituting equations and into equation and simplifying yields

$$F(x) = \frac{\theta\exp[-\alpha(-\log x)^{\beta}] + \exp[-\gamma(-\log x)^{\lambda}]}{\theta+1}, 0 < x < 1. \tag{8}$$

The accompanying PDF of the TCMUW model is

$$f(x) = \frac{\theta\alpha\beta(-\log x)^{\beta-1}\exp[-\alpha(-\log x)^{\beta}] + \gamma\lambda(-\log x)^{\lambda-1}\exp[-\gamma(-\log x)^{\lambda}]}{x(\theta+1)}. \tag{9}$$

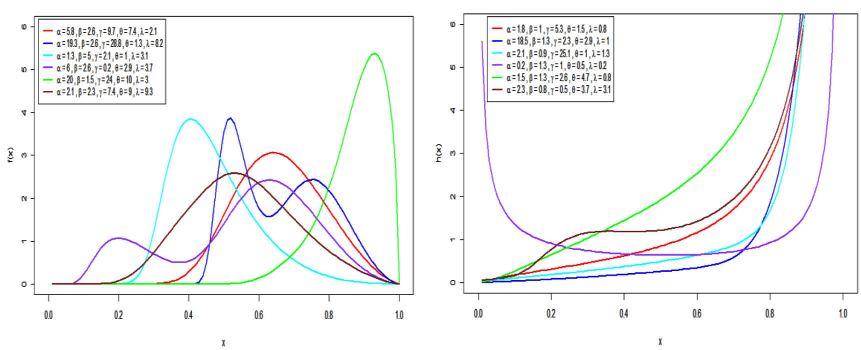

Figure 1

Pictorial view of PDF (left) and HRF (Right)

The hazard rate function (HRF) of the TCMUW model is

$$h(x)=\frac{\theta\alpha\beta(-\log x)^{\beta-1}\exp[-\alpha(-\log x)^{\beta}]+\gamma\lambda(-\log x)^{\lambda-1}\exp[-\gamma(-\log x)^{\lambda}]}{x(\theta+1)-x[\theta\exp(-\alpha(-\log x)^{\beta})+\exp(-\gamma(-\log x)^{\lambda})]}. \tag{10}$$

Figure 1 unveils that the PDF and HRF of the TCMUW model possesses interesting shapes. This makes the model a competitive one for data modeling on the (0, 1) interval.

3. Statistical Properties of TCMUW Distribution

Some properties of the TCMUW model are described here.

3.1 *Quantile Function*

For a specific statistical distribution, random data are attained utilizing the quantile function (QF). The QF is handy for describing the median, quartiles and measures of shapes of a particular distribution [7]. Suppose $Q_X(u)$ represents the QF of TCMUW model. Let $F(x)=u,$ for all $u\in(0,1).$ Using the fact that

$$Q_X=v_1Q_{X_1}(u)+v_2Q_{X_1}(u),$$

where $Q_{X_i}(u),\ i=1,2$ denote the quantile functions of the unit-Weibull models. The QF of the TCMUW model after some simplifications yields

$$Q_X(u)=\frac{\theta\exp\left[-\left(\frac{-\log(u)}{\alpha}\right)^{1/\beta}\right]+\exp\left[-\left(\frac{-\log(u)}{\gamma}\right)^{1/\lambda}\right]}{(\theta+1)},u\in(0,1). \tag{11}$$

The Bowley skewness (BS) and Moors kurtosis (MK) are both vital measures of shapes of a distribution and are computed utilizing the QF. Hence,

$$BS=\frac{[Q_X(0.75)+Q_X(0.25)-2Q_X(0.5)]}{[Q_X(0.75)-Q_X(0.25)]}$$

and

$$MK=\frac{[Q_X(0.375)-Q_X(0.125)+Q_X(0.875)-Q_X(0.625)]}{[Q_X(0.75)-Q_X(0.25)]}.$$

Figures 2 display the plots of BS for some set of parameter values. Variations in parameter values have an effect on the skewness. According to the figures, the TCMUW model can be either negatively or positively skewed.

