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The discretization of the Topp-Leone generated family of distributions

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Abstract

The idea of discretizing an existing continuous lifetime model has gained considerably attention in the theory of probability distributions. In practice, real datasets may appear continuous in nature but seem discrete by observation. Apparently, it becomes more appropriate to model such dataset by a discrete analogue of existing continuous models. This paper develops a new discrete family of distributions called the Discrete Topp-Leone Generated Family of distributions with the sole aim of introducing more tractable discrete models in lifetime data analysis. Mathematical properties of the family are presented and in particular, the discrete Topp-Leone Power Lindley (DTLPL) distribution is studied in detail. A unique feature of the DTLPL distribution is that its hazard rate function can be decreasing, increasing, bathtub and inverted bathtub shape. We adopt the maximum likelihood method to estimate the unknown parameters of the distribution and two count data sets are used to illustrate the potential of the DTLPL distribution. Empirical findings show the flexibility of the DTLPL distribution over some existing discrete models in lifetime data fittings.

Subject Classification (2010): *60E05, 62E99.*

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1. Introduction

Lifetime models play an important role in data analysis, inference, and predictions. The time of occurrence, frequencies and effects of many real-life phenomena are analyzed using lifetime statistical models. In practice, most events such as the counts of traffic accidents in a day/week/month, records of death cases of highly contagious diseases, number of failure times of a component in a device etc, can be modeled by discrete probability distributions. Although, in most real-life situation, the original data may appear continuous in nature but discrete by observation. Thus, it becomes more reasonable and convenient to model such situation by a discrete analogue of existing continuous models, preserving one or more important traits of the continuous distribution. Discretization of a continuous lifetime model is an interesting and intuitively appealing approach to derive a discrete lifetime model corresponding to the continuous one (Lai, [14]).

Thanks to Chakraborty [3] for a detailed study on the different methods of discretizing existing continuous distributions. These methods include; the discretization based on the survival function, probability mass function (infinite series), cumulative distribution function, hazard function, reverse hazard function, the difference equation analogues of Pearsonian differential equation and the two-stage composite.

Let $S_x(x)$ denote the survival function of a random variable X following a known continuous probability distribution, then the probability mass function (pmf) of a discrete random variable $Y = [X]$, where $[X]$ indicates the smallest integer part or equal to X, is defined as:

$$\begin{aligned} P(Y = k) &= \sum_{j=0}^{1} (-1)^j S_x(k + j), \\ &= S_x(k) - S_x(k+1). \end{aligned} \tag{1}$$

The method defined in equation (1) is known as the discretization based on survival function. Many discrete analogues of continuous distributions have been introduced using this method. For example, the discrete Weibull distribution by Nakagawa and Osaki [16], discrete Burr distribution by Krishna and Singh [13], discrete Lindley distribution by Gomez-Deniz and Calderin-Ojeda [7], discrete generalized exponential distribution by Nekoukhou et al. [18], inverse Rayleigh distribution by Hussain et al. [9], discrete Power Lindley distribution by Oliveira et al. [19], etc.

Supanekar and Shirke [26] considered the survival function of the Marshall-Olkin family of distributions as the parent distribution in equation (1), to develop the discrete analogue of the continuous Marshall-Olkin family of distributions. The discrete Marshall-Olkin family generalized exponential distribution by Almetwally et al. [2], discrete Marshall Olkin Weibull distribution by Opone et al. [23] were developed using this framework. Jayakumar and Sankaran [10] also introduced a new discrete family of distributions using the truncated discrete Mittag-Leffler distribution. This family of discrete distributions is seen as a generalization of the discrete Marshall Olkin family distributions in Supanekar and Shirke [26].

The one-parameter Topp-Leone distribution on the other hand, have attracted a wide application due to its unique feature of exhibiting a bathtub hazard rate properties. Sangsanit and Bodhisuwan [25] developed the Topp-Leone generated class as a family of distributions based on the Topp-Leone distribution, Elgarhy et al. [4] introduced the type II Topp-Leone generated family of distributions, Reyad et al. [24] proposed the Topp-Leone odd Lindley-G family of distributions, Opone et al. [22] developed the Topp-Leone Power Lindley distribution, Tuoyo et al. [27] proposed the Topp-Leone Weibull distribution, Opone and Iwerumor [20] used the Marshall-Olkin method to develop the Marshall-Olkin Topp-Leone distribution, Opone and Osemwenkhae [21] considered the transmuted Marshall-Olkin Topp-Leone distribution.

