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Estimating percentiles of time-to-failure distribution obtained from a Weibull accelerated degradation model

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Abstract

We propose a nonparametric kernel estimation method to estimate the percentiles of the time-to-failure distribution under the usual use condition obtained from a Weibull accelerated degradation model. We discuss some of the well-known parametric methods that used to estimate the time-to-failure distribution and its percentile under the usual use condition including ordinary least square method and maximum likelihood method. The different exiting methods were compared with the kernel method through simulation by using the mean square error and the bootstrap confidence interval length. In general, when the distributional assumption is available, the maximum likelihood estimator performs better than the oher two estimators, while the kernel estimator performs better than the other two estimators when the distributional assumption is not available. Application to real data set was disscuced.

Subject Classification: *(2010) 62N05, 62G05.*

Keywords: *Accelerated degradation model, Kernel method, Maximum likelihood, Ordinary least squares, Reliability, Weibull distribution.*

1. Introduction

In accelerated degradation test, the units are measured at a higher a stress. The test is conducted under different levels of stress to estimate the time to failure distribution and its percentiles under the usual use condition. [1]

Alodat and Al-Haj Ebrahem [2] used the rank sampling to estimate the parameters of a linear degradation model. Al-Haj Ebrahem et. al [3] used the

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Bayesian method to estimate the percentile of a simple linear degradation model. Al-Haj Ebrahem et. al. [4] used the nonparametric classical kernel method to estimate the time-to-failure distribution and its percentiles for a linear degradation model. Ba Dakhn et. al. [5] used a semi-parametric method. Eidous et. al. [6] used the double kernel method. Al-Momani et. al. [7] used variable scale kernel method.For more details see I. U. Siloko [8].

We need to estimate the percentiles of the time-to-failure distribution under the usual use condition obtained from a Weibull accelerated degradation model using kernel method. We compare the kernel method with OLS and MLE methods using simulation.

In Section 2, the model and the time-to-failure distribution are presented. In Section 3, the we introduce the kernel method to estimate the percentiles of the time-to-failure distribution uder the usual use condition. In Sections 4 and 5, we present the estimators of the percentiles using OLS and MLE methods respectively. Simulation results are given in Section 6. In Section 7 we present an application to a real data set . Conclusions are given in Section 8.

2. Accelerated Degradation Model Using a Weibull Distribution

The linear accelerated degradation model is given by:

$$y_{ijk} = \beta_{ij} s_i t_{ijk} + \epsilon_{ijk} \ , i = 1, \ldots, r, j = 1, \ldots, n \text{ and } k = 1, \ldots, m$$

where y_{ijk} is the degradation of unit j at time t_{ijk} under stress level i, β_{ij} is the random effect parameter, s_i is the i^{th}stress value, r is the number stress levels, n is the number of units, m is the number of inspections for unit j under stress level i and t_{ijk} is the k^{th} time ϵ_{ijk}'s are error terms , ϵ_{ijk}'s and β_{ij} are assumed to be independent. In the accelerated degradation model, the failure is assumed to occur at time T when the degradation path reaches D_f, so we can write $D_f = \beta s_i T$. Thus, for the stress level , the distribution function of the time-to-failure is given by:

$$F_T\,(t) = P\,(T \le t) = P\,\left(\frac{D_f}{\beta s_i} \le t\right) = P\,\left(\frac{\beta s_i}{D_f} > \frac{1}{t}\right) = 1 - P\,\left(\beta \le \frac{D_f}{t s_i}\right)$$

$$= 1 - G_{\beta_i}\left[\left(\frac{D_f}{t\, s_i}\right)\right]$$

We need to caculate the $100p^{\text{th}}$ percentile of the time-to-failure distribution under the usual use condition, t_{p0}, by solving $F_T(t_{p0}) = p$ for t_{p0}.