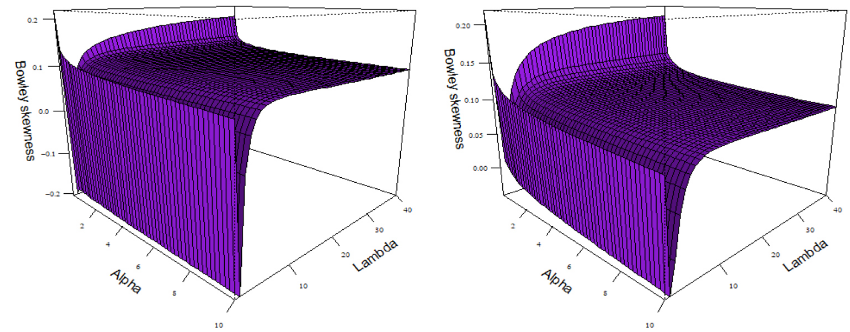

Figure 2

Bowley skewness of the TCMUW distribution

Figures 3 display the MK plots for some specific values of the parameters. Variations in parameter values have an effect on the kurtosis. MK plots revealed that the TCMUW model can either have a less or high peak than the normal curve.

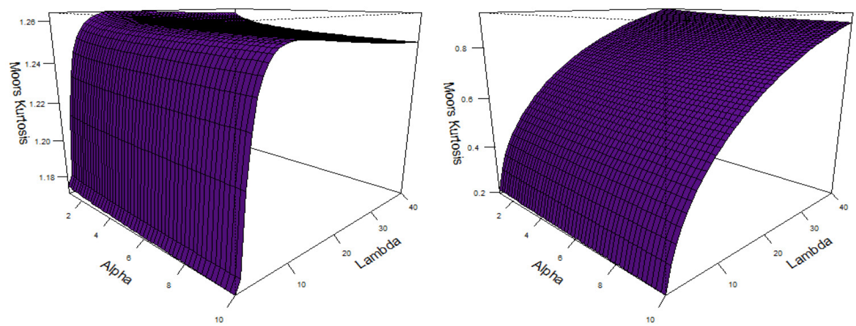

Figure 3

Moor kurtosis of the TCMUW distribution

3.2 *Moment*

In the analysis of statistical data, moments are very important. The mean, standard deviation, variance, kurtosis, and skewness among others can be established using the moments. The TCMUW model's k^{th} moments can be expressed as

$$\mu'_k = \frac{\theta}{\theta+1}\mu'_{k_1} + \frac{1}{\theta+1}\mu'_{k_2}, \tag{12}$$

where

$$\mu'_{k_1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\alpha^{k/\beta}} \Gamma\left(\frac{k}{\beta}+1\right) \text{ and } \mu'_{k_2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\gamma^{k/\lambda}} \Gamma\left(\frac{k}{\lambda}+1\right).$$

Substituting μ'_{k_1} and μ'_{k_2} into equation (12), the k^{th} moment of the TCMUW model is

$$\mu'_k = \frac{\theta}{\theta+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\alpha^{k/\beta}} \Gamma\left(\frac{k}{\beta}+1\right) + \frac{1}{\theta+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\gamma^{k/\lambda}} \Gamma\left(\frac{k}{\lambda}+1\right). \tag{13}$$

Table 1 presents estimates of the first four moments, standard deviation (∂_1), coefficient of variation (∂_2), coefficient of skewness (∂_3), and coefficient of kurtosis (∂_4) for:

I : $\alpha = 15, \beta = 6, \gamma = 35, \theta = 1, \lambda = 0.5$,
II : $\alpha = 15, \beta = 6, \gamma = 35, \theta = 1, \lambda = 0.3$,
III : $\alpha = 15, \beta = 6, \gamma = 35, \theta = 1, \lambda = 45$ and
IV : $\alpha = 15\beta = 6, \gamma = 35, \theta = 1, \lambda = 10$.

