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Showing the effects of comorbidity of Diabetes Mellitus on diabetics' population from a simple linear mathematical model

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Abstract

It is an established fact that many infectious diseases, including COVID-19, AIDS are difficult to treat in the presence of various other comorbid diseases. It is, therefore, imperative to examine the impact of comorbidity in a noninfectious disease like diabetes. This study examines the effect of comorbidity of Diabetes Mellitus on the population dynamics of individuals with and without complications by modifying an existing simple linear mathematical model. A parameter was added to account for the rate of developing comorbid diabetes in the existing mathematical model. The stability nature of the model was determined and found to be asymptotically stable. The problem was solved using analytical and numerical methods of Runge-Kutta of order four, Euler method, and Fundamental matrix. Numerical solutions compare favorably with the analytic solutions. Graphical comparisons were made between the solutions of the original and modified models. The model predicts a continued increase in the number of diabetics, but effective and simultaneous management of Diabetes Mellitus, its complications, and comorbidities can significantly reduce the number of diabetics with complications. Managing any additional medical conditions that a diabetic may have is crucial for effective Diabetes Mellitus management.

Subject Classification: *(2010) 93A30, 65L20, 93B35, 34A30.*

Keywords: *Mathematical model, Comorbidity, Diabetes Mellitus.*

1. Introduction

A metabolic endocrine illness called Diabetes Mellitus develops when the body is unable to metabolize glucose (sugar) because of flaws in insulin secretion, action, or both. Hyperglycemia, a hazardous rise in blood sugar, is the result of this. When a blood sugar test after an eight-hour fast consistently reveals readings above 125mg% (7.0mmol/L), diabetes is often diagnosed. Fasting blood sugar (FBS) readings between 110 and 125 (6.1-6.9 mmol/L) are indicative of impaired fasting glucose tolerance, which frequently precedes fully developed diabetes [1]. The normal range of systemic blood glucose levels is 4.5 to 5.5 mmol/L [2]. Insulin sensitivity is impaired in people with type 2 diabetes, which prevents glucose from entering cells. Though it is increasingly being diagnosed in children and teenagers, type 2 diabetes is more common than type 1 and primarily affects adults [3].

The increasing prevalence of *Diabetes Mellitus* worldwide is comparable to the spread of SARS-CoV-2 and affects people of all genders, ages, and ethnicities. As of 2019, it was estimated that around 9.3% of the world's population, or 463 million adults aged 20 to 79, have diabetes. However, if current trends persist, this number is expected to rise to 578 million (10.2%) by 2030 and 700 million (10.9%) by 2045. Moreover, it has

been projected that diabetes and its associated complications will result in 4.2 million deaths within this period [3]. Comorbidity refers to the presence of two or more medical conditions simultaneously in an individual, but the coexistence of conditions associated with diabetes does not necessarily indicate causality. High blood pressure, high cholesterol, heart disease, stroke, depression, loss of vision, and kidney-related issues are common comorbidities associated with diabetes [4, 5, 6]. Adults with type 2 diabetes and high body mass index (BMI) have been observed to have an increased risk of developing cancer, particularly in women for breast and endometrial cancer and in both sexes for colorectal and intrahepatic (liver) cholangiocarcinoma [7]. A complication is an additional problem that develops due to a condition being worsened by a disease, injury, or therapy. Comorbidities can also increase the risk of developing diabetic complications or other health issues. The burden of caregiving on households, families, and the community as a whole increases with multiple conditions, placing additional demands on those who are affected, their families, and the community. Accurately estimating the prevalence of diabetes is challenging since it often goes undiagnosed for a prolonged period of time [8, 9]. Poor blood glucose control significantly increases the risk of complications and mortality [10]. The presence of comorbidities, including Diabetes Mellitus, can lead to a rapid and severe progression of COVID-19, resulting in increased mortality rates [11, 12]. The virus can induce diabetes by worsening insulin resistance or by targeting the pancreatic islets [13]. Diabetic patients have a higher susceptibility to COVID-19 [14]. Clinical outcomes were worse for patients with any comorbidity than for those without [9, 15].

This research aims to develop a mathematical approach to address the challenges associated with the treatment and management of diabetes, including the factor of comorbidity. Mathematical modeling utilizes symbols, numbers, and letters to represent variables and their relationships to describe real-world systems. Previous studies have explored the diabetic population with and without complications [16, 17, 18, 19], and in [20], a mathematical model for diabetes complications and control was developed, analyzed for stability, and numerically solved to provide insights into the concurrent treatment of diabetes and its complications. To solve the model presented in [18], [21] utilized the fundamental matrix method, which employs matrix differential equation theory. This research uses the same method to obtain an analytical solution for the mathematical model of diabetes complications and control, incorporating comorbidity.