The Weibull Distribution

The probability density function (pdf) a Weibull distribution is given by:

$$g(x) = \begin{cases} 0, x < 0 \\ \dfrac{\gamma}{\alpha}\left(\dfrac{x}{\alpha}\right)^{\gamma-1} e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}, x \ge 0 \end{cases}$$

The cumulative distribution function (CDF) is given by:

$$G(x) = \begin{cases} 0, x < 0 \\ 1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}, x \ge 0 \end{cases}$$

Assuming that the random effect parameter β has a Weibull distribution, the distribution function of the time-to-failure is given by:

$$F(t) = P\,(T \leq t) = \begin{cases} 0, t < 0 \\ exp\left[-\left(\frac{D_f}{t\alpha s_i}\right)^{\gamma}\right], t \geq 0 \end{cases}$$

The probability distribution function of T is given by:

$$f(t) = \begin{cases} 0, t < 0 \\ \frac{\gamma}{t}\left(\frac{D_f}{t\alpha s_i}\right)^{\gamma} e^{-\left(\frac{D_f}{t\alpha s_i}\right)^{\gamma}}, t \geq 0 \end{cases}$$

To find the $100\,p^{th}$percentile of the time-to-failure distribution t_{pi}at the $i-th$ stress level we need to solve $F_T\,(t_{pi}) = p$ with respect to t_{pi}. Hence,

$$t_{pi} = \frac{D_f}{\alpha s_i(-\ln p)^{\frac{1}{\gamma}}}.$$

3. Estimating Percentiles Under Usual Use Condition Using the Kernel Method

For the stress level i, the classical kernel density estimator of $g_{\beta i}(b)$ is given by:

$$\hat{g}_{\beta_i}(b,h) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{b-\beta_{ij}}{h}\right)$$

we choose *K(u)* to be the Gaussian density function, so for stress level i, $\hat{g}_{\beta i}(b,h)$ can be written as

$$\hat{g}_{\beta_i}(b,h) = \frac{1}{nh}\sum_{j=1}^{n}\frac{1}{\sqrt{2\pi}h} e^{-\frac{1}{2h^2}(b-\beta_{ij})^2}$$

It follows that, for stress level i, the proposed kernel estimator is,

$$\hat{F}_T(t) = 1 - \hat{G}_{\beta i}\left(\frac{D_f}{ts_i}\right) = 1 - \frac{1}{n}\sum_{j=1}^{n}\int_{-\infty}^{\frac{D_f}{ts_i}}\frac{1}{\sqrt{2\pi}h} e^{-\frac{1}{2h^2}(b-\beta_{ij})^2}\,db$$

$$= 1 - \frac{1}{n}\sum_{j=1}^{n}\Phi\left(\frac{D_f - \beta_{ij}ts_i}{ts_i h}\right),$$

To obtain the kernel density estimate value of the $100p^{\text{th}}$ percentile at each stress level i, $\hat{t}_{pi\,KE}$ we need to solve numerically $\hat{F}_T(\hat{t}_{pi\,KE}) = p$ for $\hat{t}_{pi\,KE}$.

4. Estimating Percentiles Under Usual Use Condition Using the OLS Method

To obtain the OLS estimator of the $100\,p^{th}$percentile for each stress level i, $\hat{t}_{pi\,OLS}$ we assume that $\beta_{ij} \sim weibull(\gamma_i, \alpha_i)$.

Solving $F_T\,(t_{ip}) = p$, we get $t_{pi} = \frac{D_f}{s_i\alpha_i(-\ln p)^{\frac{1}{\gamma_i}}}$

For the i^{th}stress, the OLS estimator of t_{ip} is given by:

$$\hat{t}_{pi\,OLS} = \frac{D_f}{s_i\hat{\alpha}_i(-\ln p)^{\frac{1}{\hat{\gamma}_i}}}$$

where $\hat{\gamma}_i, \hat{\alpha}_i$ are the OLS of γ_i, α_i, respectively. To obtain $\hat{\gamma}_i$ and $\hat{\alpha}_i$ we need to minimize $Q(\gamma_i, \alpha_i)$, with respect to γ_i and α_i,where

$$Q(\gamma_i, \alpha_i) = \sum_{j=1}^{n}\sum_{k=1}^{m}\left(y_{ijk} - E(y_{ijk})\right)^2$$