The estimates for $\partial_1, \partial_2, \partial_3$ and ∂_4 are attained by making use of

$$\partial_1 = \sqrt{\mu'_2 - \mu^2},$$

$$\partial_2 = \frac{\sigma}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$\partial_3 = \frac{\mu'_3 - 3\mu'_2\mu + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}$$

and

$$\partial_4 = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}.$$

The ∂_3 values in Table 1 shows that both negatively-skewed and positively-skewed datasets can be modeled by the TCMUW model. The TCMUW distribution ∂_4 values demonstrate that, based on the values of the parameter, it may be less or more peaked than the normal curve because the ∂_4 can be less than three or more than three. This further affirms that the TCMUW distribution is highly flexible for modeling data.

Table 1

First four values of moments, as well as $\partial_1, \partial_2, \partial_3$ and ∂_4

μ'_k	I	II	III	IV
μ'_1	0.7780	0.7788	0.4796	0.5363
μ'_2	0.6560	0.6575	0.2383	0.2912
μ'_3	0.5880	0.5903	0.1228	0.1601
μ'_4	0.5494	0.5525	0.0656	0.0893
∂_1	0.2252	0.2259	0.0910	0.0591
∂_2	0.2895	0.2901	0.1897	0.1102
∂_3	-0.1147	-0.1143	0.7810	1.0347
∂_4	1.1391	1.1377	2.6481	4.4632

4. Parameter Estimation

The TCMUW model parameters are estimated here employing maximum likelihood (ML) approach. If $x_1, x_2, \ldots, x_n$ are random observations of size n from the TCMUW model, the log-likelihood function is

$$\ell = \sum_{i=1}^{n} \log[\theta\alpha\beta(-\log x_i)^{\beta-1} \exp[-\alpha(-\log x_i)^{\beta}] + \gamma\lambda(-\log x_i)^{\lambda-1} \exp[-\gamma(-\log x_i)^{\lambda}]] \\ -\sum_{i=1}^{n} \log[(\theta+1)x_i]. \tag{14}$$

Numerical values of the TCMUW model parameters are attained by maximizing equation (14). Different symbolic software such as MATLAB, MATHEMATICA, MAPLE, SAS or R can be used for the numerical optimization.

5. Simulation Study

To study the ML estimators of the parameters, simulation exercises are undertaken utilizing two sets of parameter combinations. The two sets of parameter combinations are: $\mathrm{I}: \alpha = 2.3,\ \beta = 3.5,\ \gamma = 1.2,\ \lambda = 3.2,\ \theta = 0.5$ and $\mathrm{II}: \alpha = 3.5,\ \beta = 4.1,\ \gamma = 2.3,\ \lambda = 4.5,\ \theta = 0.2.$ The experiments are replicated 5000 times using sample sizes $n = 20,\ 60,\ 100,\ 140,\ 180,\ 300,\ 600$ and $800.$ The samples were drawn utilizing the QF. The performances of estimators are examined employing the absolute bias (AB), root mean square error (RMSE) and coverage probability (CP). From Table 2, the performance measures intimates that the ML estimators are well behaved.