2. Methodology

The existing model as presented in [18], analyzed and solved in [20] is given as:

$$C'(t) = I - (\rho + \theta)C(t) + \rho N(t), C(0) = C_0 \tag{1}$$

$$N'(t) = 2I - (\nu + \delta)C(t) - \mu N(t), N(0) = N_0 \tag{2}$$

The parameters are defined as given in Table 1.

2.1 *Formulation of the new Model*

Let C(t) and D(t) denote the number of diabetics with and without complications, respectively, at a given time t. N(t) represents the total population of diabetics, and the incidence of diabetes is represented by I, which is divided into I1 for diabetes without complications and I2 for diabetes with complications. The mathematical model includes several parameters, such as the natural mortality rate (μ), the probability of developing complications (ρ), the rate of complication management, the rate of severe disability among diabetics with complications (v), and the mortality rate due to complications.

The number of diabetics without complications will decrease due to natural mortality (μD) and the development of complications (ρD), but increase due to effective complication management (γC). The decline in the number of diabetics with complications can be attributed to natural mortality (μC), the rate at which patients become severely disabled (vC) due to poorly controlled complications, and the mortality rate from complications. Effective complication management also contributes to the decrease in the number of diabetics with complications.

Moreover, the development of comorbidities increases the difficulty in managing diabetes, leading to a transition from diabetics without complications to diabetics with complications. Additionally, the number of diabetics with complications increases as C(t) increases.

Figure 1 is the schematic diagram depicting the formulation of the new model. The following system of Ordinary differential equations represents the rates of change:

$$D'(t) = \frac{dD(t)}{dt} = I_1 - (\rho + \mu)D(t) - \tau D(t) + \gamma C(t) \tag{3}$$

$$C'(t) = \frac{dC(t)}{dt} = I_2 + \rho D(t) - (\gamma + \mu + \nu + \delta)C(t) + \tau C(t) \tag{4}$$

$$N(t) = D(t) + C(t) \tag{5}$$

$$N'(t) = D'(t) + C'(t) \tag{6}$$

$$\theta = \gamma + \mu + \nu + \delta \tag{7}$$

Assuming that $I_1 = I_2 = I$, since new cases are usually not diagnosed with or without complication in mind, although as at the time of diagnosis many might have develop a complication of the disease. Hence

$$N'(t) = 2I - (\nu + \delta - 2\tau)C(t) - (\mu + \tau)N(t) \tag{8}$$

$$C'(t) = I - (\rho + \theta - \tau)C(t) + \rho N(t) \tag{9}$$

So, for the new Model, the set of linear initial value ordinary differential equations is as follows:

$$C'(t) = I - (\rho + \theta - \tau)C(t) + \rho N(t), \qquad C(0) = C_0 \tag{10}$$

$$N'(t) = 2I - (\nu + \delta - 2\tau)C(t) - (\mu + \tau)N(t), \qquad N(0) = N_0 \tag{11}$$

Represented thus in matrix form as:

$$\begin{pmatrix} C(t) \\ N(t) \end{pmatrix}' = \begin{pmatrix} -(\rho+\theta-\tau) & \rho \\ -(\nu+\delta) & -(\mu+\tau) \end{pmatrix} \begin{pmatrix} C(t) \\ N(t) \end{pmatrix} + \begin{pmatrix} I \\ 2I \end{pmatrix}, \begin{pmatrix} C_0 \\ N_0 \end{pmatrix} \tag{12}$$

2.2 *The Eigenvalues*

Finding the Eigenvalues from Equation (12)

