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On characterization of a continuous distribution through lower record values

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Abstract

The characterization results via conditional expectation of lower record and truncation moments are considered for a new family of probability distribution. Several probability distributions can be characterized by the proposed method.

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1. Introduction

A sequence of observations that are smaller or larger than the previously recorded observations is known as records. Record values have distinct elements in the recorded studies. Every change in the study for Y indicates that a record is noted. The records are found naturally in several areas of study e.g., sports, medical sciences, engineering, and traffic among others.

Let Y_1, Y_2, Y_3 .... be a sequence of independent and identically distributed (iid) random variables $(r.v.)$. having cumulative density function (CDF) $F(y)$ and probability density function (PDF) $f(y)$. The joint PDF $f(y_1, y_2, \dots, y_r)$ of the r lower record values is

$$f(y_1, y_2, \dots, y_r) = \prod_{i=1}^{r-1} f(y_r)\frac{f(y_i)}{F(y_i)}, -\infty < y_1 < y_2 < \cdots < y_r < \infty, \tag{1}$$

while joint PDF of $Y_{L(r)}$ and $Y_{L(s)}$ is

$$f_{L(r,s)}(y, z) = \frac{1}{\Gamma(r)\Gamma(s-r)}[-\ln(F(y))]^{r-1}[-\ln(F(z)) + \ln(F(y))]^{s-r-1}$$

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$$\times \frac{f(y)}{F(y)} f(z), -\infty < y_r < y_s < \infty. \tag{2}$$

Then, the conditional PDF of $(Y_{L(s)}|Y_{L(r)} = y)$ is defined as

$$f(Y_{L(s)}|Y_{L(r)} = y) = \frac{1}{\Gamma(s-r)}[-\ln(F(z)) + \ln(F(y))]^{s-r-1}\frac{f(z)}{F(y)}, \text{ for } 1 \leq r < s \tag{3}$$

The statistical inference and applications of record values were pioneered by Chandler [1]. Later, profound literature, theory, and methodology on record values have been grown over various realms, see Ahsanullah [2], Arnold et al. [3], and Nevzorov [4] for instance.

In literature, a lot of work has been concerned with upper record values from a continuous distribution. The lower record values possess their own significance, in specific problems of investigation e.g., in the stock market, freestyle swimming, and marathon walk (see. Paul [5]).

The characterization of continuous distributions has been proposed by many authors. Some of them, Malinowska and Szynal [6], Shawky and Bakoban [7]. Faizan and Khan [8], Khan and Faizan [9], Faizan et al. [10], Kotb and Ahsanullah [11] and Khan [12] are considered in detail. The goal of the current study is to find characterizations of the proposed distribution using lower record values.

Let Y be a continuous $r.v.$ with $CDFF(y)$ and $PDFf(y)$ on the support (u, v), where u and v may be finite or infinite, are given respectively,

$$F(y) = [k(y)]^{\delta}, \delta > 0 \tag{4}$$

$$f(y) = \delta[k(y)]^{\delta-1}k'(y), \delta > 0 \tag{5}$$

where $k(y)$ is an increasing and differential function with $k(y) \to 0$ as $y \to u^+$ and $k(y) \to 1$ as $y \to v^-$ for all $y \in (u, v)$.

Noting that (4) and (5) hold the following relation.

$$f(y)k(y) = \delta k'(y)F(y). \tag{6}$$

This family includes the most noted lifetime distributions such as the power, inverse power, reflected exponential, inverse Weibull, inverse exponential, Gumbel, Burr Type II to Burr Type XI distributions which are given in Table 1.