Table 2

Simulation results

Parameter	N	I			II		
		AB	RMSE	CP	AB	RMSE	CP
α	20	1.7854	2.1788	0.9940	3.8210	4.1025	0.9950
	60	1.4224	1.9451	0.9860	3.1246	3.6621	0.9780
	100	1.2327	1.8081	0.9660	2.7542	3.3918	0.9690
	140	1.0675	1.6936	0.9460	2.5301	3.1991	0.9680
	180	0.9191	1.5714	0.9380	2.4365	3.1484	0.9580
	300	0.7808	1.5045	0.8870	2.2193	2.9825	0.9450
	600	0.6261	1.4174	0.8740	1.9394	2.8390	0.9300
	800	0.5749	1.4088	0.8530	1.7895	2.7031	0.9320
β	20	0.4678	1.0959	0.9860	0.8026	1.6703	0.9790
	60	0.4266	0.9400	0.9860	0.7052	1.6677	0.9770
	100	0.3562	0.8790	0.9760	0.6898	1.5447	0.9640
	140	0.2896	0.8304	0.9780	0.6781	1.4709	0.9660
	180	0.2482	0.8265	0.9730	0.6279	1.4394	0.9540
	300	0.1803	0.7786	0.9700	0.6159	1.3999	0.9120
	600	0.0496	0.7506	0.9640	0.5754	1.3590	0.9160
	800	0.0415	0.7390	0.9640	0.5455	1.3295	0.9110
γ	20	0.1405	0.7314	0.9800	0.2163	1.3000	0.9810
	60	0.1386	0.6153	0.9810	0.2087	0.9426	0.9750
	100	0.1379	0.4678	0.9900	0.2004	0.6141	0.9680
	140	0.1073	0.3898	0.9840	0.1844	0.5560	0.9790
	180	0.0983	0.3317	0.9880	0.1340	0.4617	0.9730
	300	0.0899	0.3048	0.9900	0.1310	0.4251	0.980
	600	0.0772	0.2910	0.9850	0.0951	0.3224	0.9790
	800	0.0500	0.2795	0.9820	0.0669	0.3089	0.9820
λ	20	1.1820	1.5743	0.9890	1.9214	2.5467	0.9660
	60	1.0486	1.4123	0.9880	1.5943	2.1979	0.9800
	100	1.0260	1.3633	0.9850	1.3311	1.8978	0.9780
	140	0.9696	1.3074	0.9840	1.2312	1.7563	0.9800
	180	0.9234	1.2587	0.9770	1.0621	1.5452	0.9910
	300	0.8550	1.1721	0.9590	0.9628	1.3535	0.9830
	600	0.7580	1.0448	0.8940	0.6180	0.9820	0.9850
	800	0.6849	0.9575	0.8620	0.5586	0.8728	0.9680

Contd...

θ	20	0.6330	0.9412	0.9710	0.5525	0.6377	0.9850
	60	0.5456	0.9341	0.9840	0.5403	0.6343	0.9680
	100	0.5243	0.9274	0.9840	0.5377	0.6318	0.9660
	140	0.5203	0.8719	0.9820	0.5123	0.6163	0.9760
	180	0.5179	0.8437	0.9720	0.5053	0.6160	0.9710
	300	0.5145	0.8253	0.9780	0.5011	0.6010	0.9640
	600	0.5071	0.8248	0.9530	0.4189	0.5655	0.9620
	800	0.4917	0.7924	0.9470	0.4051	0.5614	0.9570

6. Applications

Applicability of the TCMUW model is shown employing two published data. The TCMUW model performance is juxtaposed with that of other models. These models are: unit Burr-XII (UBXII) [14], unit Gompertz (UG) [18], unit Rayleigh (UR) [5], Topp-Leone (TL) [1], truncated Cauchy power inverse exponential (TCPIE) [7], unit Burr-III (UBIII) [8], new unit-Lindley (NUL) [16] and unit-Weibull (UW) [15] distributions. Information criterion (IC) like Akaike IC ($\wp_1$), Bayesian IC ($\wp_2$), and Corrected AIC ($\wp_3$), are employed in the identification of the best model. Also, Kolmogorov-Smirnov ($\wp_4$), Anderson-Darling ($\wp_5$) [4, 13], and [24] Cramér-von Mises ($\wp_6$) tests are adopted to select the most appropriate model. The first dataset is COVID-19 death rate of patients from 1st November to 26th December, 2020 in Canada. The second dataset is recovery rate from 3rd March to 7th May, 2020 of Spain's COVID-19 patients. The two datasets have been used by Nasiru et al. [19].