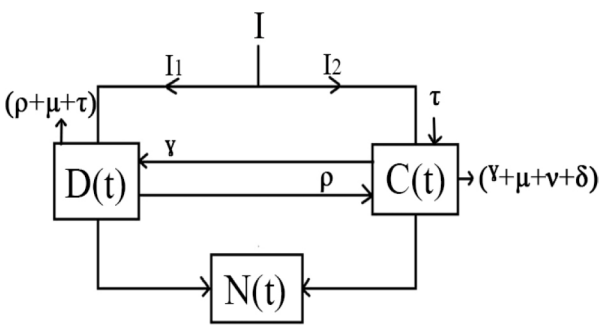

Figure 1

Schematic diagram of the model

Table 1

Parameter definitions, values, and sources

Parameter	Definition of parameters	Value	Source(s)
Δ	Mortality rate due is complications	0.05	Boutayeb et al., [16].
ρ	Probability of developing a complication	0.85	Akinsola and Oluvo, [18, 20].
υ	Rate at which patients with complications become severely disabled	0.05	Boutayeb et al., [16].
γ	Rate at which complications are controlled	0.50	Informed from stability analysis.
μ	Natural mortality rate	0.02	Boutayeb et al., [16].
τ	Rate of developing comorbidity of diabetes	0.16	Du et al., [22]; AIHW, [23].
I	Incidence of *Diabetes Mellitus*	6×10^6	Boutayeb et al., [16]; Sarah et al., [24].
I_1	Incidence of *Diabetes Mellitus* without complications		
I_2	Incidence of *Diabetes Mellitus* with complications		
$D(t)$	Number of people with *Diabetes Mellitus* (diabetics) without complications		
$D(0)$	Assumed initial value of population of Diabetics without complication	1.7×10^7	Akinsola and Oluvo, [18, 20].
$C(t)$	Number of people with *Diabetes Mellitus* (diabetics) with complications		
$C(0)$	Assumed initial value of population of Diabetics with complications	9.5×10^7	Akinsola and Oluvo, [18, 20].

$$\begin{vmatrix} -(\rho+\theta-\tau)-\lambda & \rho \\ -(\nu+\delta) & -(\mu+\tau)-\lambda \end{vmatrix} = 0 \tag{13}$$

$$\lambda^2 + (\rho+\theta+\mu)\lambda + \rho(\nu+\delta+\mu-\tau) + (\mu+\tau)(\theta-\tau) = 0 \tag{14}$$

$$\psi = \rho + \theta + \mu \tag{15}$$

$$\phi = (\theta - \tau)(\tau + \mu) + \rho(\nu + \delta + \mu - \tau) \tag{16}$$

$$\lambda^2 + \psi\lambda + \phi = 0 \tag{17}$$

The eigenvalues are determined by the roots of the quadratic equation (17).

$$\lambda_1 = \frac{-\psi + \sqrt{\psi^2 - 4\phi}}{2} \tag{18}$$

$$\lambda_2 = \frac{-\psi - \sqrt{\psi^2 - 4\phi}}{2} \tag{19}$$

Table 2 lists the model's Eigenvalues and stability characteristics for a range of specified and indicted values.

Table 2

Summary of the Eigenvalues and nature of stability of equilibrium point for different cases at various values of the rate of developing comorbidity (τ) in the model.

Cases	τ	λ_1	λ_2	**Nature of Critical point**
Case 1	0.08	−0.06160772612	−1.428392274	Asymptotically Stable
Case 2	0.16	−0.03350509489	−1.456494905	Asymptotically Stable
Case 3	0.32	−0.04431932702	−1.534319327	Asymptotically Stable
Case 4	0.42	−0.1047205423	−1.594720542	Asymptotically Stable

2.3 *Analytic solution of the model using Fundamental matrix formula method*

Suppose A(t) and f(t) are continuous on an open interval L that includes the point t0, then the initial value problem has a unique solution x(t) on the entire interval L for any initial vector x0.

$$\mathbf{X}' = A(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{x_0} \tag{20}$$

The formula represents the entire solution x(t).

$$\mathbf{X}(t) = \Phi(t)\Phi^{-1}(t_0)X_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)f(s)ds \tag{21}$$

where $\Phi(t)$ is the fundamental matrix. This is also called variation of constants formula.

Let

$$\lambda_1 = -\kappa_1 \tag{22}$$

$$\lambda_2 = -\kappa_2 \tag{23}$$

Finding the corresponding Eigenvectors by substituting (22) into (13) yields

$$\begin{pmatrix} -(\rho+\theta-\tau)-(\lambda_1) & \rho \\ -(\nu+\ \delta-2\tau) & -(\mu+\tau)-(\lambda_1) \end{pmatrix}\begin{pmatrix} C \\ N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

$$(\rho+\theta-\tau-\kappa_1)C = \rho N \tag{25}$$

Taking $$C = \rho \tag{26}$$

$$N = (\rho+\theta-\tau-\kappa_1) \tag{27}$$

$$\Phi_1(t) = e^{-\hat{e}t}\begin{pmatrix} \rho \\ \rho+\theta-\tau-\kappa_1 \end{pmatrix} \tag{28}$$