Now, $E(y_{ijk}) = E(\beta_{ij}s_i t_{ijk}) = s_i t_{ijk}E(\beta_{ij}) = s_i t_{ijk}\alpha_i\Gamma\left(1+\frac{1}{\gamma_i}\right)$

Since $\beta_{ij} \sim weibull(\gamma_i, \alpha_i)$, $E(\beta_{ij}) = \alpha_i\Gamma\left(1+\frac{1}{\gamma_i}\right)$

Thus,

$$Q(\gamma_i, \alpha_i) = \sum_{j=1}^{n}\sum_{k=1}^{m}\left(y_{ijk} - \left(s_i t_{ijk}\alpha_i\Gamma\left(1+\frac{1}{\gamma_i}\right)\right)\right)^2$$

So, to find the values of $\hat{\alpha}_i$ and $\hat{\gamma}_i$, we need to solve numerically the following two equations for α_i and γ_i,

$$\frac{\partial Q(\gamma_i,\alpha_i)}{\partial \alpha_i} = 0 \text{ , and } \frac{\partial Q(\gamma_i,\alpha_i)}{\partial \gamma_i} = 0,$$

where,

$$\frac{\partial Q(\gamma_i,\alpha_i)}{\partial \alpha_i} = -2\sum_{j=1}^{n}\sum_{k=1}^{m}\left(\left(y_{ijk} - \left(s_i t_{ijk}\alpha_i\Gamma\left(1+\frac{1}{\gamma_i}\right)\right)\right)\left(s_i t_{ijk}\Gamma\left(1+\frac{1}{\gamma_i}\right)\right)\right) = 0,$$

We get,

$$\hat{\alpha}_i = \frac{\sum_{j=1}^{n}\sum_{k=1}^{m} y_{ijk}t_{ijk}}{s_i\Gamma\left(1+\frac{1}{\hat{\gamma}_i}\right)\sum_{j=1}^{n}\sum_{k=1}^{m}(t_{ijk}^2)},$$

$$\frac{\partial Q(\gamma_i,\alpha_i)}{\partial \gamma_i} =$$

$$-2\sum_{j=1}^{n}\sum_{k=1}^{m}\left(\left(y_{ijk} - \left(s_i t_{ijk}\alpha_i\Gamma\left(1+\frac{1}{\gamma_i}\right)\right)\right)\left(\alpha_i s_i t_{ijk}\frac{\partial}{\partial \gamma_i}\Gamma\left(1+\frac{1}{\gamma_i}\right)\right)\right) = 0$$

We get,

$$\sum_{j=1}^{n}\sum_{k=1}^{m} y_{ijk}t_{ijk} - \sum_{j=1}^{n}\sum_{k=1}^{m} s_i t_{ijk}^2\hat{\alpha}_i\Gamma\left(1+\frac{1}{\hat{\gamma}_i}\right) = 0$$

Now, to obtain the estimate value of the $100p^{\text{th}}$ percentile under the usual use condition, $\hat{t}_{p0\,OLS}$, we need to fit the linear regression model, $\hat{t}_{pi\,OLS} = \delta_0 + \delta_1 s_i + \varepsilon_i$, where s_i is the i^{th} stress value, i = 1, 2, …, *r*. Thus, $\hat{t}_{po\,OLS} = \hat{\delta}_0 + \hat{\delta}_1 s_0$, where s_0 is the stress value under the usual use condition.

