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The Marshall-Olkin Type II exponentiated half logistic family of distributions with properties and applications

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Abstract

A new Marshall-Olkin Type II Exponentiated Half Logistic-G (MO-TIIEHL-G) distribution is developed. Some properties and special models together with estimates of the parameters are presented. The effectiveness of the estimation process is evaluated using Monte Carlo simulation. Lastly, by applying a special case to actual data sets, the flexibility and significance of the new family of models are shown.

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1. Introduction

Generalizations of the existing continuous distributions have been considered by different scholars in the literature. The importance of generalizing those distributions is that they provide a superior fit than existing ones in modelling real life data. Some examples of those generalized distributions include Marshall-Olkin Fréchet (MOFR) by Krishna et al. [12], Marshall-Olkin Extended Lomax (MOEL) by Ghitany et al. [9], Marshall-Olkin extended uniform (MOEU) by Josea and Krishnab [11] and Marshall-Olkin Half Logistic-G (MOHL-G) by Makubate et al.

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[15]. Ahmad and Almetwally [1] proposed the Marshall-Olkin generalized Pareto distribution.

The Marshall-Olkin-G (MO-G) transformation (Marshall and Olkin [14]) is given by

$$F_{MO-G}(y;\delta,\underline{\Psi}) = 1 - \frac{\delta\overline{G}(y;\underline{\Psi})}{1-\overline{\delta}\overline{G}(y;\underline{\Psi})} \tag{1}$$

and

$$f_{MO-G}(y;\delta,\underline{\Psi}) = \frac{\delta g(y;\underline{\Psi})}{[1-\overline{\delta}\overline{G}(y;\underline{\Psi})]^2}, \tag{2}$$

respectively, for $\delta > 0$, $\overline{\delta} = 1-\delta$, with equations 1 and 2 being the cumuative distribution function (cdf) and probability density function (pdf) with the baseline cdf $G(y;\underline{\Psi})$ and parameter vector $\underline{\Psi}$. (See Marshall and Olkin [14] for additional details).

Cordeiro et al. [7] introduced the exponentiated half logistic distribution, and Chamunorwa at el. [5] studied an extension with the half logistic sub-family. Al-Mofleh et al. [2] developed and studied the type II exponentiated half logistic-G (TIIEHL-G) with the cdf and pdf

$$F_{TIIEHL-G}(y;\gamma,\lambda,\underline{\Psi}) = 1 - B_G(y;\lambda,\gamma,\underline{\Psi}), \tag{3}$$

and

$$f_{TIIEHL-G}(y;\gamma,\lambda,\underline{\Psi}) = \frac{2\gamma\lambda g(y;\underline{\Psi})(G(y;\underline{\Psi}))^{\lambda-1}[1-(G(y;\underline{\Psi}))^{\lambda}]^{\gamma-1}}{[1+(G(y;\underline{\Psi}))^{\lambda}]^{\gamma+1}}, \tag{4}$$

where

$$B_G(y;\lambda,\gamma,\underline{\Psi}) = \left[\frac{1-(G(y;\underline{\Psi}))^{\lambda}}{1+(G(y;\underline{\Psi}))^{\lambda}}\right]^{\gamma}, \tag{5}$$

for $y > 0$, $\gamma, \lambda > 0$ and baseline vector $\underline{\Psi}$.

The underlying motivations for constructing this new generalized distribution is to achieve skewness for models which are symmetric, to develop special models which exhibit various hazard rate function (hrf) shapes, to come up with heavy-tailed distributions, and to obtain better fits than other generalized distributions with the same parent cdf.

This paper is organized as follows: Section 2, contain the new distribution and its sub-families. Some results are given in Section 3. Maximum likelihood estimates are presented in Section 4 and special cases in Section 5. Monte Carlo simulation results are presented in Section

6. Section 7 contains applications to a few actual data sets. We end with conclusions in Section 8.

2. The Generalized Distribution

By using the Marshall and Olkin [14] generalized distribution with the TIIEHL-G distribution of Al-Mofleh et al. [2] as baseline cdf, the proposed distribution has the cdf and pdf

$$F_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi}) = 1-\frac{\delta(B_G^*(y;\gamma,\underline{\Psi}))}{1-\bar{\delta}B_G^*(y;\gamma,\underline{\Psi})} \tag{6}$$

and

$$f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi}) = \frac{2\delta\gamma g(y;\underline{\Psi})[\bar{G}(y;\underline{\Psi})]^{\gamma-1}}{[1+G(y;\underline{\Psi})]^{\gamma+1}[1-\bar{\delta}B_G^*(y;\gamma,\underline{\Psi})]^2}, \tag{7}$$

respectively, for $y>0$, $\delta,\gamma>0$ and parameter vector $\underline{\Psi}$, where

$$B_G^*(y;\gamma,\underline{\Psi}) = \left[\frac{\bar{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]^{\gamma}.$$

2.1 *Sub-Families*

Sub-families are given below:

- By letting $\delta=1$, we have the TIIEHL-G distribution with the cdf

$$F(y;\gamma,\underline{\Psi}) = B_G^*(y;\gamma,\underline{\Psi}),$$

  for $\gamma>0$, and baseline vector $\underline{\Psi}$.

- If $\gamma=1$, then the resulting cdf becomes

$$F(y;\delta,\underline{\Psi}) = 1-\frac{\delta\left(\left[\frac{\bar{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]\right)}{1-\bar{\delta}\left[\frac{\bar{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]},$$

  for $\delta>0$, and baseline vector $\underline{\Psi}$.

- When $\gamma=\delta=1$, the MO-TIIEHL-G distribution has the reduced cdf

$$F(y;\underline{\Psi}) = \frac{2G(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}$$

  for parameter vector $\underline{\Psi}$.

3. Some Properties

We present some properties of the proposed distribution in this section.

3.1 *MO-TIIEHL-G Density Expansion*

Linear representation via the following generalized binomial expansions

$$(1+q)^{-b}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b+i)}{\Gamma(b)i!}q^i,\quad \text{for } |q|<1, \text{ and } b>0,$$

$$(1-q)^b=\sum_{s=0}^{\infty}\binom{b}{s}(-1)^s q^s \text{ and } (1+q)^n=\sum_{i=0}^{\infty}\binom{n}{i}q^i,$$

is presented. The pdf in equation (7) can be re-written as

$$f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})=\frac{f_{TIIEHL-G}(y;\gamma,\underline{\Psi})}{\delta\left[1-\frac{\delta-1}{\delta}F_{TIIEHL-G}(y;\gamma,\underline{\Psi})\right]^2}, \tag{8}$$

where $F_{TIIEHL-G}(y;\gamma,\underline{\Psi})$ and $f_{TIIEHL-G}(y;\gamma,\underline{\Psi})$ are given in equation (3) and (4), respectively when $\lambda=1$. For $\delta\in(0,1)$, we have

$$f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})=f_{TIIEHL-G}(y;\gamma,\underline{\Psi})\sum_{t=0}^{\infty}\sum_{k=0}^{t}V_{t,k}\times(F_{TIIEHL-G}(y;\gamma,\underline{\Psi}))^{t-k}, \tag{9}$$

where $V_{t,k}=V_{t,k}(\delta)=\delta(t+1)(1-\delta)^t(-1)^{t-k}$, then we obtain MO-TIIEHL-G pdf as

$$\begin{aligned}f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi}) &= \sum_{t,i,l,m,p=0}^{\infty}\sum_{k=0}^{t}\frac{2\gamma(-1)^{i+l+m+p}}{p+1}V_{t,k}\binom{-(\gamma(t-k+1)+1)}{i}\\ &\quad\times\binom{\gamma(t-k+1)-1}{l}\binom{i+l}{m}\binom{m}{p}\times(p+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^p \\ &= \sum_{p=0}^{\infty}V^*_{p+1}g^*_{p+1}(y;\underline{\Psi}),\end{aligned} \tag{10}$$

where

$$V^*_{p+1}=\sum_{t,i,l,m=0}^{\infty}\sum_{k=0}^{t}\frac{2\gamma(-1)^{i+l+m+p}}{p+1}V_{t,k}\binom{-(\gamma(t-k+1)+1)}{i}\times\binom{\gamma(t-k+1)-1}{l}\binom{i+l}{m}\binom{m}{p} \tag{11}$$

and

$$g^*_{p+1}(y;\underline{\Psi})=(p+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^p.$$

Hence, when $0<\delta<1,$ the MO-TIIEHL-G pdf is an infinite linear combination of the exponentiated-G (Exp-G) densities with power parameter (p+1).

Moreover, when $\delta>1,$ we can write,

$$f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})=f_{TIIEHL-G}(y;\gamma,\underline{\Psi})\times\sum_{t=0}^{\infty}w_t(F_{TIIEHL-G}(y;\gamma,\underline{\Psi}))^t, \tag{12}$$

where $w_t=w_t(\delta)=\frac{(t+1)(1-1/\delta)}{\delta}$.

On the application of the generalized binomial expansions presented in the appendix, we obtain

$$\begin{aligned} f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi}) &= \sum_{t,i,l,m,p=0}^{\infty}\frac{2\gamma(-1)^{i+l+m+p}}{p+1}w_t\binom{-(\gamma(t+1)+1)}{i} \\ &\times\binom{\gamma(t+1)-1}{l}\binom{i+l}{m}\binom{m}{p}\times(p+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^p \\ &=\sum_{p=0}^{\infty}w^*_{p+1}g^*_{p+1}(y;\underline{\Psi}), \end{aligned} \tag{13}$$

where

$$w^*_{p+1}=\sum_{t,i,l,m=0}^{\infty}\frac{2\gamma(-1)^{i+l+m+p}}{p+1}w_t\binom{-(\gamma(t+1)+1)}{i}\times\binom{\gamma(t+1)-1}{l}\binom{i+l}{m}\binom{m}{p}, \tag{14}$$

and

$$g^*_{p+1}(y;\underline{\Psi})=(p+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^p, \tag{15}$$

respectively.