In this paper, we construct a new discrete family of distributions called the discrete Topp-Leone generated family of distributions and study its properties. In Section 2, we introduce the discrete analogue of the continuous Topp-Leone generated family of distributions reported in Sangsanit and Bodhisuwan [25] and present some of its mathematical properties. In Section 3, some members of the new discrete Topp-Leone generated family of distribution are obtained. The discrete Topp-Leone Power Lindley (DTLPL) distribution which is a member of the new discrete family of distributions is introduced and its mathematical properties are derived in Section 4. In Section 5, we employed two count data sets (over-dispersed and equi-dispersed) to illustrate the potentiality of the family of distributions in modelling count data sets. The last Section concludes the paper.

2. The Discrete Topp-Leone Generated (DTL-G) Family of Distributions

Sangsanit and Bodhisuwan [25] introduced the Topp-Leone generated family of distributions with survival function defined by

$$S(x) = 1 - [G(x)]^{\alpha}[2 - G(x)]^{\alpha}, \quad x > 0, \alpha > 0, \tag{2}$$

where $G(x)$ is a parent distribution function.

Using this methodology, Sangsanit and Bodhisuwan [25] obtained the Topp-Leone generalized exponential distribution by considering the distribution function of the generalized exponential distribution proposed by Gupta and Kundu [8] as the parent distribution in equation (2).

Let X be a discrete random variable associated with a continuous random variable following the Topp-Leone generated family of distributions, we define the probability mass function (pmf) of the new discrete family of distributions by substituting equation (2) into (1) as

$$P(X = x) = P_x = [G(x+1)]^{\alpha}[2 - G(x+1)]^{\alpha} - [G(x)]^{\alpha}[2 - G(x)]^{\alpha}, \quad x > 0, \alpha > 0. \tag{3}$$

Lemma 2.1: *The probability mass function of the discrete Topp-Leone generated family of distributions can be rewritten as two infinite and one finite mixtures (with weight* $2^{\alpha-k}, k = 0, 1, 2, \ldots$) *of their parent distribution.*

Proof: Suppose $\alpha > 0$, is a real non-integer, then using the binomial series expansion, the first term in equation (3) can be expressed as

$$[G(x+1)]^{\alpha}[2 - G(x+1)]^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k 2^{\alpha-k}[G(x+1)]^{\alpha+k}, \tag{4}$$

since $\alpha > 0$, is a real non-integer, $[G(x+1)]^{\alpha+k}$ can be rewritten as

$$[G(x+1)]^{\alpha+k} = [1 - (1 - G(x+1))]^{\alpha+k} = \sum_{m=0}^{\infty}\binom{\alpha+k}{m}(-1)^m (1 - G(x+1))^m,$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{\alpha+k}{m}\binom{m}{n}(-1)^{m+n}(G(x+1))^n,$$

so that equation (4) now becomes,

$$[G(x+1)]^{\alpha}[2 - G(x+1)]^{\alpha} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{\alpha}{k}\binom{\alpha+k}{m}\binom{m}{n}(-1)^{k+m+n}[G(x+1)]^n,$$

using similar approach for the second term in equation (3), the pmf of the DTL-G family of distributions can be expressed as

$$P_x = \sum_{k,m=0}^{\infty}\sum_{n=0}^{m}\psi_{k,m,n}[G^n(x+1) - G^n(x)], \tag{5}$$

where $\psi_{k,m,n} = \binom{\alpha}{k}\binom{\alpha+k}{m}\binom{m}{n}(-1)^{k+m+n}$

which completes the proof.

