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Conjugate and objective priors for the parameter and cumulative function under bathtub and increased shapes of hazard function

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Abstract

We present a full Bayesian inference for the parameter and the associated reliability function of the Exponentiated Gamma distribution. This distribution has the advantage of being uni-parametric and its hazard function takes on increasing and bathtub shapes. We show that dealing with Jeffreys, uniform and gamma priors for the parameter then the corresponding posterior distributions result in a Gamma. Besides, the posterior distributions of cumulative function for a fixed lifetime are Negative Log-Gamma (NLG) distributions under transformation of parameter and also when Jeffreys, uniform and NGL priors are

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assigned to the cumulative function. This way, we show that the families of gamma and NLG distributions provide conjugate prior distribution for the parameters of interest. We also construct credible and HPD intervals in a simple closed-form. We further derive the predictive distribution and interval of future observation assuming gamma posterior distribution for the parameter.

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1. Introduction

Lifetime distributions which can accommodate increasing and bathtub-shaped hazard function have been widely used in engineering and medicine.

Gupta et al. [4] introduced an interesting uni-parametric lifetime distribution, Exponentiated gamma (EG), which hazard function has these same shape curves as well as exponential power [14] and Chen distributions [2]. However, the exponential power and Chen distributions have two-parameters to estimate. Therefore, the Exponentiated gamma distribution can be a good alternative to gamma, Weibull, exponential power and Chen distributions, after all, from the practical point of view, it is always important to consider parsimonious models with as few parameters as possible [12].

Khan and Kumar [6], Feroze and Aslam [3], Singh et al. [13] and others have also proposed the estimation procedures for the Exponentiated gamma distribution under complete and censored data.

It is known that the Bayesian approach has a reputation for making mathematics difficult and almost always requiring approximation and simulation methods to obtain the estimators and marginal posterior distributions. The Bayesian approach applied in the cited papers presents complex formulas for estimators and the authors do not provide known posterior distributions for the parameter and reliability. Our paper demonstrates that it is possible to derive simple closed-forms for the parameter and reliability, known posterior distributions for the parameters under each proposed prior, that is, it provides conjugate priors, closed-form for predictive distribution. In addition, it allows prior information to be taken into account in the analysis through prior predictive distribution.

In this paper, we present an exact Bayesian inference for the parameter θ and the associated reliability function $R = P\{T > t_o\}$ for a fixed lifetime t_o , hence not requiring the use of any approximation and computation algorithms as MCMC to obtain the posterior distributions and summary a posteriori, which can also be considered other advantage of this distribution. We also obtain conjugate prior distributions for the parameter of EG and for its cumulative function $W = 1 - R$.

Firstly, a fully Bayesian analysis is developed under Jeffreys [5], Uniform (flat) and vague priors for estimation of parameter θ. We show that resulting posterior is a gamma distribution for any of these priors. This way, we prove that the family of gamma distributions provides conjugate prior distribution for the parameter θ of EG distribution. Jeffreys prior and MLE result the same estimator for θ. We also demonstrate that the three classes of priors lead to a Negative Log-gamma (NLG) posterior distribution for the parameter W, through reparameterization.

Secondly, we carried out the Bayesian analysis by considering Jeffreys, uniform in interval [0, 1] and NLG prior distributions for the parameter W. The NLG distribution seems to be very appropriate to describe uncertainty in W, since it is very flexible: it can assume a wide range of shapes by varying the values of the shape parameters [7]. In this case, the resulting posterior is a NLG distribution for the three priors considered, hence, the NLG prior is a conjugate prior distribution for W under the EG distribution for the lifetime T.

In addition, the exact Bayesian estimates as posterior expectance, variance, credible and HPD intervals are also explicitly derived for the parameters θ, W and $R = 1 - W$.

We further consider the Bayesian posterior prediction of the future observation based on the present sample. To perform an inference about a future observation, the predictive distribution is determined using gamma posterior distribution for the parameter θ. The predictive density, cumulative functions and corresponding interval for a future observation are derived.

A simulation study is also performed to determine which prior provides better performance under absence of information for the parameters θ and W.

The Exponentiated Gamma distribution with parameter θ, denoted by EG, has density and corresponding cumulative function given by,

$$f(t) = \theta t e^{-t}[1 - e^{-t}(t+1)]^{\theta-1}, \tag{1}$$

$$F(t) = [1 - e^{-t}(t+1)]^{\theta}. \tag{2}$$

for all $t > 0$ with θ as the shape parameter.

Since the cumulative function (2) cannot be explicitly inverted then the quantiles can be obtained by numerically solving for the root of the equation given by

$$e^{-t}(t+1)+u^{1/\theta}-1=0, \tag{3}$$

where u is a random value from the Uniform[0, 1].

The survival and hazard functions associated to (1) are given, respectively, by

$$R(t)=1-[1-e^{-t}(t+1)]^{\theta} \tag{4}$$

and

$$h(t)=\theta te^{-t}[1-e^{-t}(t+1)]^{\theta-1}\{1-[1-e^{-t}(t+1)]^{\theta}\}^{-1}. \tag{5}$$

Shawky and Bakoban [8], [9], [10], [11] observed that the hazard function $h(t)$ is increasing when $\theta>1/2$ and takes bathtub shape when $\theta<1/2$.

The paper is organized as follows: In Section 2, the maximum likelihood estimator (MLE), exact confidence interval and the Bayesian inference for the parameter θ are derived. The MLE and Bayesian analysis for the cumulative $W=F(t_o)$ and reliability $R(t_o)$, with respective summaries and intervals are obtained as well in Section 3. Section 4 contains the Bayesian prediction under posterior gamma distribution for the parameter θ. The simulation study and comparisons are presented in Section 5 and a numerical example using real data set is illustrated in Section 6. Final comments are given in Section 7.