Again, the new MO-TIIEHL-G pdf is an infinite linear combination of the exponentiated-G (Exp-G) densities with power parameter (p+1). More details on derivations are given in the appendix.

3.2 *Quantile & Hazard Functions*

We present the quantile function and hrf in this subsection. The hrf of the MO-TIIEHL-G distribution is

$$h_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})=\frac{2\delta\gamma g(y;\underline{\Psi})[\bar{G}(y;\underline{\Psi})]^{\gamma-1}}{[1+G(y;\underline{\Psi})]^{\gamma+1}\left[1-\bar{\delta}\left[\frac{\bar{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]^{\gamma}\right]^2}\times\left(\frac{\delta(B^*_G(y;\gamma,\underline{\Psi}))}{1-\bar{\delta}B^*_G(y;\gamma,\underline{\Psi})}\right)^{-1}. \tag{16}$$

The quantile function (qf) is derived by inverting the non-linear equation

$$F_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi}) = 1 - \frac{\delta(B_G^*(y;\gamma,\underline{\Psi}))}{1-\bar{\delta}B_G^*(y;\gamma,\underline{\Psi})} = u$$

for $0 \leqslant u \leqslant 1$. We have

$$(1-\bar{\delta}B_G^*(y;\gamma,\underline{\Psi}))(1-u) = \delta(B_G^*(y;\gamma,\underline{\Psi})),$$

and the quantile function reduces to

$$Q_{G(u;\delta,\gamma,\underline{\Psi})} = G^{-1}\left[\frac{1-[1-u(\delta+(1-u)\bar{\delta})^{-1}]^{\frac{1}{\gamma}}}{[1-u(\delta+(1-u)\bar{\delta})^{-1}]^{\frac{1}{\gamma}}+1}\right]. \tag{17}$$

3.3 *Probability Weighted Moments (PWMs) & Related Functions*

We present the PWMs and related functions in this subsection. The $(a,r)^{th}$ PWMs for MO-TIIEHL-G distribution are defined by

$$\eta_{a,r} = E(Y^a[F(Y)]^r) = \int_{-\infty}^{\infty} y^a f_{MO-TIIEHL-G}(y)[F_{MO-TIIEHL-G}(y)]^r dy. \tag{18}$$

For $\delta \in (0,1)$,

$$f(y)[F(y)]^r = f_{TIIEHL-G}(y;\gamma,\underline{\Psi})\sum_{t=0}^{\infty}\sum_{k=0}^{t} B_{t,r,k}\left(\left[\frac{\bar{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]^{\gamma}\right)^{t+r-k}, \tag{19}$$

where $B_{t,r,k} = B_{t,r,k}(\delta) = \delta^r(1-\delta)^t(-1)^{t-k}\binom{r+t}{t}$.

Considering the generalized binomial expansions outlined in the appendix, we obtain

$$\begin{aligned} f(y)[F(y)]^r &= \sum_{t,r,l,p,s,q=0}^{\infty}\sum_{k=0}^{t} 2aB_{t,r,k}\frac{(-1)^{l+p+s+q}}{q+1}\binom{\gamma(t+r-k+1)-1}{l} \\ &\quad \times \binom{-(\gamma(t+r-k+1)+1)}{p}\binom{l+p}{s}\binom{s}{t} \times (q+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^q \\ &= \sum_{q=0}^{\infty} B_{q+1}^* h_{q+1}(y;\underline{\Psi}), \end{aligned} \tag{20}$$

where

$$B_{q+1}^* = \sum_{t,r,l,p,s=0}^{\infty}\sum_{k=0}^{j} 2\gamma B_{t,r,k}\frac{(-1)^{l+p+s+q}}{q+1}\binom{\gamma(t+r-k+1)-1}{l} \times \binom{-(\gamma(t+r-k+1)+1)}{p}\binom{l+p}{s}\binom{s}{q},$$

and $h_{q+1}(y;\underline{\Psi}) = (q+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^q$. Finally, the PWMs are given as

$$\eta_{a,r} = \sum_{q=0}^{\infty} B_{q+1}^* \int_{-\infty}^{\infty} y^a h_{q+1}(y;\underline{\Psi})dy. \tag{21}$$

Note that for $\delta > 1$, we have

$$1-\bar{\delta}\bar{F}(y;\gamma,\underline{\Psi}) = \delta\{1-(\delta-1)F(y;\gamma,\underline{\Psi})/\delta\}, \text{ and}$$

$$f(y)[F(y)]^r = f_{TIIEHL-G}(y;\gamma,\underline{\Psi})\sum_{t=0}^{\infty}\sum_{z=0}^{\infty} C_{t,r} F_{TIIEHL-G}^{t+r}(y;\gamma,\underline{\Psi}), \tag{22}$$

where $C_{t,r} = C_{t,r}(\delta) = \frac{(\delta-1)^t}{\delta^{r+t}}\binom{r+t}{t}$. Thus,

$$\begin{aligned} f(y)[F(y)]^r &= \frac{2\gamma g(y;\underline{\Psi})[\bar{G}(y;\underline{\Psi})]^{\gamma-1}}{[1+G(y;\underline{\Psi})]^{\gamma+1}} \sum_{t,z=0}^{\infty} C_{t,r}(B_G^*(y;\gamma,\underline{\Psi}))^{t+r} \\ &= \sum_{t,r,l,p,s,q=0}^{\infty} 2\gamma C_{t,r} \frac{(-1)^{l+p+s+q}}{q+1}\binom{\gamma(t+r+1)-1}{l}\times\binom{-(\gamma(t+r+1)+1)}{p}\binom{l+p}{s}\binom{s}{q} \\ &\quad \times (q+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^q \\ &= \sum_{q=0}^{\infty} C_{q+1}^* h_{q+1}(y;\underline{\Psi}), \end{aligned} \tag{23}$$

where $h_{q+1}(y;\underline{\Psi}) = (q+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^q$ and

$$C_{q+1}^* = \sum_{t,r,l,p,s=0}^{\infty} 2\gamma C_{t,r,} \frac{(-1)^{l+p+s+q}}{q+1}\binom{\gamma(t+r+1)-1}{l}\times\binom{-(\gamma(t+r+1)+1)}{p}\binom{l+p}{s}\binom{s}{q}. \tag{24}$$

Also, for this case the PWMs is

$$\eta_{a,r} = \sum_{q=0}^{\infty} C_{q+1}^* \int_{-\infty}^{\infty} y^a h_{q+1}(y;\underline{\Psi})dy. \tag{25}$$

Details of derivations follows in the appendix.

Using equation (10) for $\delta \in (0,1)$, the n^{th} moment is

$$E(Y^n) = \int_{-\infty}^{\infty} y^n f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})dy = \sum_{p=0}^{\infty} V_{p+1}^* E(X_{p+1}^n), \tag{26}$$

where $E(X_{p+1}^n)$ is the n^{th} moment of X_{p+1}, an Exp-G distributed random variable with power parameter $p+1$ and V_{p+1}^* is given in equation (11). The n^{th} incomplete moment of the MO-TIIEHL-G distribution is

$$I_Y(q) = \int_{-\infty}^{q} y^n f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})dy = \sum_{p=0}^{\infty} V_{p+1}^* I_{p+1}(q),$$

where $I_{p+1}(q) = \int_{-\infty}^{q} y^n (p+1)[G(y;\underline{\Psi})]^p g(y;\underline{\Psi})dy$.

Also, when $\delta > 1$, the n^{th} n^{th} moment is

$$E(Y^n) = \int_{-\infty}^{\infty} y^n f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})dy = \sum_{p=0}^{\infty} w_{p+1}^{*} E(X_{p+1}^{n}), \tag{27}$$

where $E(X_{p+1}^{n})$ is the n^{th} moment of X_{p+1}, and w_{p+1}^{*} is as given in equation (14). The corresponding n^{th} incomplete moment in this case is

$$I_Y(q) = \int_{-\infty}^{q} y^n f_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi})\,dy = \sum_{p=0}^{\infty} w_{p+1}^{*} I_{p+1}(q),$$

where $I_{p+1}(q) = \int_{-\infty}^{q} y^n (p+1)[G(y;\underline{\Psi})]^p g(y;\underline{\Psi})dy.$

3.4 *Stochastic Orders*

Stochastic ordering for the MO-TIIEHL-G distributions are presented in this section. Consider two random variables Z and T with functions $F_Z(k)$ and $F_T(k)$, respectively, and $\overline{F}_Z(k) = 1 - F_Z(k)$ the survival function. Note that Z is stochastically smaller than T (denoted by $Z <_s T$) if $\overline{F}_Z(k) \leq$ for all k or $F_Z(k) \geq F_T(k)$ for all k. Hazard rate order (denoted by $Z <_{hk} T$) and likelihood ratio order (denoted by $Z <_{lk} T$) are stronger and given by $h_Z(k) \geq h_T(k)$ for all k, and $\frac{f_Z(k)}{f_T(k)}$ decreasing in k, (see Shaked and Shanthikumar [20] for additional information).