2.1 *Distribution, Survival and Hazard(Failure) Rate Functions*

The cumulative distribution function, survival function, hazard rate function, reversed hazard rate function and the second rate of failure of the DTL-G family of distributions are respectively defined by

$$F(x,\alpha)=1-S(x,\alpha)+P(X=x)=[G(x+1)]^{\alpha}[2-G(x+1)]^{\alpha}, \tag{6}$$

$$S(x,\alpha)=1-[G(x)]^{\alpha}[2-G(x)]^{\alpha}, \tag{7}$$

$$h(x,\alpha)=\frac{P(X=x)}{S(x,\alpha)}=\frac{[G(x+1)]^{\alpha}[2-G(x+1)]^{\alpha}-[G(x)]^{\alpha}[2-G(x)]^{\alpha}}{1-[G(x)]^{\alpha}[2-G(x)]^{\alpha}}, \tag{8}$$

$$r^{*}(x,\alpha)=\frac{P(X=x)}{F(x,\alpha)}=1-\frac{[G(x)]^{\alpha}[2-G(x)]^{\alpha}}{[G(x+1)]^{\alpha}[2-G(x+1)]^{\alpha}}, \tag{9}$$

and

$$r^{**}(x,\alpha)=log\left(\frac{S(X=x)}{S(x+1,\alpha)}\right)=log\left(\frac{1-[G(x)]^{\alpha}[2-G(x)]^{\alpha}}{1-[G(x+1)]^{\alpha}[2-G(x+1)]^{\alpha}}\right). \tag{10}$$

2.2 *Probability Generating Function, Moments and Quantiles*

Lemma 2.2: *Let X be a discrete random variable belonging to the DTL-G family of distributons, then the probability generating function denoted by* $G_X(S)$, *of X is defined by*

$$G_X(S)=E(S^X)=1+(S-1)\sum_{x=1}^{\infty}S^{x-1}\{1-[G(x)]^{\alpha}[2-G(x)]^{\alpha}\}.$$

Proof: It is straight forward from the definition of a probability generating function.

Lemma 2.3: *The mean and variance of the DTL-G family of distributions are respectively defined by*

$$\mu=G'_X(1)=\sum_{x=1}^{\infty}\{1-[G(x)]^{\alpha}[2-G(x)]^{\alpha}\},$$

and

$$\sigma^2=\sum_{x=1}^{\infty}(2x-1)\{1-[G(x)]^{\alpha}[2-G(x)]^{\alpha}\}-\{\sum_{x=1}^{\infty}\{1-[G(x)]^{\alpha}[2-G(x)]^{\alpha}\}\}^2.$$

Proof: The proof follows from lemma 2.2

Lemma 2.4: *The recurrence relation for generating probabilities of the DTL-G family of distributions is given by*

$$P_{x+1} = \frac{[G(x+2)]^{\alpha}[2-G(x+2)]^{\alpha} - [G(x+1)]^{\alpha}[2-G(x+1)]^{\alpha}}{[G(x+1)]^{\alpha}[2-G(x+1)]^{\alpha} - [G(x)]^{\alpha}[2-G(x)]^{\alpha}} P_x, \quad x = 0,1,2,\ldots$$

where,

$$P_0 = [G(1)]^{\alpha}[2-G(1)]^{\alpha}$$

Proof: It is straight forward.

Lemma 2.5: *The quantile function of the DTL-G family of distributions denoted by* $Q_x(u)$ *is given by*

$$Q_x(u) = G^{-1}\left[1-\sqrt{(1-u^{\frac{1}{\alpha}})}\right]-1.$$

Proof: The proof is obtained by solving the system of non-linear equation $F^{-1}(u) = Q_x(u)$.

The corresponding median of the DTL-G family of distributions is thus obtained as

$$Q_x(0.5) = G^{-1}\left[1-\sqrt{(1-(0.5)^{\frac{1}{\alpha}})}\right]-1.$$

3. Some Members of the DTL-Generated Family of Distributions

3.1 *Discrete Topp-Leone Weibull Distribution*

The cumulative distribution function of the Weibull distribution with shape parameter λ and scale parameter β is given by $G(x) = 1-e^{-\beta x^{\lambda}}$.