2. Estimation of the parameter θ

In this section, it is discussed the punctual and interval estimation for the parameter θ under the maximum likelihood and Bayesian methods.

2.1 *Maximum likelihood estimation*

Suppose we have independent identically distributed lifetimes T_1, T_2, …, T_n from the exponentiated gamma (1), then the likelihood function is given by:

$$L(\theta)\propto\theta^{n}\prod_{i=1}^{n}[1-e^{-t_i}(t_i+1)]^{\theta-1}=\theta^{n}\exp\left(-\theta\log\left(\frac{1}{c}\right)\right), \tag{6}$$

where $c=\prod_{i=1}^{n}[1-e^{-t_i}(t_i+1)]<1.$

The maximum likelihood estimator (MLE) for θ is $\hat{\theta}=\dfrac{n}{\log\left(\frac{1}{c}\right)}$ and the Fisher information for θ is given by $I(\theta)=\dfrac{n}{\theta^2}$.

For EG distribution is possible to obtain exact confidence intervals and hypothesis tests from the pivotal quantative $2\theta\log\left(\frac{1}{c}\right) \sim \chi^2_{(2n)}$, for example, a two-sided $100(1-\gamma)\%$ confidence interval for θ is easily obtained as:

$$\frac{\chi^2_{\gamma/2}(2n)}{2\log\left(\frac{1}{c}\right)} \le \theta \le \frac{\chi^2_{1-\gamma/2}(2n)}{2\log\left(\frac{1}{c}\right)} \tag{7}$$

where $\chi^2_p(2n)$ is the p-th quantile of $\chi^2_{(2n)}$.

2.2 *Bayesian estimation*

For Bayesian analysis, we assume three types of prior distributions for θ like Uniform, Gamma and Jeffreys priors.

Proposition 1:

a) Under Jeffreys prior $\pi_J(\theta) = \sqrt{I(\theta)}$ for the parameter θ, resulting in

$$\pi_J(\theta) \propto \frac{1}{\theta}, \tag{8}$$

the corresponding posterior is the Gamma distribution with parameters $\alpha^* = n$ and $\beta^* = \log\frac{1}{c}$.

Table 1

Estimator and variance of the parameter θ for each estimation method

Method	Prior	Estimator $\hat{\theta}$	Variance
MLE	—	$\frac{n}{\log\left(\frac{1}{c}\right)}$	$\frac{n}{[\log(c)]^2}$
Uniform prior	$\pi(\theta) \propto 1$	$\frac{n+1}{\log\left(\frac{1}{c}\right)}$	$\frac{n+1}{[\log(c)]^2}$
Gamma prior	$\pi(\theta) \propto \theta^{\alpha-1}e^{-\beta\theta}$	$\frac{n+\alpha}{\beta+\log\frac{1}{c}}$	$\frac{n+\alpha}{[\beta-\log(c)]^2}$
Jeffreys prior	$\pi(\theta) \propto \frac{1}{\theta}$	$\frac{n}{\log\left(\frac{1}{c}\right)}$	$\frac{n}{[\log(c)]^2}$

b) If the prior for θ is the $Gamma(\alpha, \beta)$ distribution $(\alpha > 0$ and $\beta > 0)$ with density

$$\pi_G(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \tag{9}$$

for all $\theta > 0$, then the corresponding posterior is a gamma distribution with parameters $\alpha^* = n + \alpha$ and $\beta^* = \beta + \log \frac{1}{c}$.

c) By assuming an uniform prior for θ then the posterior is a gamma distribution with parameters $\alpha^* = n + 1$ and $\beta^* = \log \frac{1}{c}$.

From the Proposition 1, it is noted that the prior gamma and posterior belong to the same family of densities implying a conjugate analysis. Besides, when the hyperparameters α and β tends to zero then the posterior distribution is the same by using a Jeffreys prior and hence the Bayesian estimators under both priors are the same that MLE as well. We can observe that the posterior under $Gamma(\alpha, \beta)$ prior is equal to the posterior under uniform prior if $\alpha = 1$ and $\beta = 0$. Table 1 shows the priors along with corresponding posterior means and variances and MLE summaries. Note that the maximum likelihood estimator obtained in (7) is equal to the Bayesian estimator obtained under the Jeffreys prior with the same variance.

3. Estimation of the reliability $R = R(t_o)$

In this Section, the maximum likelihood and Bayesian approaches are applied to estimate the reliability $R = R(t_o)$ of the EG distribution for some fixed value t_o.

3.1 *Maximum likelihood estimation of* $R = R(t_o)$

Because the maximum likelihood estimator is invariant under transformation then the MLE of reliability R is given by $\hat{R} = 1 - [1 - e^{-t_o} (t_o + 1)]^{\hat{\theta}}$, where $\hat{\theta} = \frac{n}{\log\left(\frac{1}{c}\right)}$ is the MLE of the parameter θ.

Now, from the $100(1-\gamma)\%$ confidence interval for θ given in (7) we obtain the exact $100(1-\gamma)\%$ confidence interval for R given by

$$1 - \exp\left(-\frac{\chi^2{}_{\gamma/2}(2n)}{2v\log(c)}\right) \le R \le 1 - \exp\left(-\frac{\chi^2{}_{1-\gamma/2}(2n)}{2v\log(c)}\right), \tag{10}$$

where $v = 1/\log[1-e^{-t_o}\ (t_o+1)] < 0$ for t_o fixed value.