Let Y_1 and Y_2 be independent random variables each following MO-TIIEHL-G $(\delta_1,\gamma,\underline{\Psi})$ and MO-TIIEHL-G $(\delta_2,\gamma,\underline{\Psi})$ distributions, then the ratio of the densities $f_1(y;\delta_1,\gamma,\underline{\Psi})$ and $f_2(y;\delta_2,\gamma,\underline{\Psi})$ is

$$\frac{f_1(y;\delta_1,\gamma,\underline{\Psi})}{f_2(y;\delta_2,\gamma,\underline{\Psi})} = \frac{\delta_2}{\delta_1}\left(\frac{\left[1-\overline{\delta}_2\left[\frac{\overline{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]^{\gamma}\right]}{\left[1-\overline{\delta}_1\left[\frac{\overline{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]^{\gamma}\right]}\right)^2. \tag{28}$$

Now, by differentiating equation (28) with respect to y, we get

$$\frac{d}{dy}\left(\frac{f_1(y;\delta_1,\gamma,\underline{\Psi})}{f_2(y;\delta_2,\gamma,\underline{\Psi})}\right) = \frac{\delta_2}{\delta_1}(\delta_1-\delta_2)\frac{[1-\overline{\delta}_2 B_G^{*}(y;\gamma,\underline{\Psi})]}{[1-\overline{\delta}_1 B_G^{*}(y;\gamma,\underline{\Psi})]^3}\times\frac{2\gamma g(y;\underline{\Psi})[\overline{G}(y;\underline{\Psi})]^{\gamma-1}}{[1+G(y;\underline{\Psi})]^{\gamma+1}}. \tag{29}$$

Clearly equation (29) ≤ 0 if $\delta_1 \leq \delta_2$. Hence, $Y_1 <_{lk} Y_2$, $Y_1 <_{hk} Y_2$ so that $Y_1 <_s Y_2$, and we conclude that Y_1 and Y_2 are stochastically ordered.

3.5 *Density of Order Statistics and Entropy*

The density of the i^{th} order statistic and R'enyi entropy are derived in this subsection.

3.5.1 Order Statistics

The pdf of the i^{th} order statistic is

$$f_{i:n}(y)=f_{TIIEHL-G}(y;\gamma,\underline{\Psi})\sum_{m=0}^{n-i}\frac{\delta n!(-1)^m\binom{n-i}{m}}{(i-1)!(n-i)!}\frac{F_{TIIEHL-G}^{m+i-1}(y;\gamma,\underline{\Psi})}{[1-\bar{\delta}F_{TIIEHL-G}(y;\gamma,\underline{\Psi})]^{m+i-1}}. \tag{30}$$

For $\delta\in(0,1)$, we have

$$f_{i:n}(y)=f_{TIIEHL-G}(y;\gamma,\underline{\Psi})\sum_{t=0}^{\infty}\sum_{m=0}^{n-i}\sum_{k=0}^{t}D_{t,m,k}(F_{TIIEHL-G}(y;\gamma,\underline{\Psi}))^{t+m-k+i-1}, \tag{31}$$

with $D_{t,m,k}=D_{t,m,k}(\delta)=\frac{\delta n!(-1)^m(1-\delta)^t(-1)^{t-k}}{(i-1)!(n-i)!}\binom{t}{k}\binom{n-i}{m}\binom{m+i+t}{t}.$

For $\delta\in(0,1)$ and using equation (31), we get

$$\begin{aligned}f_{i:n}(y)&=\sum_{t,l,p,r,s=0}^{\infty}\sum_{m=0}^{n-i}\sum_{k=0}^{t}\frac{2\gamma(-1)^{l+p+r+s}}{s+1}D_{t,m,k}\binom{\gamma(t+m-k+i)-1}{l}\\&\quad\times\binom{-(\gamma(t+m-k+i)+1)}{p}\binom{l+p}{r}\binom{r}{s}\times(s+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^s \qquad (32)\\&=\sum_{s=0}^{\infty}D^*_{s+1}g^*_{s+1}(y;\underline{\Psi}),\end{aligned}$$

where

$$D^*_{s+1}=\sum_{t,l,p,r=0}^{\infty}\sum_{m=0}^{n-i}\sum_{k=0}^{t}\frac{2\gamma(-1)^{l+p+r+s}}{s+1}D_{t,m,k}\binom{\gamma(t+m-k+i)-1}{l}\times\binom{-(\gamma(t+m-k+i)+1)}{p}\binom{l+p}{r}\binom{r}{s}$$

and $g^*_{s+1}(y;\underline{\Psi})=(s+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^s.$

It should be noted that, for $\delta>1$, we can rewrite

$1-\bar{\delta}F_{TIIEHL-G}(y;\gamma,\underline{\Psi})=\delta\{1-(\delta-1)F_{TIIEHL-G}(y;\gamma,\underline{\Psi})/\delta\}$, so that

$$f_{i:n}(y)=f_{TIIEHL-G}(y;\gamma,\underline{\Psi})\sum_{t=0}^{\infty}\sum_{m=0}^{n-i}A_{t,m}(F_{TIIEHL-G}(y;\gamma,\underline{\Psi}))^{t+m+i-1}, \tag{33}$$

where

$$A_{t,m}=A_{t,m}(\delta)=\frac{(-1)^m(\delta-1)^t n!}{\delta^{m+t+i}(i-1)!(n-i)!}\binom{n-i}{m}\binom{t+m+i}{j}.$$

By using equation (7), we have

$$\begin{aligned}f_{i:n}(y)&=\sum_{t,l,p,r,s=0}^{\infty}\sum_{m=0}^{n-i}\frac{2\gamma(-1)^{l+p+r+s}}{s+1}\binom{A_{t,m}\gamma(t+m+i)-1}{l}\times\binom{-(\gamma(t+m+i)+1)}{p}\binom{l+p}{r}\binom{r}{s}\\&\quad\times(s+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^s \qquad (34)\end{aligned}$$

$$= \sum_{s=0}^{\infty} A^*_{s+1} g^*_{s+1}(y;\underline{\Psi}),$$

where $g^*_{s+1}(y;\underline{\Psi}) = (s+1)g(y;\underline{\Psi})[G(y;\underline{\Psi})]^s$ and

$$A^*_{s+1} = \sum_{t,l,p,r=0}^{\infty} \sum_{m=0}^{n-i} \frac{2\gamma(-1)^{l+p+r+s}}{s+1} \binom{A_{t,m}\gamma(t+m+i)-1}{l}. \tag{35}$$

Hence for $\delta > 1$, the density function of the i^{th} order statistic is indeed an infinite linear combination of the Exp-G densities with power parameter $(s+1)$ and baseline vector $\underline{\Psi}$. Details on derivations follows in the appendix.

3.5.2 R'enyi Entropy

R'enyi entropy [19] for a pdf $f(y)$ of a random variable Y is

$$I_R(\Lambda) = (1-\Lambda)^{-1} \log[\textstyle\int_0^\infty f^\Lambda(y)dy] \text{ for } \Lambda > 0 \text{ and } \Lambda \neq 1.$$

On applying expansion $(1-z)^{-b} = \sum_{t=0}^{\infty} \frac{\Gamma(b+t)}{\Gamma(b)t!} z^t$, for $|z|<1$ and $b>0$, and for $\delta \in (0,1)$, we have

$$\begin{aligned} f^\Lambda_{MO-TIIEHL-G}(y;\underline{\Psi}) = &\frac{\delta^\Lambda f^\Lambda_{TIIEHL-G}(y;\gamma,\underline{\Psi})}{\Gamma(2\Lambda)} \sum_{t=0}^{\infty} (1-\delta)^t \Gamma(2\Lambda+t) \\ &\times \frac{[1-F_{TIIEHL-G}(y;\gamma,\underline{\Psi})]^t}{t!}. \end{aligned} \tag{36}$$

Now, if $\delta \in (0,1)$, R'enyi entropy is

$$I_R(\Lambda) = (1-\Lambda)^{-1} \log[\sum_{t=0}^{\infty} a_t \textstyle\int_0^\infty f^\Lambda_{TIIEHL-G}(y;\gamma,\underline{\Psi})[1-F_{TIIEHL-G}(y;\gamma,\underline{\Psi})]^t dx], \tag{37}$$

where $a_t = a_t(\delta) = \frac{\delta^\Lambda (1-\delta)^t \Gamma(2\Lambda+t)}{t\Gamma(2\Lambda)!}$. Now,

$$I_R(\Lambda) = (1-\Lambda)^{-1} \log\left[\sum_{t=0}^{\infty} a_t \int_0^\infty \frac{(2\gamma)^\Lambda g^\Lambda(y;\underline{\Psi})[\bar{G}(y;\underline{\Psi})]^{(\gamma-1)\Lambda}}{[1+G(y;\underline{\Psi})]^{(\gamma+1)\Lambda}} \times \left(1-\left[\frac{\bar{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})}\right]^\gamma\right)^t dy\right]. \tag{38}$$