Let $\gamma = e^{-\beta}$, $0<\gamma<1$, then the pmf of the new distribution using equation (3) is given by

$$P(X=x) = P_x = \left[1-\gamma^{2(x+1)^{\lambda}}\right]^{\alpha} - \left[1-\gamma^{2x^{\lambda}}\right]^{\alpha}, \quad x>0, \alpha, \beta > 0.$$

Note that when $\gamma^2 = \gamma$, the distribution reduces to the exponentiated discrete Weibull distribution introduced in Nekoukhou and Bidram [17]. When $\gamma^2 = \gamma$ and $\alpha = 1$, the distribution reduces to the discrete Weibull distribution studied in Nakagawa and Osaki [16]. When $\gamma^2 = \gamma$ and $\lambda = 1$, the distribution reduces to the discrete generalized exponential distribution obtained in Nekoukhou et al. [18]. Finally, When $\gamma^2 = \gamma$ and $\alpha = \lambda = 1$, the distribution reduces to the geometric distribution (see Johnson et al. [11]).

3.2 *Discrete Topp-Leone Gumbel Distribution*

Let $G(x) = exp\left[-exp\left(-\frac{x-\mu}{\sigma}\right)\right]$ be the distribution function of the Gumbel distribution, then the pmf of the new distribution is defined using equation (3) as

$$P(X=x) = P_x = exp\left[-exp\left(-\frac{x+1-\mu}{\sigma}\right)\right]^{\alpha}\left\{2-exp\left[-exp\left(-\frac{x+1-\mu}{\sigma}\right)\right]\right\}^{\alpha}$$
$$-exp\left[-exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{\alpha}\left\{2-exp\left[-exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}, -\infty < x, \alpha, \sigma > 0, \mu < \infty. \quad (12)$$

3.3 *Discrete Topp-Leone Normal Distribution*

The pmf of the discrete Topp-Leone Normal distribution is defined by

$$P_x = \left[\Phi\left(\frac{x+1-\mu}{\sigma}\right)\left(2-\Phi\left(\frac{x+1-\mu}{\sigma}\right)\right)\right]^{\alpha} - \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\left(2-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)\right]^{\alpha},$$
$$-\infty < x, \alpha, \sigma > 0, \mu < \infty, \quad (13)$$

where $\Phi(.)$ is the distribution function of the standard Normal distribution.

Other generalizations such as the discrete Topp-Leone Burr XII, discrete Topp-Leone generalized exponential distributions can be obtained as members of the DTL-G family of distributions. In the next Section,we develop the discrete Topp-Leone power Lindley (DTLPL) distribution.

4. Discrete Topp-Leone Power Lindley distribution

Ghitany et al. [5] proposed the Power Lindley distribution as an extension of the one-parameter Lindley distribution introduced in Lindley [15] with probability density function, $f(x)$ given by

$$g(x) = \frac{\lambda\beta^2(1+x^{\lambda})x^{\lambda-1}e^{-\beta x^{\lambda}}}{1+\beta}, \quad x > 0, \lambda, \beta > 0. \quad (14)$$

The cumulative distribution function associated with equation (14) is given by

$$G(x) = 1 - \frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}, \quad x > 0, \lambda, \beta > 0. \quad (15)$$

Substituting equation (14) into equation (3), we obtain the probability mass function (pmf) of the discrete Topp-Leone Power Lindley (DTLPL) distribution as

$$P_x = \left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha} - \left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha},$$
$$x>0, \lambda, \beta, \alpha > 0. \quad (16)$$

Figure 1 displays the plots of the probability mass function of the discrete Topp Leone Power Lindley distribution at various choices of the parameters. The plots indicate that the DTLPL distribution can be left-skewed, right-skewed, monotonically decreasing, and approximately symmetric.