2. Characterization

Theorem 2.1: *Let Y be a continuous $r.v.$ with $CDFF(y)$ and $PDFf(y)$ and $y \in (u, v)$. Then for $1 \leq r < s$*

$$E\left[k(Y_{L(s)}|Y_{L(a)} = y)\right] = m_{s|a}k(y), a = r, r+1 \tag{7}$$

if and only if

$$F(y) = [k(y)]^{\delta}, \delta > 0 \tag{8}$$

where $m_{s|r} = \left(\frac{\delta}{\delta+1}\right)^{s-r}$

Proof: From (3), we have

$$= \frac{1}{\Gamma(s-r)} \int_u^y k(z)[-\ln(F(z)) + \ln(F(y))]^{s-r-1} \frac{f(z)}{F(y)} dz. \tag{9}$$

Using (4) and (5) in (9), we have

$$= \frac{1}{\Gamma(s-r)} \int_u^y k(z) \left[\delta \, ln \left(\frac{k(y)}{k(z)}\right)\right]^{s-r-1} \delta[k(z)]^{\delta-1} \frac{k'(z)}{[k(y)]^{\delta}} dz. \tag{10}$$

Let

$$\omega = \; ln \left[\left(\frac{k(y)}{k(z)}\right)\right]^{\delta}.$$

Then (10) reduces to

$$= \frac{1}{\Gamma(s-r)} \int_0^{\infty} [k(y) e^{-\omega/\delta}][\omega]^{s-r-1} e^{-w} d\omega. \tag{11}$$

On simplification (11), it confirms (7).

For the sufficiency part, we deal with,

$$\frac{1}{\Gamma(s-r)} \int_u^y k(z)[-\ln(F(z)) + \ln(F(y))]^{s-r-1} f(z) dz = k_{s|r}(y) F(y). \tag{12}$$

In view of Faizan and Khan [8], (12) yields

$$F(y) = [k(y)]^{\delta}$$

and hence (8) verifies.

Theorem 2.2: *For* $1 \leq r \leq s < t$ *and same conditions hold in previous theorem*

$$E\left[k(Y_{L(t)}|Y_{L(r)}\right] = m_{t|s} E\left[k(Y_{L(s)}|Y_{L(r)}\right] \tag{13}$$

if and only if

$$F(y) = [k(y)]^{\delta}, \delta > 0. \tag{14}$$

Proof: In consideration of Theorem 2.1,

$$E\left[k(Y_{L(t)}|Y_{L(r)}\right] = m_{t|r} k(y) \text{ and } E\left[k(Y_{L(s)}|Y_{L(r)}\right] = m_{s|r} k(y).$$

Now,

$$m_{t|r} = \left(\frac{\delta}{\delta+1}\right)^{t-r} = \left(\frac{\delta}{\delta+1}\right)^{t-s} \left(\frac{\delta}{\delta+1}\right)^{s-r} = m_{t|s} m_{s|r}$$

Therefore,

$$\begin{aligned} &= m_{t|r} k(y) \\ &= m_{t|s} m_{s|r} k(y) \\ &= m_{t|s}[m_{s|r} k(y)] \\ &= m_{t|s} E\left[k(Y_{L(s)}|Y_{L(r)}\right] \end{aligned}$$

It follows (13).

For sufficiency part

$$\frac{1}{\Gamma(t-r)} \int_u^y k(z)[-\ln(F(z)) + \ln(F(y))]^{t-r-1} f(z) dz$$
$$= m_{t|s} \frac{1}{\Gamma(s-r)} \int_u^y k(z)[-\ln(F(z)) + \ln(F(y))]^{s-r-1} f(z) dz. \tag{15}$$

Performing differentiation of (15) $w.r.t.y$. It provides the sufficiency part
Note: Equation (13) reduces to (7) at $r = s$.
Characterization result based on truncated moment is presented below.

Theorem 2.3: *As stated in Theorem 2.1 and further $f'(y)$ and $E(Y|Y \leq y)$ exist, then*

$$Expectation[k(Y|Y \leq y)] = \Psi(y)\Phi(y), y \geq 0 \tag{16}$$

where

$$\Psi(y) = \frac{1}{\delta+1}\left[\frac{[k(y)]^2}{k'(y)}\right] \text{ and } \Phi(y) = \frac{f(y)}{F(y)}$$

if and only if

$$f(y) = \delta[k(y)]^{\delta-1}k'(y), \ \ \delta > 0\,. \tag{17}$$

Proof: If Y follows (5), then we have

$$= \frac{1}{F(y)}\int_u^y k(u)f(u)du$$

$$= \frac{1}{F(y)}\int_u^y k(u)\delta[k(u)]^{\delta-1}k'(u)du. \tag{18}$$

Integrating (18) right hand side by taking $t = [k(u)]^\delta$

$$= \frac{1}{F(y)}\left\{\frac{\delta}{\delta+1}[k(y)]^{\delta+1}\right\}. \tag{19}$$