3.2 *Bayesian estimation of W = F(t_o) and R = R(t_o)*

We derive posterior distributions $p(W \mid \boldsymbol{t})$ based on different prior distributions for the parameter θ and using the reparametrization given by $W = [1-e^{-t_o}\ (t_o+1)]^{\theta}$, as well as directly assigning a prior distribution $\pi(W)$ for W. This way, Bayes estimators for R can be obtained from the posterior $p(W \mid \boldsymbol{t})$.

Proposition 2: By considering the priors for θ given in Proposition 1 then the posterior for the parameter W is a Negative Log-gamma distribution, denoted by $NLG(k,\lambda)$, with density given by

$$p(W \mid t) = \frac{\lambda^k}{\Gamma(k)}\left(\log\frac{1}{W}\right)^{k-1} W^{\lambda-1}, \quad 0 < W < 1, \tag{11}$$

where

a) $\lambda = v\log(c)$ and $k = n$ if we consider a Jeffreys prior (11) for θ;

b) $\lambda = v\log(c)$ and $k = n + 1$ if θ has a uniform prior;

c) $\lambda = v[\log(c) - \beta]$ and $k = n + \alpha$ if θ has a $Gamma(\alpha,\beta)$ prior (12).

***Proof*:** The reparametrization $W = F(t_o) = [1-e^{-t_o}\ (t_o+1)]^{\theta}$ is monotonically decreasing function of θ, so there is an one-to-one transformation between W and θ. Therefore, the unique inverse is $\theta = -v\ \log(1/W)$ and the distribution of W can be determined from the distribution of θ by transformation of variables applying the Jacobian. □

Note that the posterior distributions under Uniform and Jeffreys priors are special cases of posterior under gamma prior distribution with parameters $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (0,0)$, respectively.

Now, our purpose is to present an alternative Bayesian analysis that allows to assign a prior distribution for the parameter W. The idea is to consider the likelihood function (6) depending of parameter W and assume different prior distributions for W.

Consider the likelihood function (6) with $\theta = v\ \log(W)$, then the likelihood can be written in terms of W as

$$L(W) \propto \left(\log\frac{1}{W}\right)^{n} W^{v\log(c)}. \tag{12}$$

From the likelihood function (13) then the Fisher information for W is given by

$$I(W)=\frac{n}{W^2\left(\log\frac{1}{W}\right)^2}. \tag{13}$$

Jeffreys prior for the parameter W is derived from Fisher Information as $\pi_J(W)=\sqrt{I(W)}$ resulting in

$$\pi(W)\propto\frac{1}{W\log\frac{1}{W}},\quad 0<W<1. \tag{14}$$

Due to invariance property of the Jeffreys prior then the posterior (12) for the parameter W under Jeffreys prior for θ is the same $NLG(k,\lambda)$ distribution with parameters $\lambda=v\log(c)$ and $k=n$.

Also note that the posterior distribution for W given in Proposition 2 obtained from posterior of parameter θ under gamma prior given in (9) is equivalent to assign a negative log-gamma prior for W given by

$$\pi(W)=\frac{(-\beta v)^{\alpha}}{\Gamma(\alpha)}\left(\log\frac{1}{W}\right)^{\alpha-1}W^{-\beta v-1},\quad 0<W<1, \tag{15}$$

with parameters $-\beta v>0$ and α, where $v=1/\log[1-e^{-t_o}(t_o+1)]$ is a negative constant which depends on the mission time t_o. However, the prior distribution for W should be the same no matter what the fixed time t_o is contemplated. Therefore, an alternative procedure to obtain a posterior distribution for W is to select a prior distribution for W and combine with the likelihood (13).

Several priors could be chosen for W, the most commonly used are Beta, Uniform in the interval [0, 1] and Negative-Log Gamma distributions. The Uniform (0,1) is a particular case of *NLG* and Beta distribution does not provide a simple posterior for W. In Proposition below we prove that assigning a *NLG* distribution for W then the posterior is also a *NLG* distribution and hence it induces a "conjugate prior" distribution for the parameter W under the lifetime exponentiated gamma distribution.

Proposition 3: By considering the $NLG(\alpha,\beta)$ distribution as prior for the parameter W with density given by

$$\pi(W)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\left(\log\frac{1}{W}\right)^{\alpha-1}W^{\beta-1},\quad 0<W<1, \tag{16}$$

then the corresponding posterior is a $NLG(k,\lambda)$ distribution with parameters $\lambda = v\ \log(c)+\beta$ and $k = n+\alpha$.

***Proof*:** The posterior is obtained by multiplying the likelihood (13) and the prior (17).

Notes:

1) There are some particular cases by using the $NLG(\alpha,\beta)$ prior for W as follows:
   i) For $\alpha = 1$ and $\beta = 1$ we have a uniform prior in the interval (0, 1) for W resulting in a $NLG(k,\lambda)$ posterior distribution with parameters $\lambda = v\ \log(c)+1$ and $k = n + 1$;
   ii) For $\alpha = 0$ and $\beta = 0$ then the Jeffreys prior is also a particular case of $NLG(\alpha,\beta)$;
   iii) For $\alpha = 0$ and $\beta = 1$ then the resulting posterior for W is the same the posterior obtained from the posterior for the parameter θ given in Proposition 2 under uniform prior.
2) Note that posterior $NLG(k,\lambda)$ under uniform prior for W is different that posterior obtained from uniform prior for the parameter θ, due to this prior is not invariant.

Lemma: *If X is a random variable with Negative Log-gamma distribution with parameters* λ *and k then the random variable* $Z = 2\lambda \ln\left(\frac{1}{X}\right)$ *has a* χ^2 *distribution with* 2*k degrees of freedom*.