Similarly

$$\Phi_2(t) = e^{-\hat{e}t}\begin{pmatrix} \rho \\ \rho+\theta-\tau-\kappa_2 \end{pmatrix} \tag{29}$$

$$\Phi(t) = \begin{pmatrix} \rho e^{-\kappa_1 t} & \rho e^{-\kappa_2 t} \\ (\rho+\theta-\tau-\kappa_1)e^{-\kappa_1 t} & (\rho+\theta-\tau-\kappa_2)e^{-\kappa_2 t} \end{pmatrix} \tag{30}$$

$$\left|\Phi(t)\right| = \rho(\kappa_1-\kappa_2)e^{-(\kappa_1+\kappa_2)t} \tag{31}$$

$$\Phi(t)^{-1} = \begin{pmatrix} \frac{(\rho+\theta-\tau-\kappa_2)e^{\kappa_1 t}}{\rho(\kappa_1-\kappa_2)} & \frac{-e^{\kappa_1 t}}{(\kappa_1-\kappa_2)} \\ \frac{-(\rho+\theta-\tau-\kappa_1)e^{\kappa_2 t}}{\rho(\kappa_1-\kappa_2)} & \frac{e^{\kappa_2 t}}{(\kappa_1-\kappa_2)} \end{pmatrix} \tag{32}$$

$$\Phi(t_0)^{-1} = \Phi(0)^{-1} = \begin{pmatrix} \frac{(\rho+\theta-\tau-\kappa_2)}{\rho(\kappa_1-\kappa_2)} & \frac{-1}{(\kappa_1-\kappa_2)} \\ \frac{-(\rho+\theta-\tau-\kappa_1)}{\rho(\kappa_1-\kappa_2)} & \frac{1}{(\kappa_1-\kappa_2)} \end{pmatrix} \tag{33}$$

$$\Phi(t_0)^{-1}X_0 = \begin{pmatrix} \frac{(\rho+\theta-\tau-\kappa_2)}{\rho(\kappa_1-\kappa_2)}C_0 & -\frac{1}{(\kappa_1-\kappa_2)}N_0 \\ \frac{-(\rho+\theta-\tau-\kappa_1)}{\rho(\kappa_1-\kappa_2)}C_0 + & \frac{1}{(\kappa_1-\kappa_2)}N_0 \end{pmatrix} \tag{34}$$

$$\Phi(t)\Phi^{-1}(t_0)X_0 =$$

$$\begin{pmatrix} \rho e^{-\kappa_1 t}\left[\frac{(\rho+\theta-\tau-\kappa_2)}{\rho(\kappa_1-\kappa_2)}C_0 - \left(\frac{1}{(\kappa_1-\kappa_2)}N_0\right)\right] + \rho e^{-\kappa_2 t}\left[\frac{-(\rho+\theta-\tau-\kappa_1)}{\rho(\kappa_1-\kappa_2)}C_0 + \left(\frac{1}{(\kappa_1-\kappa_2)}N_0\right)\right] \\ (\rho+\theta-\tau-\kappa_1)e^{-\kappa_1 t}\left[\frac{(\rho+\theta-\tau-\kappa_2)}{\rho(\kappa_1-\kappa_2)}C_0 - \left(\frac{1}{(\kappa_1-\kappa_2)}N_0\right)\right] + (\rho+\theta-\tau-\kappa_2)e^{-\kappa_2 t}\left[\frac{-(\rho+\theta-\tau-\kappa_1)}{\rho(\kappa_1-\kappa_2)}C_0 + \left(\frac{1}{(\kappa_1-\kappa_2)}N_0\right)\right] \end{pmatrix} \tag{35}$$

Equation (35) is the complementary solution of the Model.

$$\Phi^{-1}(s)f(s) = \frac{\mathrm{I}}{\rho(\kappa_1-\kappa_2)}\begin{pmatrix} e^{\kappa_1 s}(-\rho+\theta-\tau-\kappa_2) \\ e^{\kappa_2 s}(\rho-\theta+\tau+\kappa_1) \end{pmatrix} \tag{36}$$

$$\int_{t_0}^{t}\Phi^{-1}(s)f(s)ds = \begin{bmatrix} \frac{(-\rho+\theta-\tau-\kappa_2)e^{\kappa_1 t}\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_1} - \frac{(-\rho+\theta-\tau-\kappa_2)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_1} \\ \frac{(\rho-\theta+\tau+\kappa_1)e^{\kappa_2 t}\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_2} - \frac{(\rho-\theta+\tau+\kappa_1)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_2} \end{bmatrix} \tag{37}$$

$$\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)f(s)ds =$$

$$\begin{pmatrix} \frac{-(\rho-\theta+\tau+\kappa_2)\mathrm{I}}{(\kappa_1-\kappa_2)\kappa_1} + \frac{e^{-\kappa_1 t}(\rho-\theta+\tau+\kappa_2)\mathrm{I}}{(\kappa_1-\kappa_2)\kappa_1} + \frac{(\rho-\theta+\tau+\kappa_1)\mathrm{I}}{(\kappa_1-\kappa_2)\kappa_2} - \frac{e^{-\kappa_2 t}(\rho-\theta+\tau+\kappa_1)\mathrm{I}}{(\kappa_1-\kappa_2)\kappa_2} \\ \frac{-(\rho+\theta-\tau-\kappa_1)(\rho-\theta+\tau+\kappa_2)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_1} + \frac{(\rho+\theta-\tau-\kappa_1)(\rho-\theta+\tau+\kappa_2)\mathrm{I}e^{-\kappa_1 t}}{\rho(\kappa_1-\kappa_2)\kappa_1} + \frac{(\rho+\theta-\tau-\kappa_2)(\rho-\theta+\tau+\kappa_1)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_2} - \frac{(\rho+\theta-\tau-\kappa_2)(\rho-\theta+\tau+\kappa_1)\mathrm{I}e^{-\kappa_2 t}}{\rho(\kappa_1-\kappa_2)\kappa_2} \end{pmatrix} \tag{38}$$