6.1 *Dataset I: COVID-19 mortality rate for Canada*

The *bbmle* package in R by Bolker [6] is utilized to find the ML estimates of the parameters. The prime values for the optimization of the fitted models are determined employing the *GenSA* package [28]. For the Canada mortality rate, the summary measures are presented in Table 3. It can be seen that the excess kurtosis is -0.4402 and coefficient of skewness is -0.0850 suggesting that the data is platykurtic and left-skewed respectively.

Table 3

Summary measures for dataset I

Minimum	Maximum	Median	Mean	Skewness	Kurtosis
0.1159	0.3347	0.2261	0.2305	-0.0850	-0.4402

The ML estimates of the models fitted are shown in Table 4 along with their standard errors and *p-values* using dataset I. From Table 4, it is seen that the parameters of the TCMUW, UBXII, UG, UR, TL, TCPIE, UBII and NUL are significant at 5% level.

Table 4

ML estimates of parameters for dataset I

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$
TCMUW	0.0017	9.8335	0.016	0.2743	10.644
SE	(0.0006)	(0.1048)	(0.0095)	(0.1229)	(1.4190)
p-value	(0.0041)	(<0.0001)	(0.0909)	(0.0256)	(<0.0001)
UBXII	0.0077	333.97			
SE	(0.0010)	(8.7989×10^{-7})			
p-value	(<0.0001)	(<0.0001)			
UG	0.0027	3.6579			
SE	(0.0017)	(0.3483)			
p-value	(0.1126)	(<0.0001)			
UR		0.4361			
SE		(0.0583)			
p-value		(<0.0001)			
TL	1.0814				
SE	(0.1445)				
p-value	(<0.0001)				
TCPIE	0.67509			0.10605	
SE	(0.1138)			(0.0241)	
p-value	(<0.0001)			(<0.0001)	
UBIII	0.0365	22.244			
SE	(0.0049)	(8.0146×10^{-6})			
p-value	(<0.0001)	(<0.0001)			
NUL				0.46732	
SE				0.04532	
p-value				<0.0001	
UW	0.05522	6.16021			
SE	(0.0194)	(0.5867)			
p-value	(0.0043)	(<0.0001)			

For the nine models considered using dataset I, Table 5 intimates that the TCMUW model is adjudged the best.

Table 5

Measures of model adequacy for dataset I

Model	ℓ	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$	$\wp_5$	$\wp_6$
TCMUW	87.47	-164.93	-154.8	-163.73	0.0685	0.2982	0.0475
p-value					(0.9554)	(0.9393)	(0.8931)
UBXII	58.67	-113.34	-109.29	-113.12	0.3215	9.0846	1.7991
p-value					(<0.0001)	(<0.0001)	(<0.0001)
UG	74.54	-145.08	-141.03	-144.86	0.1684	2.4442	0.4030
p-value					(0.0835)	(0.0532)	(0.0706)
UR	41.89	-81.778	-79.752	-81.704	0.4069	12.552	2.5923
p-value					(<0.0001)	(<0.0001)	(<0.0001)
TL	24.19	-46.375	-44.349	-46.301	0.4685	15.875	3.3539
p-value					(<0.0001)	(<0.0001)	(<0.0001)
TCPIE	0.75	2.5095	6.5602	2.736	0.5006	16.774	16.774
p-value					(<0.0001)	(<0.0001)	(<0.0001)
UBII	30.89	-57.775	-53.724	-57.549	0.43369	14.88	3.1122
p-value					(<0.0001)	(<0.0001)	(<0.0001)
NUL	50.33	-98.653	-96.627	-98.579	0.35497	9.5276	1.8698
p-value					(<0.0001)	(<0.0001)	(<0.0001)
UW	79.95	-155.91	-151.86	-155.68	0.14532	1.489	0.23885
p-value					(0.1877)	(0.1792)	(0.2035)

Figure 4 unveils the probability-probability (P-P) plots of the models fitted to dataset I. It is noted that the TCMUW model offers the most reasonable fit for the dataset I.