5. Estimating Percentiles Under Usual Use Condition Using the MLE Method

We need to obtain the MLE of the $100p^{th}$ percentile at each stress level i, assuming that $\beta_{ij} \sim weibull(\gamma_i, \alpha_i)$. The PDF of the time failure is given by:

$$f(t^*, \gamma_i, \alpha_i) = \frac{\gamma_i}{t^*}\left(\frac{D_f}{t^*\alpha_i s_i}\right)^{\gamma_i} e^{-\left(\frac{D_f}{t^*\alpha_i s_i}\right)^{\gamma_i}}, t^* > 0.$$

To find the MLE of γ_i and α_i, let ${t^*}_1, {t^*}_2 \ldots, {t^*}_n$ be an observed random sample from $f(t^*, \gamma_i, \alpha_i)$, then the likelihood function of $\underline{t^*} = {t^*}_1, {t^*}_2, \ldots, {t^*}_n$ is given by:

$$L(\underline{t^*}, \gamma_i, \alpha_i) = \prod_{j=1}^{n} \frac{\gamma_i}{t_j^*} \left(\frac{D_f}{t_j^* \alpha_i s_i}\right)^{\gamma_i} e^{-\left(\frac{D_f}{t_j^* \alpha_i s_i}\right)^{\gamma_i}}$$

$$= (\gamma_i)^n \left(\frac{D_f}{\alpha_i s_i}\right)^{n\gamma_i} \prod_{j=1}^{n} \left(\frac{1}{t_j^*}\right)^{\gamma_i+1} e^{-\sum_{j=1}^{n}\left(\frac{D_f}{t_j^* \alpha_i s_i}\right)^{\gamma_i}}$$

And the natural logarithm of $L(\underline{t^*}, \gamma_i, \alpha_i)$ is:

$$\ln L(\underline{t^*}, \gamma_i, \alpha_i) = \mathrm{n} \ln \gamma_i + \ln \prod_{j=1}^{n} \left(\frac{1}{t_j^*}\right)^{\gamma_i+1} + \mathrm{n}\gamma_i \ln \frac{D_f}{\alpha_i s_i} + \ln e^{-\sum_{j=1}^{n}\left(\frac{D_f}{t_j^* \alpha_i s_i}\right)^{\gamma_i}}$$

$$= \mathrm{n} \ln \gamma_i + \ln \prod_{j=1}^{n} \left(\frac{1}{t_j^*}\right)^{\gamma_i+1} + \mathrm{n}\gamma_i \ln D_f - \mathrm{n}\gamma_i \ln \alpha_i s_i - \sum_{j=1}^{n} \left(\frac{D_f}{t_j^* \alpha_i s_i}\right)^{\gamma_i}$$

$\ln L(\underline{t^*}, \gamma_i, \alpha_i)$, can be maximized by solving the non-linear equations obtained by differentiating the last equation with respect to γ_i, α_i which are respectively given as follows.

$$\frac{\partial \ln L(\underline{t^*}, \gamma_i, \alpha_i)}{\partial \alpha_i} = 0,$$

$$\frac{\partial \ln L(\underline{t^*}, \gamma_i, \alpha_i)}{\partial \gamma_i} = 0.$$

The MLE of α_i and γ_i can be obtained by solving the above two equations simultaneously with respect to α_i and γ_i. Of course, we need to use numerical methods to compute the MLE of α_i and γ_i.

To obtain the estimate value of the $100p^{\text{th}}$ percentile under the usual use condition, $\hat{t}_{p0\,MLE}$, we need to fit the linear regression model, $\hat{t}_{pi\,MLE} = \delta_0 + \delta_1 s_i + \varepsilon_i$, where s_i is the i^{th} stress value, i = 1, 2, …, r. Thus, $\hat{t}_{p0\,MLE} = \hat{\delta}_0 + \hat{\delta}_1 s_0$, where s_0 is the stress value under the usual use condition.

6. Simulation Results

In order to compare the three estimates we compute the mean squared error (MSE) and the length of a 95% percentile bootstrap confidence interval for t_{p0}, (Efron and Tibshirani, [9]).

Simulation indices are:

1. The number of stresses $r = 3$.
2. For the i^{th}stress value $s_i, i = 1, 2, \ldots, r, s_1 = 10, s_2 = 20, s_3 = 30$.
3. The stress value under the usual use condition $s_0 = 5$.
4. The sample sizes $n = 20, 40\ , 60$.
5. For the $100p^{\text{th}}$ percentile $= 0.1, 0.3, 0.5$.
6. The critical degradation level $D_f = 20$.