Equation (38) is the particular solution (integral) of the new model. Therefore the complete solution is

$$\begin{aligned} C(t) = {} & \left[\frac{(\rho+\theta-\tau-\kappa_2)C_0}{\kappa_1-\kappa_2} - \frac{\rho N_0}{\kappa_1-\kappa_2} - \frac{(\rho-\theta+\tau+\kappa_2)\mathrm{I}}{\kappa_1(\kappa_1-\kappa_2)}\right]e^{-\kappa_1 t} \\ & - \left[\frac{(\rho+\theta-\tau-\kappa_1)}{\kappa_1-\kappa_2}C_0 - \frac{\rho N_0}{\kappa_1-\kappa_2} + \frac{(\rho-\theta+\tau+\kappa_1)\mathrm{I}}{\kappa_2(\kappa_1-\kappa_2)}\right]e^{-\kappa_2 t} + \left[\frac{(\rho-\theta+\tau+\kappa_1)}{\kappa_2(\kappa_1-\kappa_2)} - \frac{(\rho-\theta+\tau+\kappa_2)}{\kappa_1(\kappa_1-\kappa_2)}\right]\mathrm{I} \end{aligned} \tag{39}$$

$$\begin{aligned} N(\mathrm{t}) = {} & (\rho+\theta-\tau-\kappa_1)e^{-\kappa_1 t}\left[\frac{(\rho+\theta-\tau-\kappa_2)}{\rho(\kappa_1-\kappa_2)}C_0 - \left(\frac{1}{(\kappa_1-\kappa_2)}N_0\right) + \frac{(\rho-\theta+\tau+\kappa_2)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_1}\right] \\ & + \frac{(\rho+\theta-\tau-\kappa_2)(\rho-\theta+\tau+\kappa_1)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_2} + (\rho+\theta-\tau-\kappa_2)e^{-\kappa_2 t} \\ & \left[\frac{-(\rho+\theta-\tau-\kappa_1)}{\rho(\kappa_1-\kappa_2)}C_0 + \left(\frac{1}{(\kappa_1-\kappa_2)}N_0\right) - \frac{(\rho-\theta+\tau+\kappa_1)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_2}\right]\frac{-(\rho+\theta-\tau-\kappa_1)(\rho-\theta+\tau+\kappa_2)\mathrm{I}}{\rho(\kappa_1-\kappa_2)\kappa_1} \end{aligned} \tag{40}$$

2.4. *Numerical Solution*

The model was solved numerically using both the Euler method and the traditional Runge-Kutta method of order four. The algorithms were coded in the mathematical computation program Maple 14, and the resulting outcomes are presented in the following tables.

Table 3
Numerical and Analytic solutions of the Model at indicated values of the rate of developing comorbidity of *Diabetes Mellitus* (τ)

Values of tau, τ	Solution C(t)	10	20	30	40	60	70	100
0.08	Analytic	$9.214586565\ 10^7$	$8.971183252\ 10^7$	$8.764173549\ 10^7$	$8.588688689\ 10^7$	$8.315973233\ 10^7$	$8.211912538\ 10^7$	$7.996865410\ 10^7$
	Euler	$9.212470467\ 10^7$	$8.967518754\ 10^7$	$8.759414601\ 10^7$	$8.583195746\ 10^7$	$8.309801224\ 10^7$	$8.205681937\ 10^7$	$7.991106711\ 10^7$
	RK4	$9.214586566\ 10^7$	$8.971183252\ 10^7$	$8.764173547\ 10^7$	$8.588688681\ 10^7$	$8.315973215\ 10^7$	$8.211912510\ 10^7$	$7.996865365\ 10^7$
0.16	Analytic	$9.28717248\ 10^7$	$9.11007582\ 10^7$	$8.963844405\ 10^7$	$8.84427082\ 10^7$	$8.67104826\ 10^7$	$8.61153856\ 10^7$	$8.51404283\ 10^7$
	Euler	$9.285372489\ 10^7$	$9.106967753\ 10^7$	$8.959819461\ 10^7$	$8.839639090\ 10^7$	$8.665874699\ 10^7$	$8.606331583\ 10^7$	$8.509274657\ 10^7$
	RK4	$9.287172485\ 10^7$	$9.110075820\ 10^7$	$8.963844051\ 10^7$	$8.844270819\ 10^7$	$8.671048257\ 10^7$	$8.611538568\ 10^7$	$8.514042842\ 10^7$
0.32	Analytic	$9.43410459\ 10^7$	$9.39466406\ 10^7$	$9.37799127\ 10^7$	$9.380923941\ 10^7$	$9.43514324\ 10^7$	$9.48210897\ 10^7$	$9.68251724\ 10^7$
	Euler	$9.432771787\ 10^7$	$9.392371333\ 10^7$	$9.375031651\ 10^7$	$9.377525751\ 10^7$	$9.431354269\ 10^7$	$9.478292301\ 10^7$	$9.678980583\ 10^7$
	RK4	$9.434104597\ 10^7$	$9.394664041\ 10^7$	$9.377991247\ 10^7$	$9.380923925\ 10^7$	$9.435143208\ 10^7$	$9.482108943\ 10^7$	$9.682517212\ 10^7$