The required posterior mean and variance, credible and HPD intervals for the reliability function $R = R(t_o)$ *is given in the Proposition* 4.

Proposition 4: Since the posterior distribution for W is $NLG(k,\lambda)$ given in Prepositions 3 and 5 then:

a) The posterior distribution for the reliability function $R = R(t_o)$ at every given mission time t_o, under the considered priors for θ and W, is given by

$$p(R|\boldsymbol{t}) = \frac{\lambda^k}{\Gamma(k)}\left(\log\frac{1}{1-R}\right)^{k-1}(1-R)^{\lambda-1}\ ,\quad 0<R<1, \tag{17}$$

   where λ and k are the posterior parameters corresponding for each proposed prior;

b) The posterior mean and variance for $R = R(t_o)$ are given by

$$E(R|\boldsymbol{t}) = 1-\left(1+\frac{1}{\lambda}\right)^{-k} \text{ and } var(R|\boldsymbol{t}) = \left(1+\frac{2}{\lambda}\right)^{-k} - \left(1+\frac{1}{\lambda}\right)^{-2k}, \tag{18}$$

respectively;

c) The $100(1-\gamma)\%$ credible interval for R, denoted by (R_L, R_U), is given by

$$R_L = 1-\exp\left\{-\frac{\chi^2_{\gamma/2}(2k)}{2\lambda}\right\} \text{ and } R_U = 1-\exp\left\{-\frac{\chi^2_{1-\gamma/2}(2k)}{2\lambda}\right\}; \tag{19}$$

d) The $100(1-\gamma)\%$ HPD for R is obtained by solution of system of equations given by

$$1-\gamma = G\left(2\lambda \ln\left(\frac{1}{1-H_U}\right)\right) - G\left(2\lambda \ln\left(\frac{1}{1-H_L}\right)\right), \tag{20}$$

and

$$\left(\frac{\log(1-H_L)}{\log(1-H_U)}\right)^{k-1}\left(\frac{1-H_L}{1-H_U}\right)^{\lambda-1} - 1 = 0. \tag{21}$$

Proof: For the proof (a) and (b) of Proposition just use transformation $R(t_o) = 1 - W$ and the expectance and variance of a random variable $X \sim NLG(\lambda, k)$ distribution given by

$$E(X) = \left(1+\frac{1}{\lambda}\right)^{-k} \text{ and } var(X) = \left(1+\frac{2}{\lambda}\right)^{-k} - \left(1+\frac{1}{\lambda}\right)^{-2k}. \tag{22}$$

By Lemma we can construct the exact $100(1-\gamma)\%$ credible interval (W_L, W_U) for W and hence for R. □

Note that with Jeffreys prior for parameters then the Bayesian (credible) interval $100(1-\gamma)\%$ for R is identical to the confidence interval given in (13).

Table 2 presents the posterior mean and variance for $R = R(t_o)$ by considering several priors for the parameters θ and W.

4. Predictive functions of a future observation

If the posterior density function of parameter θ is one of the gamma distribution given in Proposition 2 then the posterior prediction density of a future observation y given the observed data is given by

Table 2

Estimators and variance of the parameter R for each prior considered

Prior	Estimator	Variance
Uniform for θ	$1-\left(1+\frac{1}{v\log c}\right)^{-(n+1)}$	$\left(1+\frac{2}{v\log c}\right)^{-(n+1)}-\left(1+\frac{1}{v\log c}\right)^{-2(n+1)}$
Gamma for θ	$1-\left(1+\frac{1}{v(\log c-\beta)}\right)^{-(n+\alpha)}$	$\left(1+\frac{2}{v(\log c-\beta)}\right)^{-(n+\alpha)}-\left(1+\frac{1}{v(\log c-\beta)}\right)^{-2(n+\alpha)}$
Jeffreys	$1-\left(1+\frac{1}{v\log c}\right)^{-n}$	$\left(1+\frac{2}{v\log c}\right)^{-n}-\left(1+\frac{1}{v\log c}\right)^{-2n}$
Uniform for W	$1-\left(1+\frac{1}{v\log c+1}\right)^{-(n+1)}$	$\left(1+\frac{2}{v\log c+1}\right)^{-(n+1)}-\left(1+\frac{1}{v\log c+1}\right)^{-2(n+1)}$
NLG for W	$1-\left(1+\frac{1}{v\log c+b}\right)^{-(n+a)}$	$\left(1+\frac{2}{v\log c+b}\right)^{-(n+a)}-\left(1+\frac{1}{v\log c+b}\right)^{-2(n+a)}$

$$f(y|\boldsymbol{t})=\frac{\alpha^*(\beta^*)^{\alpha^*}}{(\beta^*-\ln(1-e^{-y}(y+1)))^{\alpha^*+1}}\frac{ye^{-y}}{1-e^{-y}(y+1)}, \tag{23}$$

where α^* and β^* are parameters given according to the prior distribution used.

If the posterior density function of parameter θ is one of the gamma distribution given in Proposition 2 then the posterior predictive distribution function of a future observation y given the observed data is given by

$$F(y|\boldsymbol{t})=\left(\frac{\beta^*}{\beta^*-\ln(1-e^{-y}(y+1))}\right)^{\alpha^*}, \tag{24}$$

where α^* and β^* are parameters given according to the prior distribution used.

Another important problem is to construct a two sided predictive interval of the future observation y given the observed data. Therefore, we can obtain a two sided prediction interval (y_L, y_U) for y by solving the following two equations

$$F(y_L|t) = \frac{\gamma}{2} \text{ and } F(y_U|t) = 1 - \frac{\gamma}{2} \quad (25)$$

It is not possible to obtain the solutions analytically, then we need to apply suitable numerical techniques for solving a system of nonlinear equations resulted from (44).