Simulation results are presented in Tables (1-4).

We study and compare the three estimators for $t_{0.5}$ when the assumption of the random effects distribution is violated. Simulation results are presented in Table 5.

Table 1
MSE of the Estimates When n = 20

t_p	Kernel Estimator	MLE	OLS
	$\mathbf{MSE}_{KE}$	$\mathbf{MSE}_{MLE}$	$\mathbf{MSE}_{OLS}$
$t_{0.1}$	0.8642	0.8325	0.8413
$t_{0.3}$	0.9061	0.8611	0.8775
$t_{0.5}$	0.9610	0.9200	0.9451

Table 2
MSE of the Estimates When n=40

t_p	Kernel Estimator	MLE	OLS
	$\mathbf{MSE}_{KE}$	$\mathbf{MSE}_{MLE}$	$\mathbf{MSE}_{OLS}$
$t_{0.1}$	0.8405	0.8115	0.8201
$t_{0.3}$	0.8941	0.8514	0.8767
$t_{0.5}$	0.9427	0.9022	0.9201

Table 3
MSE of the Estimates When n=60

t_p	Kernel Estimator	MLE	OLS
	$\mathbf{MSE}_{KE}$	$\mathbf{MSE}_{MLE}$	$\mathbf{MSE}_{OLS}$
$t_{0.1}$	0.8377	0.7918	0.8007
$t_{0.3}$	0.8631	0.8356	0.8478
$t_{0.5}$	0.9407	0.8811	0.9124

Table 4
A 95% Bootstrap C.I. Using Different Estimation Methods

n	t_p	A 95% Bootstrap C.I.			$length\ of\ C.I.$		
		kernel	MLE	OLS	kernel	MLE	OLS
20	$t_{0.1}$ $t_{0.3}$ $t_{0.5}$	(0.1512, 0.1759) (0.1602, 0.1874) (0.1664, 0.1966)	(0.1513, 0.1765) (0.1604, 0.1889) (0.1639, 0.1949)	(0.1509, 0.1750) (0.1591, 0.1848) (0.1678, 0.1979)	0.0247 0.0272 0.0302	0.0251 0.0285 0.0309	0.0240 0.0257 0.0300
40	$t_{0.1}$ $t_{0.3}$ $t_{0.5}$	(0.1559, 0.1735) (0.1638, 0.1833) (0.1699, 0.1911)	(0.1559, 0.1742) (0.1636, 0.1834) (0.1675, 0.1891)	(0.1555, 0.1725) (0.1639, 0.1821) (0.1705, 0.1933)	0.0175 0.0194 0.0212	0.0182 0.0197 0.0216	0.0170 0.0182 0.0228

Contd...

60	$t_{0.1}$	(0.1583, 0.1725)	(0.1582, 0.1736)	(0.1573, 0.1718)	0.0142	0.0154	0.0145
	$t_{0.3}$	(0.1662, 0.1815)	(0.1653, 0.1819)	(0.1668, 0.1832)	0.0152	0.0166	0.0163
	$t_{0.5}$	(0.1716, 0.1888)	(0.1693, 0.1869)	(0.1722, 0.1911)	0.0171	0.0175	0.0189

Table 5
Estimating the Median $t_{0.5}$

Estimates	MSE	C.I.	Length of C.I.
$\hat{t}_{pi\,KE}$	4.1755	(0.5510, 1.6358)	1.0848
$\hat{t}_{pi\,OLS}$	4.4938	(1.1025, 2.9888)	1.8863
$\hat{t}_{pi\,MLE}$	4.3221	(0.9979,2.1472)	1.1493

From simulation results in Tables (1-4) we conclude:

1) The MSE of $\hat{t}_{p0\,KE}$, $\hat{t}_{p0\,OLS}$ and $\hat{t}_{p0\,MLE}$ decreases when n increases.
2) The MSE of $\hat{t}_{p0\,KE}$, $\hat{t}_{p0\,OLS}$ and $\hat{t}_{p0\,MLE}$ increases when p increases.
3) I term of MSE the MLE is nearly the best one.
4) In general, the length of 95% bootstrap C.I. for $t_{p0\,KE}$, $t_{p0\,OLS}$ and $t_{p0\,MLE}$ decreases when n increases and increases when p increases.