Considering expansions presented in the appendix, R'enyi entropy reduces to

$$I_R(\Lambda) = (1-\Lambda)^{-1} \log \left[\sum_{t,i,k,l,m,p=0}^{\infty} a_t \frac{(2\gamma)^{\Lambda}(-1)^{i+k+l+m+p}}{\left(\frac{p}{\Lambda}+1\right)^{\Lambda}} \binom{t}{i} \right.$$
$$\times \binom{\gamma(\Lambda+i)-\Lambda}{k} \binom{-(\gamma(\Lambda+i)+\Lambda)}{l} \binom{k+l}{m} \binom{m}{p} \tag{39}$$
$$\left. \times \int_0^{\infty} \left(\left[\frac{p}{\Lambda}+1\right] [G(y;\underline{\Psi})]^{\frac{p}{\Lambda}} g(y;\underline{\Psi}) \right)^{\Lambda} dy \right]$$
$$= (1-\Lambda)^{-1} \log \left[\sum_{p=0}^{\infty} a_p^* \exp((1-\Lambda) I_{REG}) \right],$$

where $I_{REG} = (1-\Lambda)^{-1} \log \left[\int_0^{\infty} \left(\left[\frac{p}{\Lambda}+1\right] [G(y;\underline{\Psi})]^{\frac{p}{\Lambda}} g(y;\underline{\Psi}) \right)^{\Lambda} dy \right]$ is R'enyi entropy of Exp-G densities with power parameter $\frac{p}{\Lambda}+1$ and

$$a_p^* = \sum_{t,i,k,l,m=0}^{\infty} at \frac{(2\gamma)^{\Lambda}(-1)^{t+k+l+m+p}}{\left(\frac{p}{\Lambda}+1\right)^{\Lambda}} \binom{t}{i} \times \binom{\gamma(\Lambda+i)-\Lambda}{k} \binom{-(\gamma(\Lambda+i)+\Lambda)}{l} \binom{k+l}{m} \binom{m}{p}. \tag{40}$$

For $\delta > 1$, we have

$$f^{\Lambda}_{MO-TIIEHL-G}(y;\delta,\gamma,\underline{\Psi}) = \frac{f^{\Lambda}_{TIIEHL-G}(y;\gamma,\underline{\Psi})}{\delta^{v}\Gamma(2\Lambda)} \sum_{t=0}^{\infty} (\delta-1)^t \Gamma(2\Lambda+t) \times \frac{[F_{TIIEHL-G}(y;\gamma,\underline{\Psi})]^t}{t!} \tag{41}$$

and R'enyi entropy is written as

$$I_R(\Lambda) = (1-\Lambda)^{-1} \log \left[\sum_{t=0}^{\infty} b_t \int_0^{\infty} \frac{(2\gamma)^{\Lambda} g^{\Lambda}(y;\underline{\Psi}) [\bar{G}(y;\underline{\Psi})]^{(\gamma-1)\Lambda}}{[1+G(y;\underline{\Psi})]^{(\gamma+1)\Lambda}} \times \left(\left[\frac{\bar{G}(y;\underline{\Psi})}{1+G(y;\underline{\Psi})} \right]^{\gamma} \right)^t dy \right], \tag{22}$$

where $b_t = b_t(\delta) = \frac{(\delta-1)^t \Gamma(2v+t)}{t!\delta^{v}\Gamma(2\Lambda)}$.

Consequently, we obtain R'enyi entropy as:

$$I_R(\Lambda) = (1-\Lambda)^{-1} \log \left[\sum_{t,k,l,m,p=0}^{\infty} b_t \frac{(2\gamma)^{\Lambda}(-1)^{k+l+m+p}}{\left(\frac{p}{\Lambda}+1\right)^{\Lambda}} \right.$$
$$\times \binom{\gamma(\Lambda+t)-\Lambda}{k} \binom{-(\gamma(\Lambda+t)+\Lambda)}{l} \binom{k+l}{m} \binom{m}{p} \tag{43}$$
$$\left. \times \int_0^{\infty} \left(\left[\frac{p}{\Lambda}+1\right] [G(y;\underline{\Psi})]^{\frac{p}{\Lambda}} g(y;\underline{\Psi}) \right)^{\Lambda} dy \right]$$

$$= (1-\Lambda)^{-1}\log\left[\sum_{p=0}^{\infty} b_p^{*}\exp((1-\Lambda)I_{REG})\right],$$

where $I_{REG} = (1-\Lambda)^{-1}\log\left[\int_0^{\infty}\left(\left[\frac{p}{\Lambda}+1\right][G(y;\underline{\Psi})]^{\frac{p}{\Lambda}} g(y;\underline{\Psi})\right)^{\Lambda} dy\right]$ is R'enyi entropy of Exp-G densities with power parameter $\frac{p}{\Lambda}+1$ and

$$b_p^{*} = \sum_{t,k,l,m=0}^{\infty} b_t \frac{(2\gamma)^{\Lambda}(-1)^{k+l+m+p}}{\left(\frac{p}{\Lambda}+1\right)^{\Lambda}} \times \binom{\gamma(\Lambda+t)-\Lambda}{k}\binom{-(\gamma(\Lambda+t)+\Lambda)}{l}\binom{k+l}{m}\binom{m}{p}.$$

Details of derivations are given in the appendix.

4. Estimation

We obtain estimates of the parameters by maximizing the likelihood function in this section. We let $Y \sim MO-TIIEHL-G(\delta,\gamma,\underline{\Psi})$ and $\mathbf{\Phi} = (\delta,\gamma,\underline{\Psi})^T$ the corresponding model parameter vector. The log-likelihood function (ℓ_n) is

$$\ell_n(\Phi) = n\log(2\delta\gamma) + \sum_{i=1}^{n}\log[g(y_i;\underline{\Psi})] + (\gamma-1)\sum_{i=1}^{n}\log[\overline{G}(y_i;\underline{\Psi})]$$
$$-(\gamma+1)\sum_{i=1}^{n}\log[1+G(y_i;\underline{\Psi})] - 2\sum_{i=1}^{n}\log\left[1-\overline{\delta}\left[\frac{\overline{G}(y_i;\underline{\Psi})}{1+G(y_i;\underline{\Psi})}\right]^{\gamma}\right].$$

The maximum likelihood estimates (mle's) of the parameters are obtained by solving the non-linear equations $\left(\frac{\partial\ell_n}{\partial\delta},\frac{\partial\ell_n}{\partial\gamma},\frac{\partial\ell_n}{\partial\underline{\Psi}_k}\right)^T = 0$ by numerical methods.

5. Some Special Cases

Some particular cases of the MO-TIIEHL-G distribution for the specified baseline cdf G are presented in the following subsections.

5.1 *Marshall-Olkin Type II Exponentiated Half Logistic-Log-Logistic (MO-TIIEHL-LLo) Distribution*

Suppose we take the baseline distribution with the cdf $G(y;c) = 1-(1+y^c)^{-1}$ and pdf $g(y;c) = cy^{c-1}(1+y^c)^{-2}$, respectively, for $c, y > 0$, then the MO-TIIEHL-LLo distribution has cdf and pdf

$$F(y;\delta,\gamma,c)=1-\frac{\delta\left(\left[\frac{(1+y^c)^{-1}}{2-(1+y^c)^{-1}}\right]^{\gamma}\right)}{1-\bar{\delta}\left[\frac{(1+y^c)^{-1}}{2-(1+y^c)^{-1}}\right]^{\gamma}}, \tag{44}$$

and

$$f(y;\delta,\gamma,c)=\frac{2\delta\gamma c y^{c-1}(1+y)^{-2}(1+y^c)^{-1}]^{\gamma-1}}{[2-(1+y^c)^{-1}]^{\gamma+1}\left[1-\bar{\delta}\left[\frac{(1+y^c)^{-1}}{2-(1+y^c)^{-1}}\right]^{\gamma}\right]^2}, \tag{45}$$

respectively, for $\delta,\gamma,c>0$.

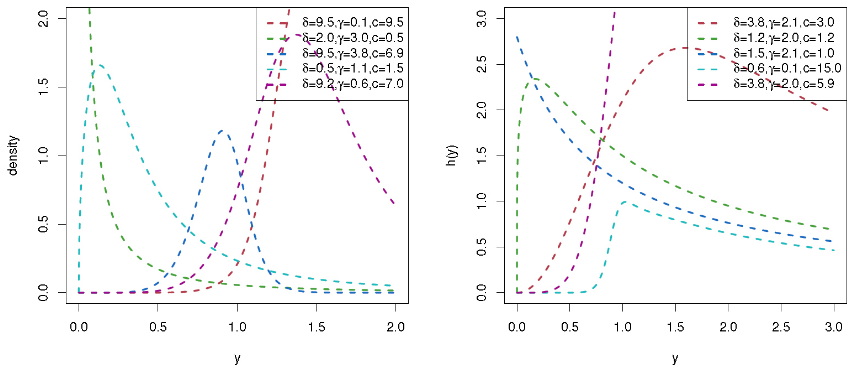

Figure 1

Graphs for pdf and hrf of MO-TIIEHL-LLo Distribution

In Figure 1, the shapes of the MO-TIIEHL-LLo distribution can be right-skewed, left-skewed, almost symmetric, J and reverse-J. The graphs of the hrf are increasing, decreasing, and upside down bathtub shapes. Table 1 present some quantile values of the MO-TIIEHL-LLo distribution for selected parameter values. 3D plots for MO-TIIEHL-LLo distribution are given in Figures 2 and 3. It can be seen that the MO-TIIEHL-LLo distribution is useful for data with varying levels of kurtosis, and skewness can either be positive or negative.