Table 4

Numerical and Analytic solutions of the Model at indicated values of the rate of developing comorbidity of *Diabetes Mellitus* τ)

Values of tau, τ	Solution N(t)	10	20	30	40	60	70	100
0.08	Analytic	1.126380026 10^8	1.132539036 10^8	1.138502590 10^8	1.144292870 10^8	1.155428080 10^8	1.160804236 10^8	1.176313914 10^8
	Euler	1.126391148 10^8	1.132558697 10^8	1.138528719 10^8	1.144323820 10^8	1.155465054 10^8	1.160842926 10^8	1.176354717 10^8
	RK4	1.126380023 10^8	1.132539032 10^8	1.138502584 10^8	1.144292860 10^8	1.155428069 10^8	1.160804222 10^8	1.176313902 10^8
0.16	Analytic	1.132386337 10^8	1.144127702 10^8	1.155308055 10^8	1.16599989 10^8	1.186161168 10^8	1.195731603 10^8	1.222875220 10^8
	Euler	1.132418819 10^8	1.144184178 10^8	1.155381766 10^8	1.166085594 10^8	1.186258940 10^8	1.195831351 10^8	1.222971549 10^8
	RK4	1.132386336 10^8	1.144127700 10^8	1.155308054 10^8	1.165999990 10^8	1.186161170 10^8	1.195731607 10^8	1.222875227 10^8
0.32	Analytic	1.144607559 10^8	1.168115056 10^8	1.190689621 10^8	1.212474669 10^8	1.254150982 10^8	1.274238473 10^8	1.332403909 10^8
	Euler	1.144663018 10^8	1.168209072 10^8	1.190808904 10^8	1.212608850 10^8	1.254292835 10^8	1.274376519 10^8	1.332513842 10^8
	RK4	1.144607556 10^8	1.168115049 10^8	1.190689615 10^8	1.212474663 10^8	1.254150972 10^8	1.274238465 10^8	1.332403905 10^8

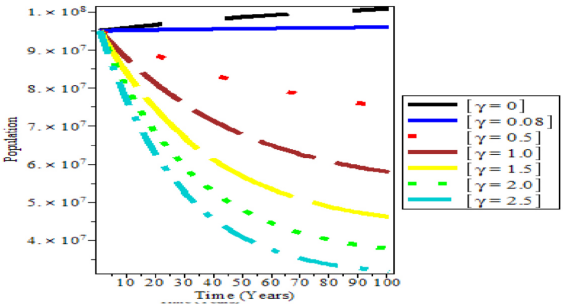

Figure 2

The number of diabetics with complications, $C(t)$, when the rate at which complications are controlled is at various values in the old model of [20].

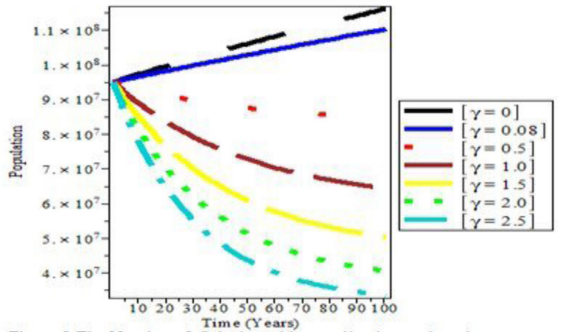

Figure 3

Population of diabetics with complications, $C(t)$, at specified values of the rate at which complications are controlled is in the new model with co morbidity term.