5. Simulation study

The focus on this section is to carry out a Monte Carlo simulation in order to check the performance of weak priors for the parameter θ and reliability R. For this, we use the mean bias and mean squared error (MSE) for different sample sizes and values of parameter θ representing both shapes, bathtub and increase, of the hazard function. The simulation is based on 1000 samples by considering the following sample sizes n = 5,15,50,100. The statistical analysis for $R = R(t_o)$ also takes into account different values for a fixed specified value of t_o.

5.1 *Estimation of parameter* θ

Here, we consider an analysis with weak priors for the parameter θ proposed in Section 3. The true values for the parameter were taken as $\theta = 0.2, 05,\ 0.7, 1.0, 2.0$ and 7.0. and the values of the bias and MSE are reported in Figures 1 and 2, respectively. From Figures 1 and 2, it is clear that both bias and MSE of all estimators decrease when the sample size increases, as would be expected. The biases and MSE for all estimators tend more quickly to zero when n is larger than 15. The Uniform prior is not recommended when sample size is small while the measures are

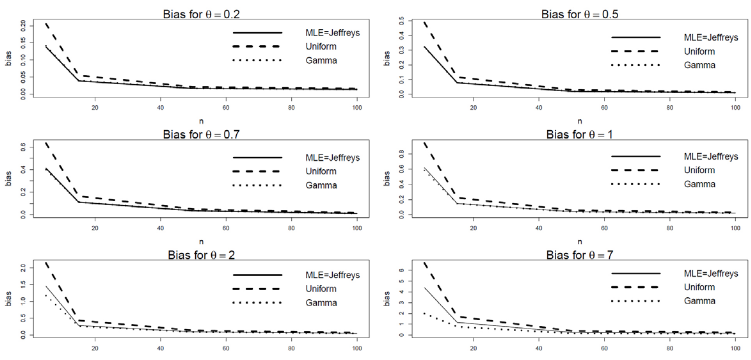

Figure 1

Plot of bias versus sample size n for estimation of θ

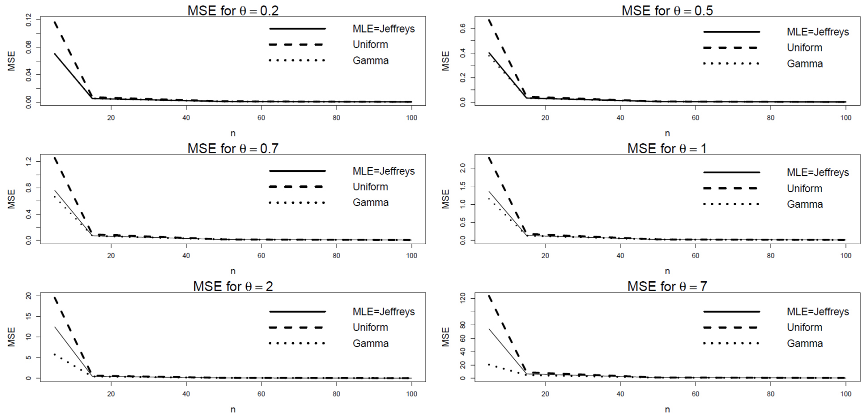

Figure 2

Plot of MSE versus sample size n for estimation of θ

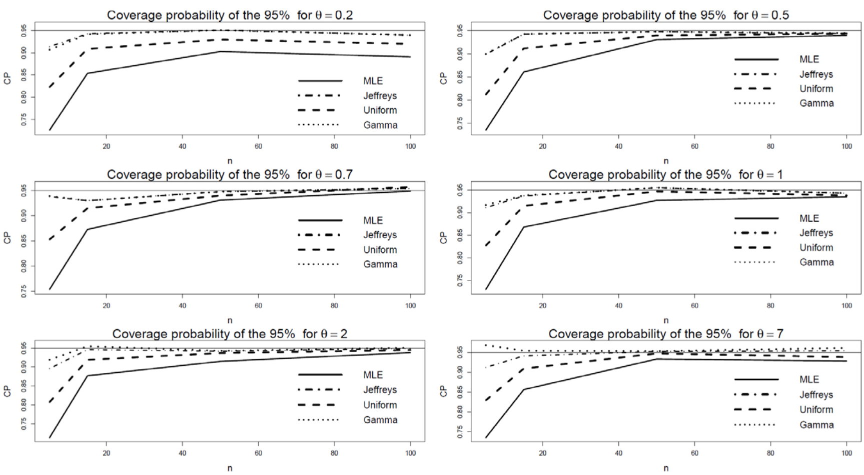

Figure 3

Plot of CP versus sample size n for estimation of θ

practically the same under gamma and Jeffreys priors as well as MLE, however, for n larger than 15, all the proposed approaches have the same performance.

A criterion for comparison of the quality of the prior distributions consists on checking the frequentist coverage probabilities of the 95% intervals arising from the MLE and prior distributions. In Figure 3, we investigate the coverage probability (CP) of 95% intervals for the parameter θ.

The simulation study indicates that the Jeffreys and gamma priors perform better than the MLE approach and uniform prior for sample size n smaller than 50. Note that, although the point estimation using the MLE and Jeffreys prior are the same, the results obtained by comparing the CP are quite different showing a better CP for the Jeffreys prior. Furthermore, Jeffreys and gamma priors present a performance quite similar for any value of n and θ.