Table 1

Quantiles for MO-TIIEHL-LLo Distribution

	(δ,γ,c)				
u	(1, 1.5, 1.3)	(0.7, 1, 1.5)	(0.4, 1, 1)	(2.1, 0.5, 1.9)	(1.5, 1, 1.2)
0.1	0.0782	0.1148	0.0222	0.4927	0.1261
0.2	0.1436	0.1971	0.0500	0.8054	0.2478
0.3	0.2134	0.2823	0.0857	1.1504	0.3883
0.4	0.2931	0.3790	0.1333	1.5783	0.5612
0.5	0.3896	0.4966	0.1999	2.1561	0.7868
0.6	0.5140	0.6508	0.2999	3.0093	1.1031
0.7	0.6886	0.8737	0.4666	4.4292	1.5942
0.8	0.9706	1.2515	0.8000	7.2997	2.4980
0.9	1.5856	2.1488	1.8000	16.1509	4.9101

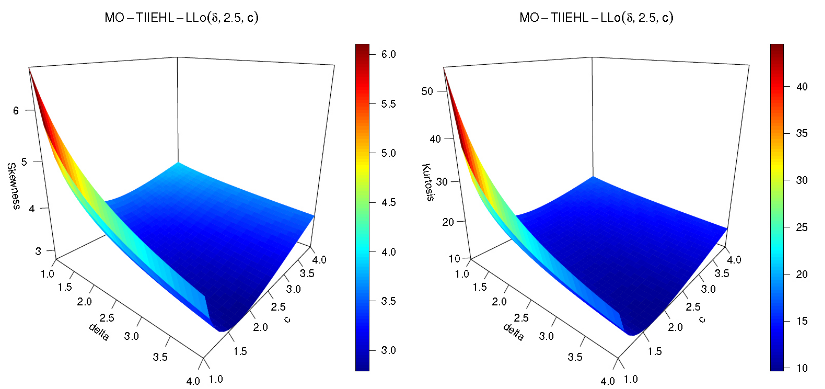

Figure 2

Graphs for Skewness and Kurtosis of MO-TIIEHL-LLo Distribution

5.2 *Marshall-Olkin Type II Exponentiated Half Logistic-Weibull (MO-TIIEHL-W) Distribution*

Consider the cdf $G(y;\beta,\lambda)=1-e^{-\beta y^{\lambda}}$ and pdf $g(y;\beta,\lambda)=\lambda\beta y^{\lambda-1} e^{-\beta y^{\lambda}}$, for $\lambda,\beta,y>0$, as the baseline distribution. The resulting MO-TIIEHL-W distribution has cdf and pdf

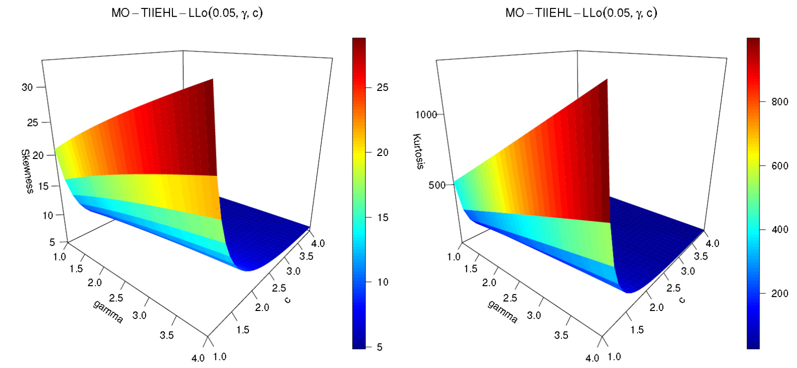

Figure 3

Graphs for Skewness and Kurtosis of MO-TIIEHL-LLo Distribution

$$F(y;\delta,\gamma,\beta,\lambda)=1-\frac{\delta\left(\left[\dfrac{e^{-\beta y^{\lambda}}}{2-e^{-\beta y^{\lambda}}}\right]^{\gamma}\right)}{1-\bar{\delta}\left[\dfrac{e^{-\beta y^{\lambda}}}{2-e^{-\beta y^{\lambda}}}\right]^{\gamma}}, \tag{46}$$

and

$$f(y;\delta,\gamma,\beta,\lambda)=\frac{2\delta\gamma\beta\lambda y^{\lambda-1}e^{-\beta y^{\lambda}}[e^{-\beta y^{\lambda}}]^{\gamma-1}}{[2-e^{-\beta y^{\lambda}}]^{\gamma+1}\left[1-\bar{\delta}\left[\dfrac{e^{-\beta y^{\lambda}}}{2-e^{-\beta y^{\lambda}}}\right]^{\gamma}\right]^{2}}, \tag{47}$$

for $\delta,\gamma,\beta,\lambda>0$.

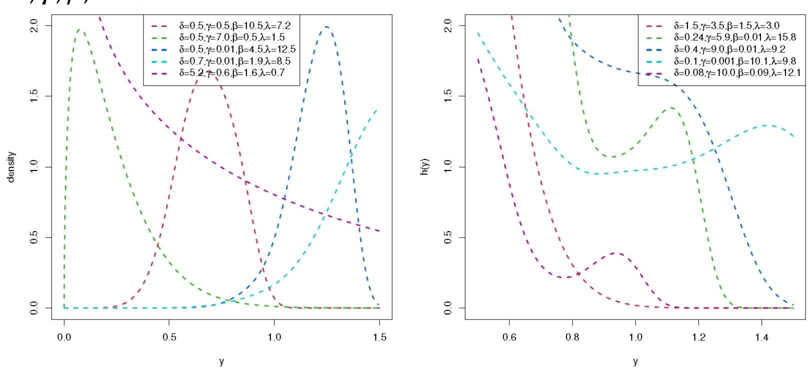

Figure 4

Graphs for pdf and hrf of the MO-TIIEHL-W Distribution

The graphs of the pdf of the MO-TIIEHL-W distribution are seen to be right-skewed, left-skewed, almost symmetric, reverse-J and J shapes. The hrf display shapes that are bathtub followed by upside down bathtub, bathtub and decreasing as indicated in Figure 4. Quantiles for the MO-TIIEHL-W distribution are given in Table 2.

Figures 5 and 6 shows that skewness and kurtosis for MO-TIIEHL-W distribution can be decreasing as δ and γ increases while holding β and λ constant. For fixing δ and γ, skewness and kurtosis for MO-TIIEHL-W is decreasing as β and λ increases.

Table 2
Some Quantiles for MO-TIIEHL-W Distribution

	$(\delta,\gamma,\beta,\lambda)$				
u	(1, 1.5, 1.3, 2)	(0.7, 1, 1.5, 0.5)	(2.4, 1, 0.6, 1.5)	(2.1, 0.5, 3, 1.2)	(1.5, 1, 1.2, 0.4)
0.1	0.1658	0.0007	0.3517	0.1183	0.0011
0.2	0.2436	0.0031	0.5761	0.2278	0.0078
0.3	0.3112	0.0087	0.7820	0.3445	0.0259
0.4	0.3769	0.0196	0.9864	0.4718	0.0664
0.5	0.4451	0.0400	1.1997	0.6133	0.1485
0.6	0.5199	0.0792	1.4333	0.7750	0.3127
0.7	0.6075	0.1584	1.7043	0.9679	0.6525
0.8	0.7200	0.3406	2.0475	1.2176	1.4345
0.9	0.8931	0.9001	2.5673	1.6045	3.8037

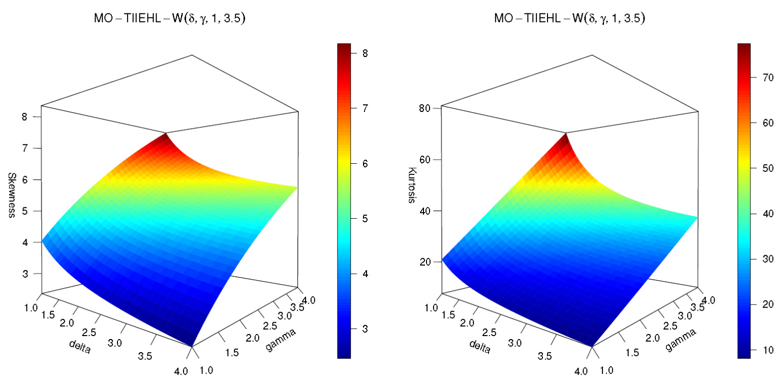

Figure 5
Graphs for Skewness and Kurtosis of MO-TIIEHL-W Distribution

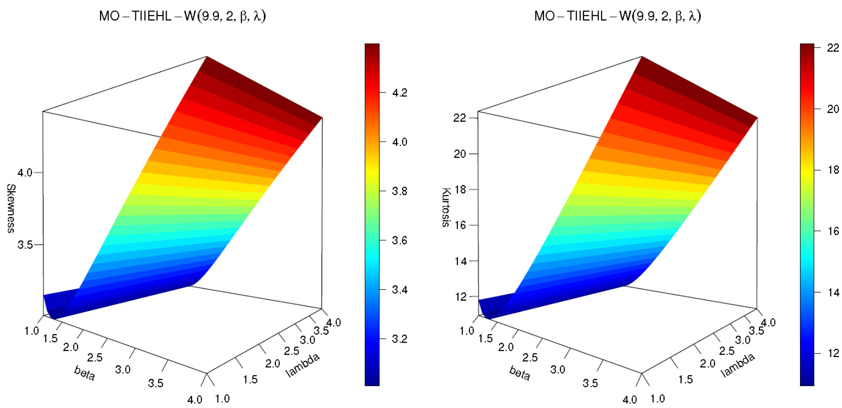

Figure 6

Graphs for Skewness and Kurtosis of MO-TIIEHL-W Distribution

5.3 *Marshall-Olkin Type II Exponentiated Half Logistic-Power (MO-TIIEHL-P) Distribution*

Let the baseline cdf be given by $G(y;\theta,k) = (\theta y)^k$ with pdf $g(\theta,k) = k\theta^k y^{k-1}$, for $y \in (0,\frac{1}{\theta})$ and $\theta, k > 0$. Then the resulting cdf and pdf of the MO-TIIEHL-P distribution are

$$F(y;\delta,\gamma,\theta,k) = 1 - \frac{\delta\left(\left[\frac{1-(\theta y)^k}{1+(\theta y)^k}\right]^\gamma\right)}{1-\bar{\delta}\left[\frac{1-(\theta y)^k}{1+(\theta y)^k}\right]^\gamma}, \tag{48}$$

and

$$f(y;\delta,\gamma,\theta,k) = \frac{2\delta\gamma\lambda\theta^k k y^{k-1}[1-(\theta y)^k]^{\gamma-1}}{[1+(\theta y)^k]^{\gamma+1}\left[1-\bar{\delta}\left[\frac{1-(\theta y)^k}{1+(\theta y)^k}\right]^\gamma\right]^2}, \tag{49}$$

respectively, for $\delta,\gamma,\theta,k > 0$.