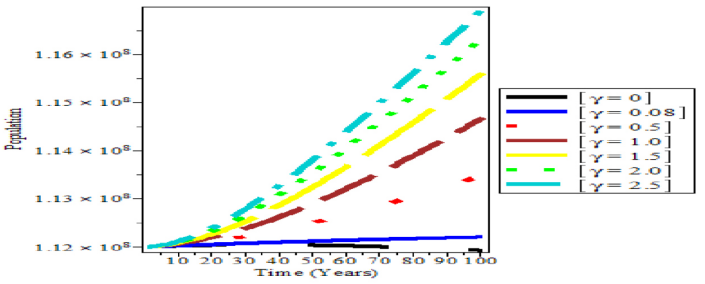

Figure 4

The number of diabetics with the without complications, $N(t)$, when the rate at which complications are controlled is at various in the old model of [20].

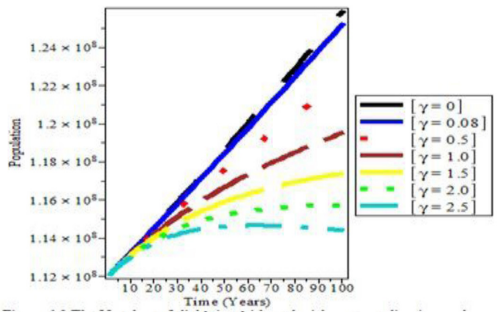

Figure 5

The number of diabetics with and without complications, $N(t)$, when the rate at which complications are controlled is at various values in the new model.

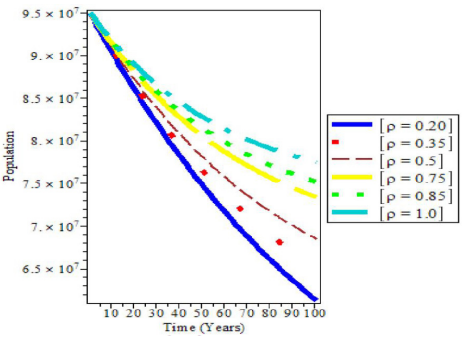

Figure 6

The number of diabetics with complications, $C(t)$, when the probability of developing a complications is at various values in the old model of [20].

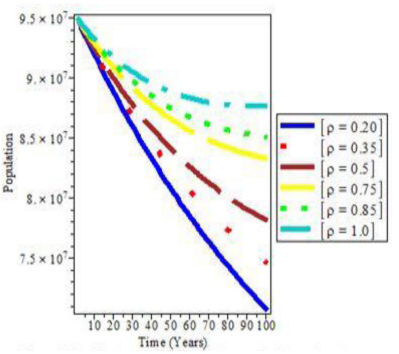

Figure 7

The number of diabetics with complications, $C(t)$, when the probability of developing a complication is at various values in the new model.

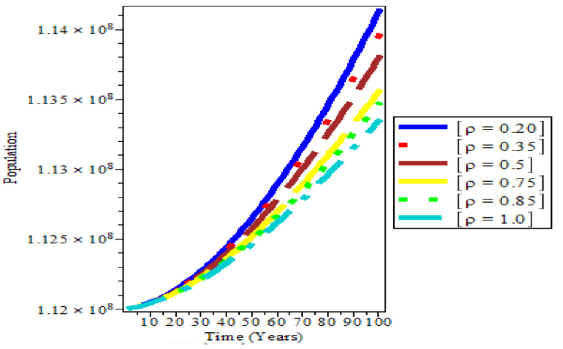

Figure 8

The number of diabetics with and without complications, $N(t)$, when the probability of developing complications is at various values in the old model of [20].

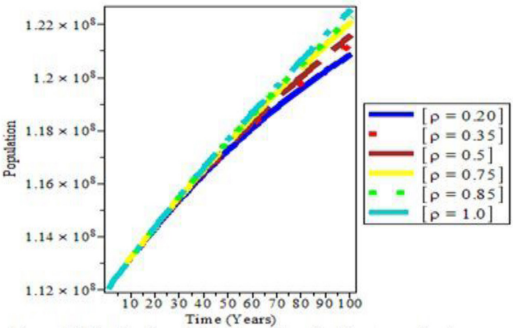

Figure 9

The number of diabetics with and without complications, $N(t)$, when the probability of developing complications is at various values in the new model.

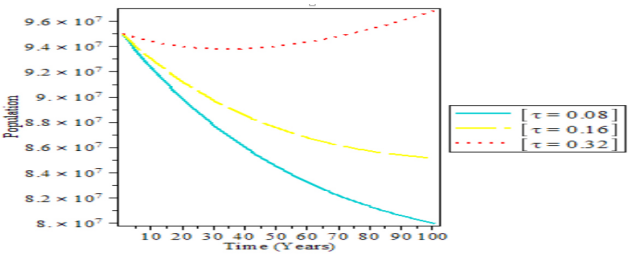

Figure 10

The number of diabetics with complications C(t), when the rate of developing comorbidity of *Diabetes Mellitus* is at various values in the Model.