5.2 *Estimation of parameter* $R = R(t)$ *under weak priors*

In this section, we check the performance of proposed priors for θ given in Section 3 and for reliability $W = F(T_o)$ proposed in Section 4 to estimate R. Since the reliability is a function of lifetime t, different values for t were considered in the simulation in order to also evaluate the behavior of the estimates when the reliability increase or decrease.

In Figures 4 and 5 are displayed the results from the biases and MSE for several parameter combinations at sample sizes, n = 5,15,50,100 with $\theta = 0.2$ and $\theta = 2$, representing bathtub and increase shapes of hazard function, respectively.

From these figures, we can conclude that the choice of priors to be used depends on the reliability value $R(t_o)$ and the sample size n. We observe that, for $R(t_o)$ assuming values close to 1, it means t_o is small, then the bias and MSE are higher when n is small and there is rather significant large difference under $U[0, 1]$ prior assigned for the parameter $R(t_o)$ while under others priors provide similar results. By other hands, when the values of $R(t_o)$ decrease (t_o is large) then the bias and MSE decrease as well and they

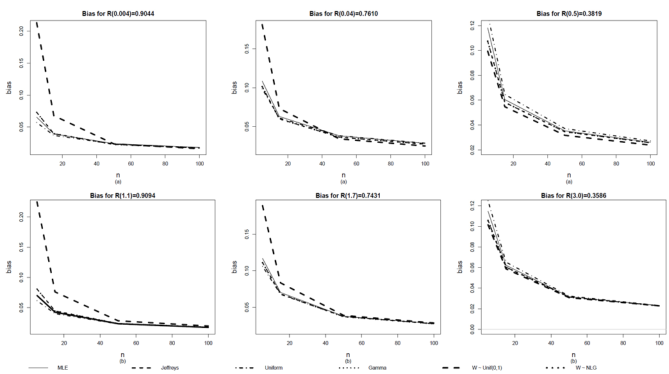

Figure 4

Plot of bias versus sample size *n* for estimation of *R*(t_o) by considering (a) $\theta = 0.2$ and (b) $\theta = 2$

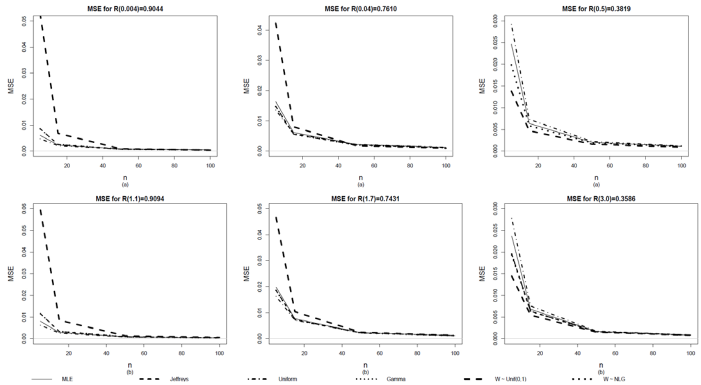

Figure 5

Plot of MSE versus sample size *n* for estimation of $R(t_o)$ by considering (a) $\theta = 0.2$ and (b) $\theta = 2$

are quite similar for each prior and MLE used when n increases. However, the $U[0, 1]$ prior assigned for the parameter $R(t_o)$ produces worse results in comparison with others priors when n is small.

We can also observe that for values of n between 15 and 50, different priors and MLE tend to yield close results each other.

Now, when $n > 50$ then MLE and Bayesian approaches based on any prior distribution provide no difference in the estimation with the bias and MSE practically zero, independently of value of reliability $R(t_o)$.

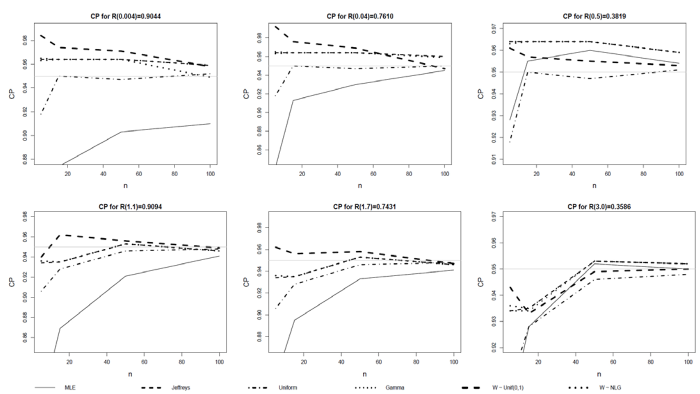

Figure 6

Plot of CP versus sample size *n* for estimation of $R(t_o)$ by considering (a) $\theta = 0.2$ and (b) $\theta = 2$

An important conclusion from the figures is that all performances of the approaches used in the analysis produce the same results, irrespective of the shape of hazard function, that is increasing or bathtub shapes.

In terms of frequentist coverage probability, Figure 6 shows the performance of MLE and prior distributions for CP of 95% intervals for the reliability $R = R(t_o)$. It shows that the CP does not reach the nominal level for both values of θ, considering a small sample size n, whereas n assuming values larger than 50, the CP tends to reach the nominal value 95%. Jeffreys, gamma and Negative Log-gamma priors for the parameter R present the same behavior for all n and both values of θ. Furthermore, the values of reliability $R = R(t_o)$ as well as parameter θ influences the performance of the priors used.