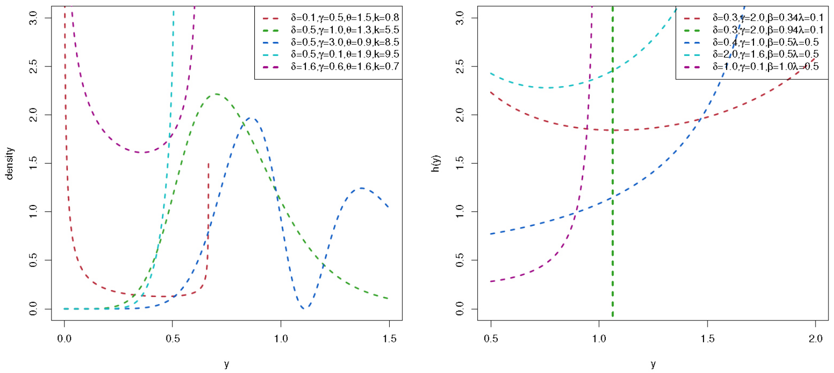

Figure 7

Graphs for pdf and hrf of MO-TIIEHL-P Distribution

Table 3

Quantiles for MO-TIIEHL-P Distribution

(δ,γ,θ,k)					
u	(0.9, 1.5, 1.3, 2)	(0.7, 1, 1.5, 3)	(2.4, 1, 0.6, 1.5)	(2.1, 4.2, 3, 1.2)	(1.5, 1, 1.2, 1.4)
0.1	3.0033	1.5618	4.0229	3.9671	2.9967
0.2	2.0209	1.1919	2.3429	2.1685	1.6791
0.3	1.5607	0.9959	1.6357	1.4953	1.1426
0.4	1.2681	0.8595	1.2183	1.1272	0.8333
0.5	1.0526	0.7508	0.9298	0.8866	0.6238
0.6	0.8780	0.6559	0.7095	0.7104	0.4669
0.7	0.7249	0.5661	0.5285	0.5694	0.3406
0.8	0.5789	0.4729	0.3689	0.4461	0.2317
0.9	0.4197	0.3609	0.2149	0.3231	0.1298

The pdf of the MO-TIIEHL-P distribution displays left-skewed, almost symmetric, U-shaped, bathtub, bimodal and J shapes, whereas the hrf gives increasing, J-shaped and bathtub shapes as shown in Figure 7.

Table 3 gives the quantiles for MO-TIIEHL-P distribution for selected parameter values. Figures 8 and 9 gives the 3D graphs for positive as well as negative skewness, and the MO-TIIEHL-P distribution can model data with various levels of kurtosis.

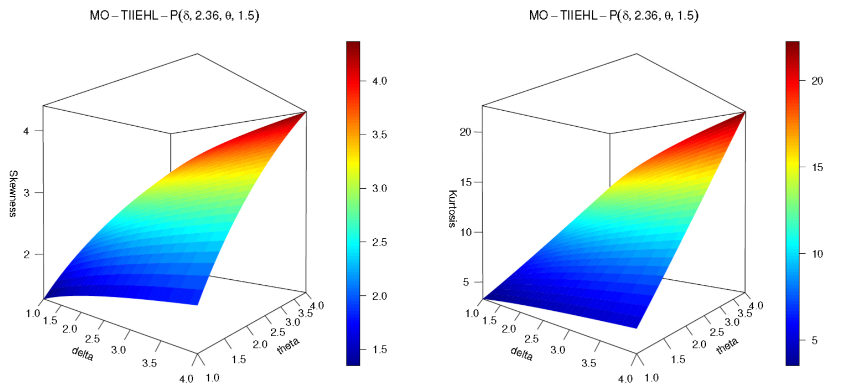

Figure 8

Graphs for Skewness and Kurtosis of MO-TIIEHL-P Distribution

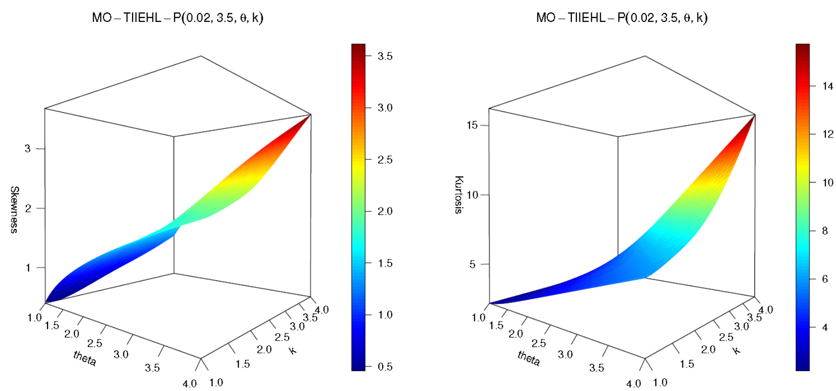

Figure 9

Graphs for Skewness and Kurtosis of MO-TIIEHL-P Distribution

6. Simulations

We examine the consistency of the estimators of the parameters of the MO-TIIEHL-LLo distribution via Monte Carlo Simulations. Equation (17) was used to obtain random samples from MO-TIIEHL-LLo distribution by the R package. The simulation was run repeatedly 1000 times each with sample sizes $n = 30, 50, 100, 200, 400, 800, 1200$. The estimated mean MLE, average bias (ABIAS) and root mean square error (RMSE) of the estimated parameter say, $\underline{\Psi}$, are computed as

$$\text{Mean} = \frac{\sum_{i=1}^{N}\underline{\Psi}_i}{N}, \quad \text{RMSE} = \sqrt{\frac{\sum_{i=1}^{N}(\underline{\Psi}_i - \underline{\Psi}^2)}{N}} \quad \text{and} \quad \text{ABIAS}\,(\underline{\Psi}) = \frac{\sum_{i=1}^{N}(\underline{\Psi}_i - \underline{\Psi})}{N},$$

respectively.

Tables 4 and 5 gives the mean MLEs, RMSEs and ABIAS. Clearly, as n increases, the mean MLEs are getting closer to the true values of the parameters, while the RMSEs and ABIAS gets smaller and moves towards zero.

Table 4

Results of Simulations for MO-TIIEHL-LLo Distribution

	(0.5, 1.2, 2.5)					(1.2, 2.6, 4.0)		
Parameters	n	Mean MLE	RMSE	ABIAS		Mean MLE	RMSE	ABIAS
δ	30	0.4757	0.1498	-0.0243		2.7889	5.2571	1.5889
	50	0.4727	0.1190	-0.0273		2.4300	1.6878	1.2301
	100	0.4733	0.0872	-0.0267		2.2948	1.1862	1.0948
	200	0.4739	0.0642	-0.0261		2.2669	1.1069	1.0669
	400	0.4792	0.0447	-0.0208		2.2647	1.0879	1.0647
	800	0.4777	0.0363	-0.0223		2.2667	1.0759	1.0667
	1200	0.4788	0.0321	-0.0212		2.2269	1.0331	1.0269
γ	30	1.5154	0.4071	0.3155		2.5327	0.6314	-0.0673
	50	1.4972	0.3568	0.2972		2.5645	0.5152	-0.0355
	100	1.4717	0.3033	0.2717		2.5344	0.3843	-0.0656
	200	1.4612	0.2763	0.2612		2.6105	0.2829	0.0105
	400	1.4501	0.2574	0.2501		2.6347	0.1413	0.0347
	800	1.4507	0.2547	0.2507		2.6201	0.1208	0.0201
	1200	1.4498	0.2526	0.2499		2.6153	0.1047	0.0153
c	30	2.6769	0.4545	0.1769		4.3952	1.1741	0.3952
	50	2.6339	0.3542	0.1339		4.2309	0.8002	0.2309
	100	2.6053	0.2483	0.1053		4.1302	0.4287	0.1302
	200	2.5881	0.1743	0.0881		4.0931	0.2921	0.0931
	400	2.5652	0.1249	0.0652		4.0554	0.2134	0.0554
	800	2.5642	0.1023	0.0642		4.0473	0.1544	0.0473
	1200	2.5607	0.0891	0.0607		4.0404	0.1294	0.0404