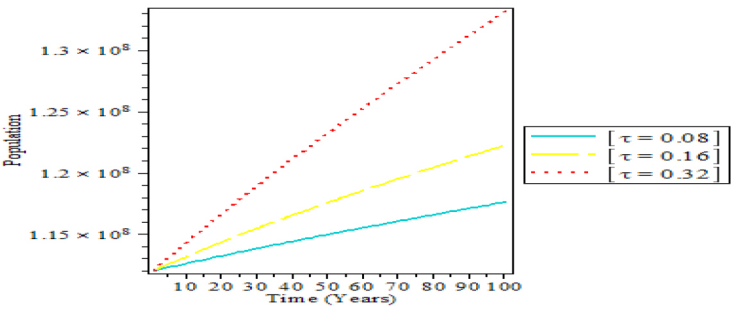

Figure 11

The number of diabetics with and without complications N(t), when the rate of developoing comorbidity of *Diabetes Mellitus* is at various values in the Model.

3. Discussions

The fourth-order Runge-Kutta method yields a better approximation of the analytical solution than the Euler method, which is to be expected. As displayed in Table 2, the model's equilibrium is asymptotically stable across various numerical values, indicating its suitability for the intended application. Tables 3 and 4 demonstrate the suitability of both the analytical and numerical solutions.

Figure 3 demonstrates that the number of diabetics with complications will decrease further as diabetes comorbidities are managed more effectively. This trend is similar to that seen in the previous model (without comorbidity) as shown in Figure 2. It is essential to diagnose and treat coexisting conditions such as hypertension and diabetes to decrease the number of diabetics with complications. Figure 4 is presented here for comparison purposes with Figure 5, although it is nearly identical to Figure 2 from [20]. Figure 5 reveals that as the rate at which complications of diabetes are controlled increases, the number of diabetics with and without complications reduces even in the presence of comorbidities. It further shows that without sufficient diabetes control, the population of diabetics shall increase geometrically. Hence a sufficient control of diabetes which can be through regular exercise, drug usage and lifestyle changes are essential. It can be deduced that the new model which incorporates comorbidity of diabetes gives more information and representation of real life situation, than the old model of Figure 4. This underscores the importance of mathematical modeling as a means to observe the cause-effect of an intervention. Timely awareness, knowledge and treatment of

diabetes and its comorbidities are important in diabetes care and management.

Figure 6, originally from [20], is presented here as Figure 3 for fair comparison. In Figure 7, it can be observed that the number of diabetics with complications increases as the risk of developing a complication from diabetes increases. To prevent the world from the impending diabetes pandemic, particularly in developing nations, which has been predicted by the International Diabetes Federation, it is essential to promote health education and conduct regular comprehensive medical check-ups that include vital signs and diabetes risk factors.

Figure 9 demonstrates that as the probability of acquiring a complication of diabetes increases, the proportion of diabetics with and without complications also increases. This pattern is more representative of the real-world scenario compared to the previous model in Figure 8. Therefore, any effort to combat this devastating epidemic should prioritize interventions that aim to delay the onset of diabetes complications.

The data in Figure 10 indicate that the number of diabetics with complications rises with an increase in the rate of comorbidities associated with Diabetes Mellitus. Therefore, it is crucial to prioritize the identification and treatment of comorbidities in diabetic patients in any intervention or care plan. Maintaining a low rate of diabetes comorbidity can be achieved through an effective blood pressure and blood sugar control program.

On the other hand, Figure 11 illustrates that as the rate of diabetes comorbidities increases, there is an increasing number of diabetics who remain unaffected. Hence, preventing and managing diabetes and its associated comorbidities should be a major focus of intervention efforts.

4. Conclusion

In this paper, an existing mathematical model on diabetics' population with and without complications was modified by adding a term on developing comorbidities of *Diabetes Mellitus*. The model was analyzed, solved analytically and numerically. The results of the modified and existing models were compared. The new model reveals that a comprehensive diagnosis and treatment of other diseases that a diabetic manifests is necessary and essential to his/her well-being. This shows that effective management and treatment of all the conditions that diabetics experience can lead to an increase in the population of diabetics without complications and a significant decrease in the number of diabetics with complications. Therefore priorities should be given to careful inquiry and

investigation in order to identify and treat other diseases condition a diabetic is encumbered with for effective management of *Diabetic mellitus*. Timely awareness, regular diagnostic tests and monitoring are essential for the control of Diabetes, its complications and comorbidities.

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