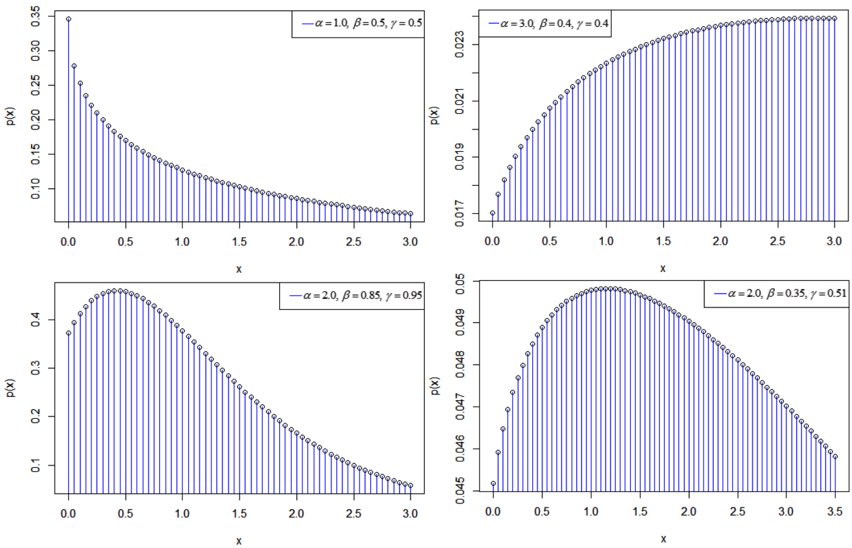

Figure 1

Probability mass function of the DTLPL distribution

4.1 *Distribution Function and Hazard Rate Function*

The cumulative distribution function, survival function, hazard rate function, reversed hazard rate function and the second rate of failure of the discrete Topp-Leone Power Lindley (DTLPL) distribution are respectively defined as

$$F(x) = \left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha}, \quad x>0, \lambda, \beta, \alpha > 0, \quad (17)$$

$$S(x)=1-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}, \tag{18}$$

$$h(x)=\frac{\left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}}{1-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}}, \tag{19}$$

$$r^{*}(x)=1-\frac{\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}}{\left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}}, \tag{20}$$

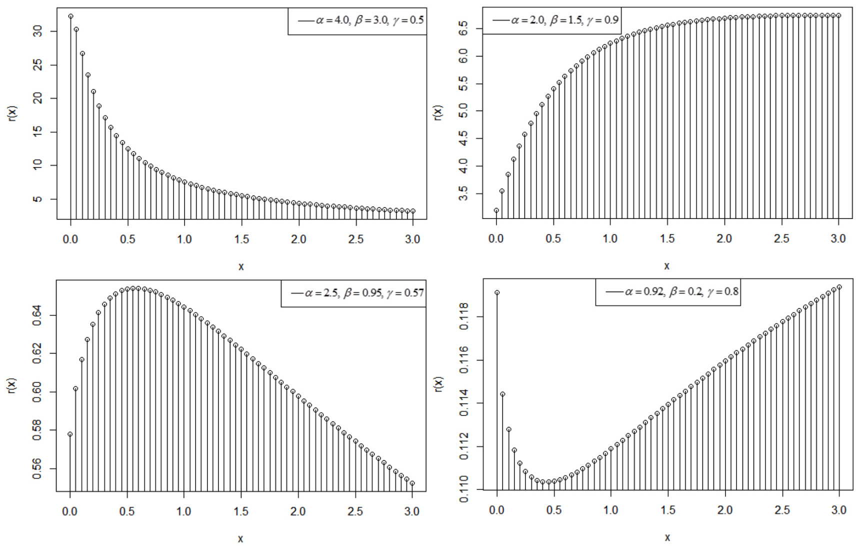

Figure 2

Hazard rate function of the DTLPL Distribution

$$r^{**}(x)=log\left[\frac{1-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}}{1-\left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}}\right]. \tag{21}$$

The plots of the hazard function of DTLPL distribution are shown in Figure 2. We observed that for varying values of the parameters of DTLPL distribution, the shape of the hazard function accommodates an increasing, decreasing, bathtub and inverted bathtub hazard rate properties.

4.2 *Probability Generating Function, Quantile, Median, Mean, Variance and Recurrence Relation*

The probability generating function of the discrete Topp-Leone Power Lindley distribution is defined by

$$G_X(S)=1+(S-1)\sum_{x=1}^{\infty}S^{x-1}\left(1-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}\right). \tag{22}$$

The mean and variance of the DTLPL distribution are obtained from equation (22) as

$$\mu=G_X^{'}(1)=\sum_{x=1}^{\infty}\left(1-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}\right),$$

and

$$\sigma^{2}=\sum_{x=1}^{\infty}(2x-1)\left(1-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}\right)-\left[\sum_{x=1}^{\infty}\left(1-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^{2}\right]^{\alpha}\right)\right]^{2}.$$

Numerical computations of the mean, variance, coefficient of skewness and coefficient of kurtosis of the DTLPL distribution are shown in Table 1.