Table 5

Results of simulations for MO-TIIEHL-LLo Distribution

		(0.6, 1.2, 2.0)				(0.5, 1.5, 2.8)		
Parameter	n	Mean MLE	RMSE	ABIAS		Mean MLE	RMSE	ABIAS
δ	30	0.5468	0.1697	-0.0532		1.3334	0.9233	0.8334
	50	0.5436	0.1376	-0.0564		1.3057	0.8556	0.8057
	100	0.5437	0.1061	-0.0563		1.2933	0.8150	0.7933
	200	0.5445	0.0845	-0.0555		1.2891	0.7995	0.7890
	400	0.5500	0.0658	-0.0499		1.2946	0.7997	0.7946
	800	0.5483	0.0602	-0.0517		1.2895	0.7918	0.7899
	1200	0.5495	0.0568	-0.0505		1.2867	0.7885	0.7867
γ	30	1.4169	0.3143	0.2169		1.7769	0.4265	0.2769
	50	1.4007	0.2637	0.2007		1.7554	0.3539	0.2554
	100	1.3806	0.2162	0.1806		1.7269	0.2857	0.2269
	200	1.3720	0.1897	0.1720		1.7138	0.2429	0.2138
	400	1.3625	0.1714	0.1625		1.6988	0.2133	0.1988
	800	1.3633	0.1681	0.1633		1.6993	0.2074	0.1993
	1200	1.3627	0.1659	0.1627		1.6979	0.2034	0.1979
c	30	2.1417	0.3578	0.1417		3.1407	0.5814	0.3407
	50	2.1076	0.2798	0.1076		3.0988	0.4737	0.2988
	100	2.0863	0.1968	0.0863		3.0811	0.3823	0.2811
	200	2.0729	0.1391	0.0729		3.0717	0.3261	0.2717
	400	2.0551	0.1006	0.0551		3.0462	0.2783	0.2462
	800	2.0543	0.0829	0.0543		3.0452	0.2636	0.2451
	1200	2.0513	0.0726	0.0513		3.0407	0.2536	0.2407

7. Applications

The MO-TIIEHL-LLo distribution was compared with several models including Marshall-Olkin extended inverse Weibull (MOIW) by Pakungwati et al. [18], type II half logistic Burr XII (TIIHLBXII) distribution by Hassan et al.[10], the inverse-power logistic-exponential (IPLE) distribution by Sobhi and Mashail [21], extended log-logistic (ExLLoG) distribution by Lima and Cordeiro [13], the exponentiated Fr'echet (EFr) by Nadarajah and Kotz [17] and the exponentiated power Lindley (EPL) distribution by Ghitany et al. [8]. The distributions are:

$$f_{MOIW}(y;\alpha,\lambda,\theta)=\frac{\alpha\lambda\theta^{-\lambda}y^{-\lambda-1}e^{-(\theta y)^{-\lambda}}}{[\alpha-(\alpha-1)e^{-(\theta y)^{-\lambda}}]^{2}}, \quad (50)$$

for $\alpha,\lambda,\theta>0$,

$$f_{TIIHLBXII}(y;\lambda,\delta,c)=\frac{2\lambda c\delta y^{c-1}(1+y^{c})^{-\delta-1}[1-(1+y^{c})^{-\delta}]^{\lambda-1}}{[1+[1-(1+y^{c})^{-\delta}]^{\lambda}]^{2}}, \quad (51)$$

for $\lambda,\delta,c>0$,

$$f_{IPLE}(y;\alpha,\lambda,\beta)=\alpha\beta\lambda y^{-\beta-1}e^{\lambda y^{-\beta}}[e^{\lambda y^{-\beta}}-1]^{\alpha-1}([e^{\lambda y^{-\beta}}-1]^{\alpha}+1)^{2}, \quad (52)$$

for $\alpha,\lambda,\beta>0$,

$$f_{EXLLoG}(y;\alpha,\beta,\lambda)=\beta\alpha\lambda(e^{\lambda}-1)^{-1}y^{\alpha-1}e^{\lambda(1+\beta y^{\alpha})^{-1}}(1+\beta y^{\alpha})^{-2}, \quad (53)$$

for $\alpha,\beta,\lambda>0$,

$$f_{EFr}(y;\alpha,\lambda,\delta)=\alpha\lambda\delta^{\lambda}[1-e^{-(\delta/y)^{\lambda}}]^{\alpha-1}y^{-(1+\lambda)}e^{-(1+\lambda)(\delta/y)^{\lambda}}, \quad (54)$$

for $\alpha,\lambda,\delta>0$, and

$$f_{EPL}(y;\alpha,\beta,\delta)=\alpha\beta\delta y^{\alpha-1}(1+y^{\alpha})e^{-\beta y^{\alpha}}\left[1-\left(1+\frac{\beta y^{\alpha}}{1+\beta}\right)e^{-\beta y^{\alpha}}\right]^{\delta-1}, \quad (55)$$

for $\alpha,\beta,\delta>0$, and $y>0$.

Plots of the fitted pdf's, the histogram of the data and probability plots (PPs) (Chambers et al. [4]) are presented in Figures 10 and 13. To obtain the probability plot (PP), $F(y_{(i)};\hat{\delta},\hat{a},\underline{\Psi})$ was plotted against $\frac{i-0.375}{n+0.25}$, $i=1,2,\cdots,n$, where $y_{(i)}$ are the ordered sample values. A measure of closeness is given by the sum of squares (SS):

$$SS=\sum_{i=1}^{n}\left[F(y_{(i)};\hat{\delta},\hat{a},\underline{\Psi})-\left(\frac{i-0.375}{n+0.25}\right)\right]^{2}.$$

Plots of the estimated cdf (ECDF), Kaplan-Meier (K-M) survival curves, Total Time on Test (TTT)-transform and estimated hrf are given in Figures 11, 12, 14 and 15.

7.1 *Amounts of Run Off Data*

The first data consists of run off amounts at Jug Bridge, Maryland (Chhikara and Folks [6]). The data is presented in the appendix.

The estimated covariance matrix for the parameters of the MO-TIIEHL-LLo model for the run off data is given by

$$\begin{bmatrix} 0.1112 & 0.0337 & -0.0711 \\ 0.0337 & 0.0630 & -0.0219 \\ -0.0711 & -0.0219 & 0.2162 \end{bmatrix},$$

and the approximate 95% confidence interval are given by $\delta\in(0.1425, 1.4499)$, $\gamma\in(0.6477, 1.6318)$, and $c\in(1.8442, 3.6669)$.

Table 6

Estimates and statistics for run off data

	Estimates					Statistics						
Model	δ	γ	c	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	KS	P-Value	SS
MO-TIIEHL-LLo	0.7963	1.1398	2.7556	29.36	35.36	36.50	39.01	0.0137	0.1005	0.0897	0.9878	0.0224
	(0.3336)	(0.2511)	(0.4649)									
	α	λ	θ									
MOIW	34.7912	2.7093	5.3799	29.39	35.39	36.54	39.05	1.3023	6.5312	0.9713	2.2×10^{-16}	7.5916
	(79.6604)	(0.5188)	(3.6903)									
	λ	δ	c									
TIIHLBXII	1.5156	1.7191	2.0510	29.58	35.58	36.73	39.24	0.0653	0.4493	0.3107	0.0160	0.6692
	(2.6838)	(2.1679)	(2.1885)									
	α	λ	β									
IPLE	$9.2290\times 10^{+01}$	6.8765×10^{-01}	2.1927×10^{-02}	29.70	35.70	36.85	39.36	8.5074	50.4401	0.9821	2.2×10^{-16}	8.3137
	(9.2252×10^{-07})	(2.0779×10^{-03})	(3.6480×10^{-03})									
	α	β	λ									
EXLLoG	$2.8054\times 10^{+00}$	$2.7639\times 10^{+00}$	5.2247×10^{-05}	29.69	35.69	68.97	39.36	8.5077	50.4408	0.9824	2.2×10^{-16}	8.3136
	(4.6752×10^{-01})	$(1.0691\times 10^{+00})$	(5.5184×10^{-02})									
	α	λ	δ									
EFr	0.0723	8.7504	0.1405	58.76	64.76	65.90	68.42	0.0944	0.6396	0.3556	0.0034	0.8265
	(0.066899)	(7.795944)	(0.012143)									
	α	β	δ									
EPL	1.1394	0.9120	0.5087	182.43	188.43	189.58	192.09	0.3467	2.0654	0.6888	9.95×10^{-11}	2.9449
	(0.1768)	(0.1394)	(0.0707)									

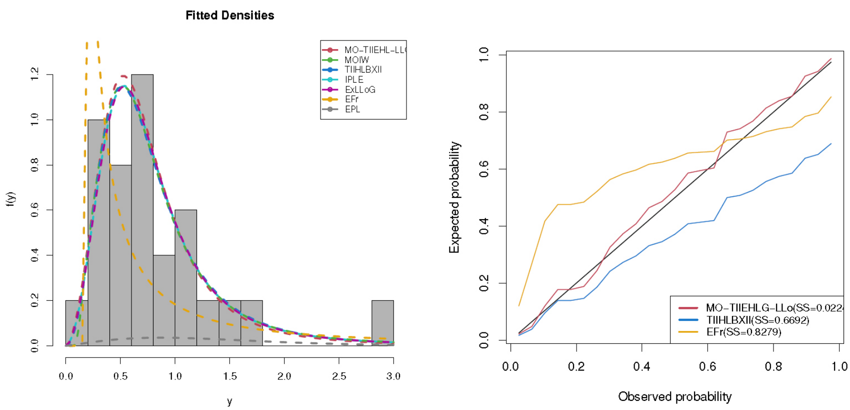

Figure 10

Fitted pdf's and PPs for run off data

Table 6 gives the parameter estimates with standard errors in parentheses, and several statistics for the run off data. From the table, the statistics: AIC, AICC and BIC are smallest for the MO-TIIEHL-LLo distribution. Again, values of SS and the statistics: A^*, W^* and KS are the least for the MO-TIIEHL-LLo distribution, which indeed showed that this model is better for this data and can be very useful.