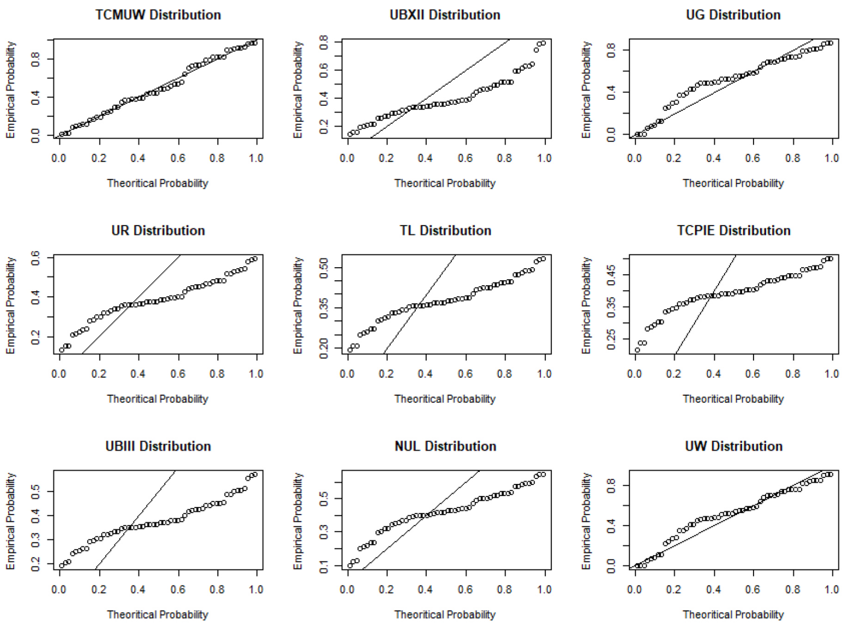

Figure 4

P-P plots for dataset I

6.2 *Dataset II: COVID-19 of Spain*

Table 6 presents the descriptive statistics of COVID-19 recovery rates of patients in Spain from March 3 to May 7, 2020. The excess kurtosis (-0.4761) and skewness (-0.6890), suggests that the data is platykurtic and left-skewed respectively.

Table 6

Summary measures for dataset II

Minimum	Maximum	Median	Mean	Skewness	Kurtosis
0.4286	0.8628	0.7533	0.7240	-0.6890	-0.4761

The ML estimates of the fitted models are given in Table 7 along with their corresponding standard errors and *p-values* using dataset II. It is observed that all the parameters of the TCMUW, UBXII, UG, UR, TL, TCPIE, UBII and NUL are significant at 5% level of significance. However, the parameter α for TCMUW is insignificant at 5% level.

Table 7
ML estimates of parameters for dataset II

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$
TCMUW	1000	4.7950	8.8389	0.8951	3.0772
SE	(983.2820)	(0.6601)	(2.6728)	(0.3615)	(0.5745)
p-value	(0.30915)	(<0.0001)	(0.00094)	(0.01328)	(<0.0001)
UBXII	10.5914	2.3612			
SE	(1.9865)	(0.1997)			
p-value	(<0.0001)	(<0.0001)			
UG	0.0027	3.8482			
SE	(0.1059)	(0.6025)			
p-value	(0.0083)	(<0.0001)			
UR		7.2290			
SE		(0.8898)			
p-value		(<0.0001)			
TL	10.528				
SE	(1.2960)				
p-value	(<0.0001)				
TCPIE	0.6816			0.3442	
SE	(0.1063)			(0.0722)	
p-value	(<0.0001)			(<0.0001)	
UBIII	5.4398	2.0613			
SE	(0.7948)	(0.1723)			
p-value	(<0.0001)	(<0.0001)			
NUL				2.99435	
SE				(0.30736)	
p-value				(<0.0001)	
UW	8.6445	2.2320			
SE	(1.6973)	(0.2036)			
p-value	(<0.0001)	(<0.0001)			

For the nine distributions considered using dataset II, Table 8 suggests that the TCMUW model offers the most reasonable fit the data set II.