6. Application to a real data

For purpose of illustration, in this section we analyze the data representing waiting time (in minutes) of 154 patients, waiting before OPD (Out Patient Diagnosis) from the 25th - 30th December, 2017 (in the 4th week of December) at Halibet Hospital. Dataset can be obtained in Berhane [1] and given by:
2 2 2 2 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 4 4 4
4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5
5 5 5 5 5 5 5 5 5 5 5 5 5 5 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 7 7 7 7 7 7 7 7 7 7 7 7
7 7 7 8 8 8 8 8 8 9 9 9 9 9 9 10 10 11 11 12 17

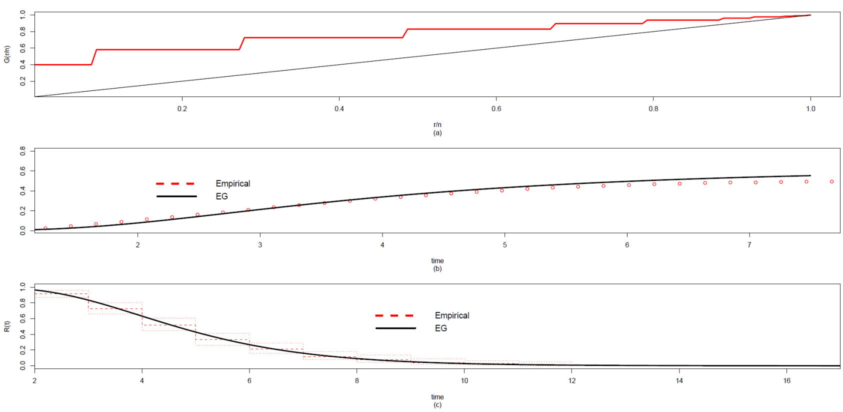

Figure 7

(a) empirical TTT-plot, (b) fitted and empirical hazard functions, (c) empirical survival function (Kaplan-Meier estimator) and fitted survival function.

Table 3

MLE and posterior mean (standard deviation in parentheses) with 95% confidence intervals for the parameter θ

Approach	Point Estimation	Intervals
MLE	8.5870 (0.6919)	(7.23, 9.94)
Jeffreys prior	8.5870 (0.6919)	(7.28, 9.99)
Gamma prior	8.5828 (0.6916)	(7.28, 9.99)
Uniform prior	8.6428 (0.6942)	(7.33, 10.05)

Firstly, in order to verify the shape of the failure hazard function, we follow a standard graphical methodology for data analysis, the total time on test (TTT) plot, presented in Figure 7-a. Based on the TTT-plot there is an indication that the hazard function has an increasing failure rate. Figure 7-b also shows a visual comparison of the empirical hazard function with the estimated EG given in (5) and Figure 7-c shows both the parametric and nonparametric estimates of the survival functions. Comparing the empirical functions with the adjusted by EG distribution we observed that the data has a good fit to the Exponentiated Gamma distribution.

Table 3 presents the posterior mean along standard deviation (between parentheses) and 95% confidence interval for the parameter θ by considering the MLE and Bayesian approaches for each prior distribution considered in this paper. From this Table we see that the results agree with simulation study, that is, there is no so significant difference among the three approaches, MLE and Bayesian based on Jeffreys and gamma priors. However, the Bayesian approach under uniform prior provides results little different as observed also in the simulation.

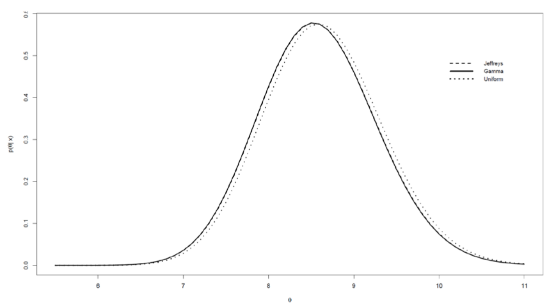

Figure 8

Posterior densities of the parameter θ for the waiting time data

Table 4

MLE and Bayesian estimators of *R(t)* for different values of time *t*

Approach	$R(3)$	$R(4)$	$R(5)$	$R(6)$
MLE	0.8515 (0.0228)	0.5617 (0.0291)	0.2984 (0.0200)	0.1395 (0.0104)
Jeffreys prior	0.8497 (0.0229)	0.5607 (0.0291)	0.2981 (0.0200)	0.1395 (0.0104)
Gamma prior	0.8496 (0.0230)	0.5605 (0.0291)	0.2980 (0.0200)	0.1394 (0.0104)
Uniform prior	0.8516 (0.0227)	0.5630 (0.0290)	0.3000 (0.0200)	0.1403 (0.0104)
$U(0, 1)$ prior	0.8481 (0.0230)	0.5611 (0.0290)	0.2991 (0.0200)	0.1402 (0.0104)
NLG prior	0.8497 (0.0229)	0.5607 (0.0291)	0.2981 (0.0200)	0.1395 (0.0104)

* standard deviation between parentheses

Table 5

95% Credible and HPD intervals of $R(t)$ for different values of time *t*

Approach	$R(3)$	$R(4)$	$R(5)$	$R(6)$
MLE	(0.8067, 0.8962)	(0.5046, 0.6188)	(0.2591, 0.3377)	(0.1191, 0.1600)
Jeffreys prior	(0.8016, 0.8914) (0.8041, 0.8934)	(0.5032, 0.6171) (0.5036, 0.6175)	(0.2596, 0.3380) (0.2591, 0.3375)	(0.1197, 0.1605) (0.1193, 0.1600)
Gamma prior	(0.8015, 0.8912) (0.8055, 0.8951)	(0.5031, 0.6169) (0.5056, 0.6198)	(0.2595, 0.3379) (0.2605, 0.3391)	(0.1197, 0.1604) (0.1200, 0.1609)
Uniform prior	(0.8039, 0.8928) (0.8064, 0.8948)	(0.5057, 0.6193) (0.5061, 0.6197)	(0.2612, 0.3396) (0.2607, 0.3391)	(0.1205, 0.1614) (0.1201, 0.1609)
$U(0, 1)$ prior	(0.8000, 0.8898) (0.8024, 0.8918)	(0.5038, 0.6174) (0.5042, 0.6177)	(0.2607, 0.3390) (0.2602, 0.3385)	(0.1204, 0.1612) (0.1200, 0.1608)
NLG prior	(0.8016, 0.8913) (0.8041, 0.8934)	(0.5032, 0.6171) (0.5036, 0.6175)	(0.2596, 0.3380) (0.2591, 0.3375)	(0.1197, 0.1605) (0.1193, 0.1600)