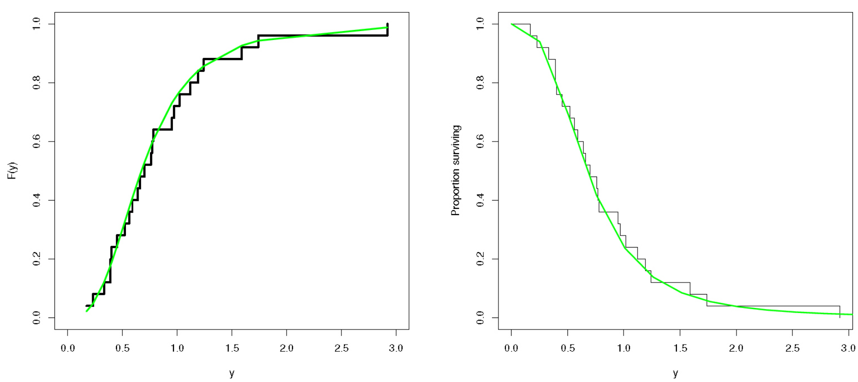

Figure 11

ECDF and K-M survival plots for run off data

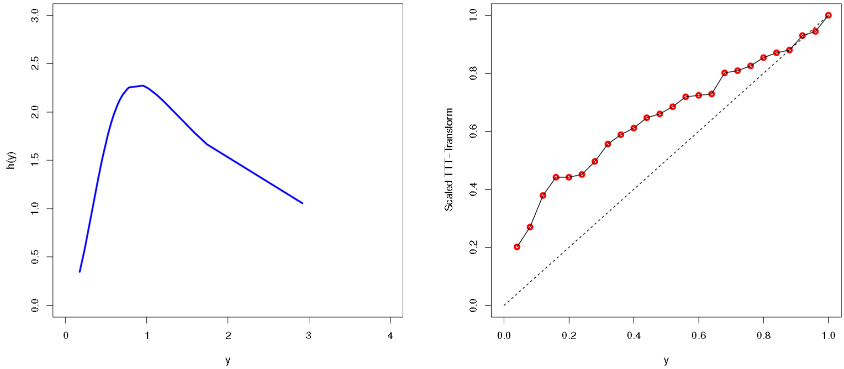

Figure 12

Estimated hrf and scaled TTT-transform plots for run off data

Figure 11 gives the ECDF and the K-M survival plots which shows that our model is the best in explaining the run off data. The estimated hrf and scaled TTT-transform plots for run off data are given in Figure 12.

7.2 *Chemotherapy Data*

The second data represents survival times (in years) of patients (Bekker et al. [3]). This data was analysed by Moakofi et al. [16]. The data is in the appendix.

The estimated covariance matrix for the parameters of the of MO-TIIEHL-LLo distribution for survival times data is given by

$$\begin{bmatrix} 0.1625 & 0.0202 & -0.0160 \\ 0.0202 & 0.0212 & -0.0090 \\ -0.0160 & -0.0090 & 0.0397 \end{bmatrix},$$

and the approximate 95% confidence interval are given by $\delta \in (0.2127, 2.2039)$, $\gamma \in (0.6067, 1.1778)$ and $c \in (1.1613, 1.9426)$.

Table 7

Estimates and statistics for chemotherapy data

	Estimates					Statistics						
Model	δ	γ	c	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	KS	P-Value	SS
MO-TIIEHL-LLo	1.4138	0.8923	1.5519	119.87	125.87	126.46	131.29	0.0819	0.5675	0.1029	0.6889	0.0792
	(0.4031)	(0.1457)	(0.1993)									
	α	λ	θ									
MOIW	41.1488	1.4620	15.5299	119.91	125.91	126.49	131.33	2.3690	11.8876	0.9757	2.2×10^{-16}	13.8904
	(1.8043)	(0.1864)	(3.1699)									
	λ	δ	c									
TIIHLBXII	2.1728	1.6315	0.9717	120.68	126.68	127.27	132.10	0.2428	1.5551	0.5933	1.8×10^{-15}	4.2042
	(2.1951)	(1.2112)	(0.5676)									
	α	λ	β									
IPLE	$1.6870 \times 10^{+02}$	6.9233×10^{-01}	6.4310×10^{-03}	120.39	126.39	126.98	131.81	15.9417	90.3303	0.9871	2.2×10^{-16}	15.4910
	(2.9995×10^{-08})	(7.8989×10^{-04})	(7.8076×10^{-04})									
	α	β	λ									
EXLLoG	1.5076	1.3168	0.0001	120.37	126.37	126.96	131.79	15.9451	90.3327	0.9871	2.2×10^{-16}	15.5114
	(0.1832)	(0.3519)	(0.0619)									
	α	λ	δ									
EFr	0.2207	1.8107	0.0669	171.10	177.10	177.69	182.52	0.1725	1.1685	0.2713	0.0020	1.1527
	(0.1116)	(0.6955)	(0.0216)									
	α	β	δ									
EPL	0.8184	0.6809	0.5091	364.85	370.85	371.43	376.27	0.4675	2.8420	0.6319	2.2×10^{-16}	4.1421
	(0.0949)	(0.0826)	(0.0527)									

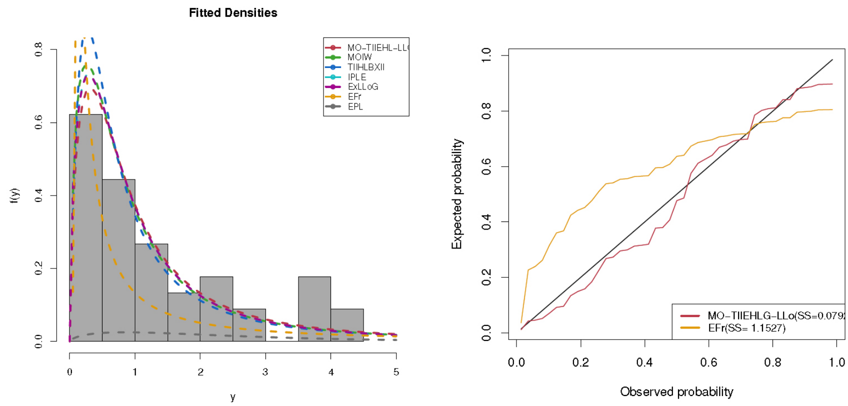

Figure 13

Fitted pdf's and PPs for chemotherapy data

Several goodness-of-fit statistics and model parameter estimates with (standard errors in parentheses) are presented in Table 7. We note that AIC, AICC and BIC for the MO-TIIEHL-LLo distribution are smallest when compared to other models listed in the table. Hence, the MO-TIIEHL-LLo distribution provide a better fit for the chemotherapy treatment data. Also, the value of the goodness-of-fit statistics: A^*, W^* and KS are smallest for chemotherapy treatment data as indicated in Table 7.

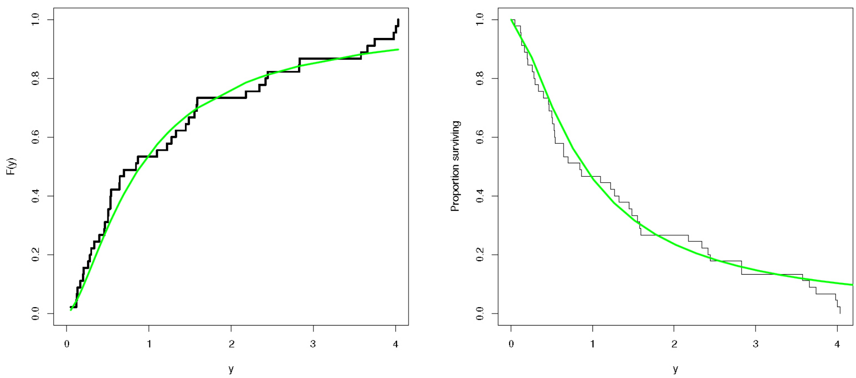

Figure 14

ECDF and K-M survival plots for chemotherapy data

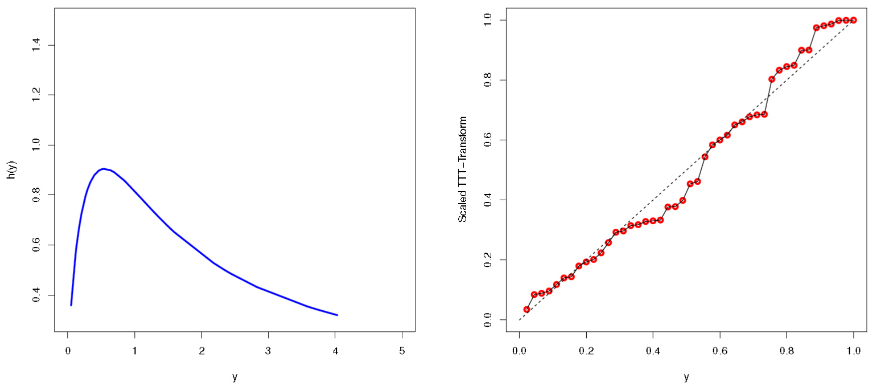

Figure 15

Estimated hrf and scaled TTT-transform plots for chemotherapy data

Figure 14 gives the ECDF and the K-M survival plots which shows that indeed our model is better in explaining the chemotherapy treatment data. The estimated hrf and scaled TTT-transform plots for chemotherapy treatment data are given in Figure 15.

8. Conclusions

We constructed and developed the Marshall-Olkin type II exponentiated half logistic-G distribution, and the parameters were estimated via maximum likelihood technique. Three particular cases were discussed in details. The applicability and flexibility of the Marshall-Olkin type II exponentiated half logistic-log-logistic distribution is given in two examples. The MO-TIIEHL-LLo distribution outperformed several non-nested models with the same number of parameters.

Appendix

https://drive.google.com/file/d/12_GfTA4THULMPk3KmakwfBjJG7F4q5mU/view?usp=sharing

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