From Table (5) we see that the MLE and the OLS estimates are doing badly when the density is violated and the kernel estimates is the best.

7. Application to Real Data

We apply the kernel estimator proposed in section three, and the estimators discussed in sections four and five to a real data set. We will use the sliding metal wear accelerated degradation data from Meeker and Escobar [1] Table C.19 page 643. The mean square error and a 95% bootstrap C.I of the $t_{0.5}$ under the usual use condition is studied .The sliding metal wear data in is presented in Table 6. Figure 1 shows this accelerated degradation data.

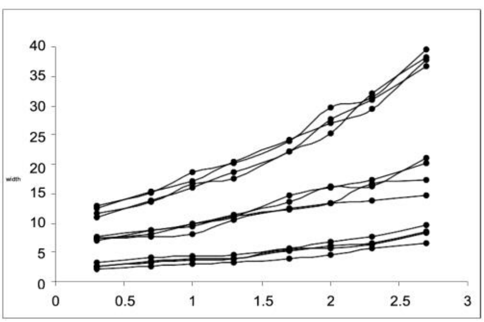

Figure 1
Plot of data

Table 6
Wear data

		Cycles							
Weight unit		**2**	**5**	**10**	**20**	**50**	**100**	**200**	**500**
10	1	3.2	4.1	4.5	4.7	5.8	6.8	7.7	9.6
	2	2.7	3.4	3.8	3.9	5.4	5.7	6.3	8.4
	3	2.1	2.7	3.1	3.3	4.0	4.6	5.7	6.6
	4	2.6	3.5	4.0	4.0	5.2	6.1	6.7	8.5
50	5	7.5	7.8	8.2	10.6	12.6	13.3	13.9	14.8
	6	7.5	8.1	9.8	10.9	14.8	16.1	17.3	20.2
	7	7.0	8.9	9.4	11.1	12.4	13.5	16.7	17.3
	8	7.8	8.9	10.0	11.5	13.7	16.2	16.2	21.0
100	9	12.5	15.4	17.2	20.5	24.1	270	29.4	37.9
	10	11.0	13.9	16.1	18.6	22.2	27.8	31.0	36.6
	11	13.0	15.1	18.6	20.2	23.9	29.7	31.5	39.6
	12	11.7	13.7	16.7	17.5	22.3	25.3	32.0	38.2

We applied the three estimation methods to the data in Table 6 with $D_f = 4$, $s_0 = 5$, $s_1 = 10$, $s_2 = 50$, $s_3 = 100$, $n = 4$, $r = 3$ and $k = 8$. In this analysis, our measurement of time is $t = \log(cycles)$.

The results are presented in Table 7.

Table 7
Results for Real Data

Estimates	MSE	C.I.	Length of C.I.
$\hat{t}_{0.5\,0\,KE}$	0.8962	(0.5412, 1.0021)	0.4609
$\hat{t}_{0.5\,0\,OLS}$	1.2183	(0.4523, 1.1224)	0.6701
$\hat{t}_{0.5\,0\,MLE}$	1.0815	(0.4328, 1.1025)	0.6697

From the results of Table 7 we conclude:

1) In general, in terms of MSE, the method of kernel density estimation of $t_{0.5\,0}$ is more efficient than the OLS and MLE methods.
2) The kernel estimation method gives a shorter interval length.

8. Conclusions

In general, in terms of the MSE and the length of the bootstrap C.I., when the distributional assumption of the random effect is available, the maximum likelihood estimator of t_{p0} performs better than the kernel estimator and the ordinary least square estimator, while the kernel estimator of t_{p0} performs better than the other estimators when the distributional assumption of the random effect is not available.

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