Table 1

Numerical Computation of the Moments of DTLPL Distribution

λ	β	α	Mean(μ)	Variance (σ^2)	Skewness (S_k)	Kurtosis (K_s)
0.5	1	15	5.4370	18.2600	1.9264	5.9202
		20	6.4470	26.5700	2.7445	12.1955
	2	15	0.4313	1.3340	3.2805	20.1469
		20	0.5624	1.2790	2.2350	7.1447
0.9	1	15	1.7370	0.9831	0.9479	1.0358
		20	1.9450	1.0680	1.1725	2.8593
	2	15	0.1323	0.2155	1.1799	2.1998
		20	0.2237	0.1816	0.9268	1.2141

Table 1 shows that for different values of the parameters of the DTLPL distribution, the mean (μ) is either greater, equal or less than the variance (σ^2) which indicates flexibility of the distribution in analyzing under-dispersed $(\mu > \sigma^2)$, equi-dispersed $(\mu \simeq \sigma^2)$ and over-dispersed $(\mu < \sigma^2)$ data sets.

The u^{th} quantile of the DTLPL distribution denoted by $Q_x(u)$ is defined by

$$Q_x(u) = \left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(\frac{(1+\beta)(\sqrt{u^{\frac{1}{\alpha}}}-1)}{e^{\beta+1}}\right)\right]^{\frac{1}{\lambda}} -1. \tag{23}$$

In particular, the median is obtained as

$$Q_x\left(\frac{1}{2}\right) = \left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(\frac{(1+\beta)(\sqrt{(0.5)^{\frac{1}{\alpha}}}-1)}{e^{\beta+1}}\right)\right]^{\frac{1}{\lambda}} -1, \tag{24}$$

where $W_{-1}(.)$ is the negative branch of the Lambert W function.

The recurrence relation for generating probabilities of the DTLPL distribution is given by

$$P_{x+1} = \frac{\left[1-\left(\frac{(1+\beta+\beta(x+2)^{\lambda})e^{-\beta(x+2)^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha} - \left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha}}{\left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha} - \left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha}} P_x, \tag{25}$$

4.3 *Parameter Estimation*

Let $x=(x_1,x_2,\cdots,x_n)$ be random samples of size n associated with the discrete Topp-Leone Power Lindley distribution with pmf defined in equation (16), then the log-likelihood function is defined by

$$\ell=\sum_{i=1}^{n} log\left\{\left[1-\left(\frac{(1+\beta+\beta(x+1)^{\lambda})e^{-\beta(x+1)^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha}-\left[1-\left(\frac{(1+\beta+\beta x^{\lambda})e^{-\beta x^{\lambda}}}{1+\beta}\right)^2\right]^{\alpha}\right\} \quad (24)$$

The maximum likelihood estimates say $\hat{\phi}=(\hat{\alpha},\hat{\lambda},\hat{\beta}),$ can be obtained by solving the system of non-linear equation $\frac{\delta\ell}{\delta\phi}=0.$ This is achieved by employing a standard numeric optimization algorithm such as the Newton-Raphson Iterative Scheme to optimize the log-likelihood function. The maximum likelihood estimates of the DTLPL distribution is obtained using the *fitdistrplus* package in R statistical program.

5. Analysis of Count Data using DTLPL Distribution

In this section, we illustrate the potentiality of the DTLPL distribution in analyzing two count data sets. The first data set records the number of European corn-borer larvae pyrausta in field corn reported in Abebe and Shanker [1]. The data set is over-dispersed with mean = 0.65 and variance = 0.85. The second data set consists of the number of deaths due to horse kicks in the Prussian army between 1875 and 1894 available in Klugman et al. [12]. The data set is equi-dispersed with mean = 0.610 and variance = 0.611.

The goodness of fit of the DTLPL distribution is compared with the exponentiated discrete Weibull (EDW) distribution due to Nekoukhou and Bidram [17], discrete Weibull (DW) distribution due to Nakagawa and Osaki [16], discrete power Lindley (DPL) distribution due to Oliveira et al. [19], discrete generalized exponential (DGE) distribution due to Nekoukhou et al. [18] and generalized geometric (GG) distribution due to Gomez-Deniz [6].