Multiply and divide by $f(y)$ in (19), we get

$$= \left[\frac{1}{\delta+1}\frac{[k(y)]^{\delta+1}}{k'(y)}\right]\frac{f(y)}{F(y)}$$

$$= \Psi(y)\Phi(y)$$

Hence, (16) is determined.
To confirm the (17). We have the following,

$$\frac{f'(y)}{f(y)} = \frac{\Psi(y)-k'(y)}{\Psi(y)} = (\delta-1)\left[\frac{k'(y)}{k(y)}\right] \text{ Ahsanullah et al. [13]} \tag{20}$$

where,

$$k'(y) = \Psi(y)\left[1-(\delta-1)\frac{k'(y)}{k(y)}\right].$$

Performing integration in (20) both sides, we obtain,

$$f(y) = g\frac{[k(y)]^\delta}{k(y)},$$

where

$$g = \delta k'(y), \text{ as } \int_{-\infty}^{\infty}(PDF)dy = 1$$

This completes the proof.

Table 1
Example based on $CDF\ F(y) = [k(y)]^{\delta}, \delta > 0.$

Distribution	$k(y)$	$F(y)$
Power	y	$y^{\delta}, y > 0$
Inverse power	$\frac{y-u}{y-v}$	$\left(\frac{y-u}{y-v}\right)^{\delta}, u < y < v$
Reflected exponential	$e^{(y-v)}$	$e^{\delta(y-v)}, -\infty < y < v$
Inverse Weibull	$e^{-y^{\theta}}$	$e^{-\delta y^{\theta}}, y > 0$
Inverse exponential	$e^{-\frac{1}{y}}$	$e^{-\frac{\delta}{y}}, y > 0$
Gumbel	$e^{-e^{-y}}$	$e^{-e^{-uy}}, y > 0$
Burr Type II	$(1+e^{-y})^{-1}$	$(1+e^{-y})^{-\delta}, -\infty < y < \infty$
Burr Type III	$(1+y^{-p})^{-1}$	$(1+y^{-p})^{-\delta}, -\infty < y < \infty$
Burr Type IV	$\left[1+\left(\frac{v-y}{y}\right)^{\frac{1}{v}}\right]^{-1}$	$\left[1+\left(\frac{c-y}{y}\right)^{\frac{1}{c}}\right]^{-\delta}, 0 < y < v$
Burr Type V	$(1+ce^{-tany})^{-1}$	$(1+ce^{-tany})^{-\delta}, -\frac{\pi}{2} < y < \frac{\pi}{2}$
Burr Type VI	$\left(1+ce^{-\delta sinhy}\right)^{-1}$	$\left(1+ce^{-\delta sinhy}\right)^{-\delta}, -\infty < y < \infty$
Burr Type VII	$\left(\frac{1+tanhy}{2}\right)$	$\left(\frac{1+tanhy}{2}\right)^{\delta}, -\infty < y < \infty$
Burr Type VIII	$\left(\frac{2}{\pi}tan^{-1}e^{y}\right)$	$\left(\frac{2}{\pi}tan^{-1}e^{y}\right)^{\delta}, -\infty < y < \infty$
Burr Type X	$\left(1-e^{-\theta y^2}\right)$	$\left(1-e^{-\theta y^2}\right)^{\lambda}, y > 0$
Burr Type XI	$\left(y-\frac{1}{2\pi}sin2\pi y\right)$	$\left(y-\frac{1}{2\pi}sin2\pi y\right)^{\delta}, 0 < y < 1$

Conclusion

Characterization is considered for the new family of distribution conditioned on lower record. The results provided in this article contribute significantly to finding the relationship between the moments.

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