* 95% HPD interval is given below the credible ones

For comparison of the priors proposed in this example, the posteriors for the parameter θ are plotted in Figure 8 which confirms the differences among the priors captured by point and interval estimates.

The aim now is to obtain the posterior estimators of the survival function $R(t)$ given in Section 4. The posterior summaries of interest and credible intervals for the survival function considering the MLE and Bayesian approaches under different prior distributions are given in Table 4 and Table 5. Differently from the estimation of parameter θ, it

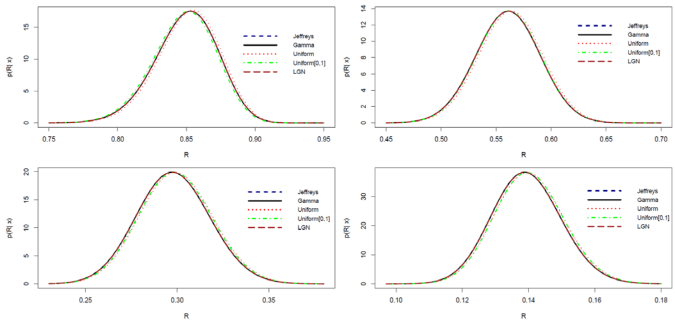

Figure 9

Plots of marginal posterior distributions of survival at t=3,4,5,6

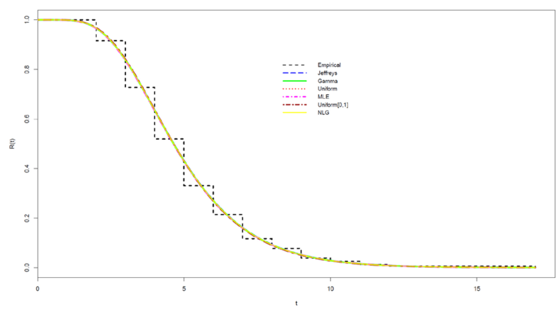

Figure 10

Empirical and estimated survival functions

is observed that the MLE and priors adopted for estimate the survival $R(t)$ in this problem have produced similar results. This can be explained by the similar shape among the graphical representations of posterior distributions shown in Figure 8, which even have a symmetrical shape.

According to the results shown in Table 4, we can also conclude, for instance, that the probability of the waiting time of patients is greater than 3 is 0.84.

Figure 9 shows the plots of marginal posterior distributions of survival at t=3,4,5,6.

The empirical and the estimate survival function using the MLE and the priors are plotted in Figure 10. This figure shows the survival functions estimated are not sensitive to the choice of the prior when considering the data in this example. In Figure 11 we present the predictive density and cumulative functions under posterior $p(\theta|t)$ based on the observed sample.

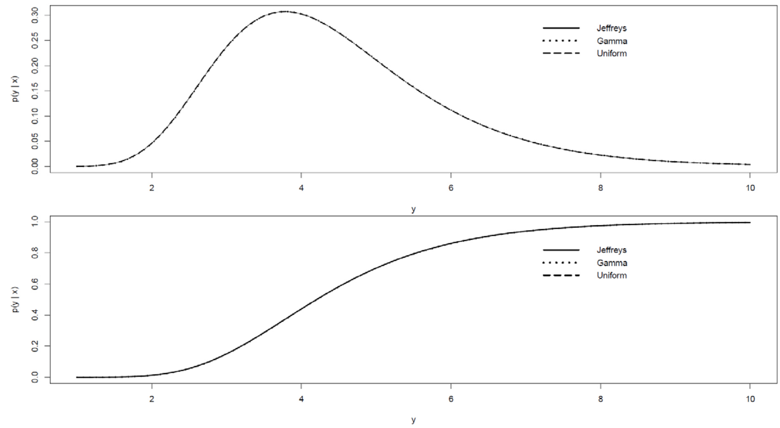

Figure 11

Plot of predictive density and cumulative functions based on posterior $p(\theta|t)$

7. Conclusions

The Exponentiated Gamma distribution could be a good alternative to the standard existing lifetime distributions with constant and monotonic hazard function as Exponential, Weibull, Gamma and others two-parameters distribution for survival data analysis since it has only one parameter and its hazard function can take increasing and bathtub shapes.

In this paper, we have shown that different formulations of priors lead to the Gamma and Negative Log-gamma posterior distributions for the parameter θ and cumulative function $W(t)$, respectively, of the EG and hence, an explicit algebraic posterior for the reliability R with respective expectance, variance, credible and HPD intervals have been derived. In this sense, obtaining known analytical forms for the posteriors is of paramount importance, since it facilitates statistical inference and there is no need to the use the Monte Carlo Markov Chain (MCMC) or approximation methods. We also shown that the use of a gamma prior for the parameter θ and a Negative Log-Gamma prior for W have taken to the conjugate prior for each parameter, respectively. Moreover, the practical relevance and applicability of the Bayesian analysis were demonstrated.

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