Table 8

Measures of model adequacy dataset II

Model	ℓ	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$	$\wp_5$	$\wp_6$
TCMUW	62.06	-114.12	-103.17	-113.12	0.06358	0.35779	0.03686
p-value					(0.9524)	(0.8889)	(0.9498)
UBXII	54.63	-105.26	-100.88	-105.07	0.1255	1.2657	0.20354
p-value					(0.2496)	(0.2437)	(0.2610)
UG	46.03	-88.057	-83.678	-87.866	0.19224	2.4709	0.36908
p-value					(0.0152)	(0.0515)	(0.0871)
UR	53.28	-104.57	-102.38	-104.5	0.14566	1.3803	0.18279
p-value					(0.1215)	(0.2078)	(0.3039)
TL	51.55	-101.11	-98.919	-101.05	0.18133	2.0475	0.27752
p-value					(0.0261)	(0.0867)	(0.1566)
TCPIE	-75.13	154.266	158.645	154.456	0.5581	22.556	4.8336
p-value					(<0.0001)	(<0.0001)	(<0.0001)
UBII	53.8	-103.59	-99.213	-103.4	0.12961	1.3725	0.22093
p-value					(0.2175)	(0.2101)	(0.2306)
NUL	37.23	-72.454	-70.264	-72.391	0.30478	6.2515	1.1165
p-value					(<0.0001)	(0.0008)	(0.0012)
UW	53.97	-103.93	-99.5523	-103.74	0.13065	1.383	0.22377
p-value					(0.2099)	(0.2070)	(0.2260)

Figure 5 displays the P-P plots for dataset II. The figure intimates that the TCMUW model is the best.

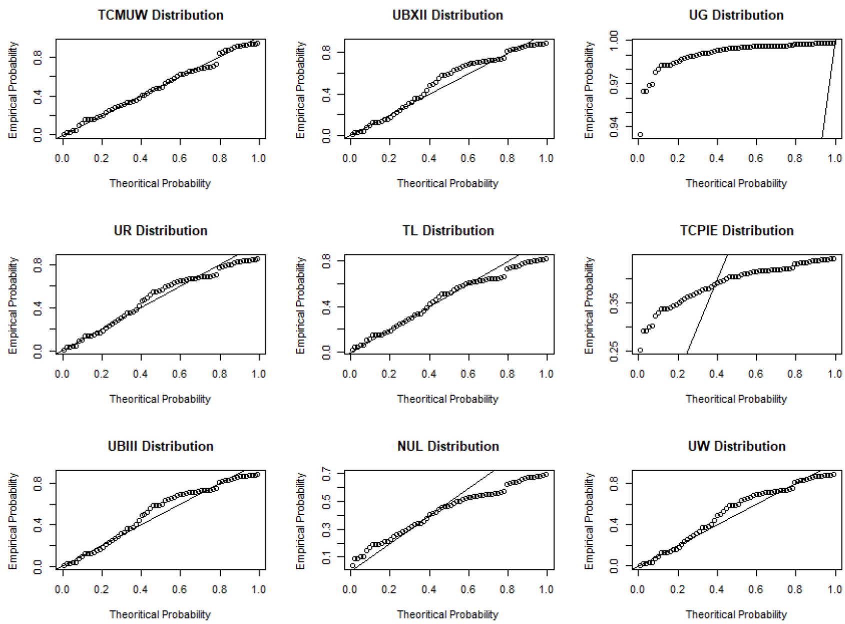

Figure 5

P-P plots for dataset II

7. Conclusions

A mixture model defined on (0, 1) interval is proposed in this study and is known as the TCMUW model. The estimates are attained employing maximum likelihood technique. The TCMUW model offered the most reasonable fit when it was fitted to two real datasets. The TCMUW model density and hazard rate functions demonstrate that the model is flexible and has the ability of modeling bimodal, negative-skewed, positive-skewed or symmetric datasets as well as datasets that are less or more peaked than the normal curve.

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