The model selection is considered based on parameter estimates, maximized log-likelihood and the expected frequencies. We represent the expected frequencies closer to the corresponding observed frequencies in bold cases. To further determine which distribution fit better for the two data sets, the chi-square goodness-of-fit statistics with the corresponding p-value is employed to evaluate the fit of each distribution. The summary statistics (mle and goodness of fit) for the data set is presented in Tables 2 and 3, respectively.

Table 2

Summary Statistics (mle and goodness of fit) for the First Data Set

Counts	**Observed**	**Expected Frequency**					
	Frequency	**DTLPL**	**EDW**	**DPL**	**DW**	**DGE**	**GG**
0	188	**187.99**	188.02	185.98	185.89	187.11	186.18
1	83	82.52	**82.69**	88.74	89.14	88.58	88.81
2	36	37.33	**36.95**	33.32	32.99	31.77	32.85
3	14	12.45	**12.61**	11.14	11.06	10.93	10.97
4	2	**3.05**	3.11	3.44	3.47	3.72	3.54
5	1	0.56	0.54	**1.00**	1.04	1.26	1.13
Estimates		$\alpha=0.403$ $\beta=0.402$ $\gamma=1.428$	$\alpha=0.375$ $\beta=1.855$ $\gamma=0.766$	$\alpha=1.088$ $\beta=0.455$ $\gamma=0.325$	$\beta=1.148$ $\gamma=0.426$	$\alpha=1.331$ $\gamma=0.338$	$\alpha=1.594$ $\gamma=0.317$
-LL		-354.96	-354.95	-355.7881	-355.9097	-356.4021	-356.07
χ^2(d.f)		0.9503(1)	0.9668(1)	1.9458(1)	2.1273(2)	2.6316(2)	2.2217(2)
$p-value$		0.3296	0.3255	0.1630	0.1447	0.1049	0.1361

Table 3

Summary Statistics (mle and goodness of fit) for the Second Data Set

Counts	**Observed**	**Expected Frequency**					
	Frequency	**DTLPL**	**EDW**	**DPL**	**DW**	**DGE**	**GG**
0	109	**108.94**	108.92	108.58	108.51	108.57	108.31
1	65	**65.48**	65.53	66.47	66.72	67.81	67.35
2	22	**20.91**	20.81	20.10	19.88	18.10	19.22
3	3	**4.08**	4.14	4.12	4.14	4.26	4.10
4	1	0.54	0.54	0.63	0.66	**0.98**	0.82
Estimates		$\alpha=0.753$ $\beta=0.619$ $\gamma=1.349$	$\alpha=0.715$ $\beta=1.649$ $\gamma=0.573$	$\alpha=1.328$ $\beta=0.507$ $\gamma=0.342$	$\beta=1.417$ $\gamma=0.458$	$\alpha=2.369$ $\gamma=0.227$	$\alpha=3.479$ $\gamma=0.196$
-LL		-206.0741	-206.0880	-206.1209	-206.1486	-206.620	-206.2909
χ^2(d.f)		0.7381(1)	0.7782(1)	0.7355(1)	0.7617(2)	1.332(2)	0.823(2)
$p-value$		0.3903	0.3777	0.3911	0.3828	0.2485	0.3643

An appropriate model for fitting count data sets corresponds to the one having a closer expected frequency to the observed frequency, the highest log-likelihood and the least χ^2-statistics values. Results obtained from both Tables was in favour of the proposed discrete Topp-Leone Power Lindley (DTLPL) distribution over the existing discrete distributions considered.

6. Conclusion

In this paper, we have developed a new discrete family of distributions called the discrete Topp-Leone generated family of distributions. Some mathematical properties of the family were derived and in particular, the discrete Topp-Leone power Lindley (DTLPL) distribution was studied in detail. It was found that the pmf of the DTLPL distribution can be decreasing, left-skewed, right-skewed unimodal and approximately symmetric shaped. On the other hand, the hazard rate function possesses a decreasing, increasing, bathtub and inverted bathtub hazard rate property. Two count data sets were used to illustrate the potential of the DTLPL distribution in lifetime data fittings. Results obtained from the two data sets revealed that the DTLPL distribution provided better fits than the competing discrete distributions.

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