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Analysis of system of fractional partial differential equations using Laplace reduced differential transform with decomposition method

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Abstract

In this article, we discuss how to solve systems of nonlinear fractional partial differential equations (NFPDEs) by combining the Laplace transform & the reduced differential transform method, often known as the Laplace reduced differential transform with decompostion method (LRDTDM). It is generally considered how to deal with these concerns. In addition, systems of FPDEs of different orders may be solved using this method. The algorithm is Efficient and effective. The convergence and existence results

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for the suggested technique are presented. To illustrate the reliability and effectiveness of calculations made with mathematical software, numerical examples are utilized.

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Keywords: *Caputo fractional derivative, Mittag-leffler function, Laplace transform method, Reduced differential transform method, Adomian decomposition method, Fractional partial differential equations.*

1. Introduction

The area of fractional calculus is one of the numerous applied mathematics disciplines. Instead of integer order, it deals with arbitrarily ordered derivatives and integrals, as well as their applications [1-3]. A system of FPDEs can be used to simulate many processes in a variety of disciplines, including fluid mechanics, biology, finance, & material science. Even with a linear FDEs, finding the precise solution is challenging, therefore approximate solutions are required. Many authors, including ErtÃ¼rk et al., who used the DT approach, pointed out the solution of the system of FPDEs [4]. The method of fractional complex transform was studied by Ghazanfari [5]. An iterative Laplace transform was given by Jafari et al. [6]. The LT and the ADM technique are combined to generate the Laplace-Adomian decomposition method (LADM) [7-9]. Numerous different kinds of non-linear equations, including classical & FDEs, can be solved using this method. This approach was used by Jafari et al.[6] to solve the linear/nonlinear fractional diffusion and wave equations. Numerous developed and explored mathematical strategies have been used to discover the precise solution of NLFDEs. For example, the ADM [10-15], beside the HAM [16], VIM [17], FDM [18], DTM [19-21], LADM [22] and so many variety of transform methods [23-25] to solved FDEs . Recently, Y. Keskin et al. [26] was introduced RDTM for PDEs & FRDTM [27]. A class of singular FDEs with two distinct orders of derivation were studied by L. Ghaffour and Z. Dahmani [28], in which the right hand side has an arbitrary singularity on a certain interval of the real axis. Furthermore, L. Ghaffour et al. emphasised recent discoveries about the existence and uniqueness of solutions. Mohamed I. Abbas provided some results on the Ulam-Hyers and generalised Ulam-Hyers-Rassias stability for linear FDEs with mixed proportional-Caputo derivatives within the scope of the Laplace transform technique [29]. The fixed point theorems of Schaefer and Banach are used to determine the existence & uniqueness of solutions for nonlinear FDEsÂ with mixed proportional-Caputo derivatives.

The main purpose of this paper is to provide a novel iterative approach for finding an analytical solution to NFPDEs. The newly developed iterative approach basically demonstrates how three effective methods can be combined and utilised to provide precise solutions to NFPDEs that arise in mathematical sciences.

2. Preliminaries

Definition 2.1: Let $\gamma \in (m-1, m)$, $m \in \mathbb{N}$ & $\gamma \in \mathbb{R}$, then Riemann-Liouville fractional partial derivative of order γ for $u(\xi,\tau)$ is defined as:

$$\mathcal{D}_\tau^\gamma u(\xi,\tau) = \frac{\partial^m}{\partial \tau^m} \int_0^\tau \frac{(\tau-\eta)^{m-\gamma-1}}{\Gamma(m-\gamma)} u(\xi,\eta) d\eta, \eta > 0. \tag{2.1}$$

Definition 2.2: Let $\gamma, \tau \in \mathbb{R}, \tau > 0$ & $u(\xi,\tau) \in C_\mu^m$. Then Caputo fractional partial derivative of $u(\xi,\tau)$ of order γ defined as,

$$\begin{cases} \mathcal{D}_\tau^\gamma u(\xi,\tau) = \int_a^\tau \frac{(\tau-\eta)^{m-\gamma-1}}{\Gamma(m-\gamma)} \frac{\partial^m u(\xi,\tau)}{\partial \tau^m} d\eta, & m-1 < \gamma \le m \in \mathbb{N}, \\ \mathcal{D}_\tau^\gamma u(\xi,\tau) = \frac{\partial^m u(\xi,\tau)}{\partial \tau^m}, & \gamma = m \in \mathbb{N}, \end{cases} \tag{2.3}$$

is called the

Theorem 2.3: *Let* $\tau, \gamma \in \mathbb{R}, \tau > 0$ *and* $m-1 < \gamma \le m \in \mathbb{N}$. *Then*

$$\begin{cases} \mathcal{I}_\tau^\gamma \mathcal{D}_\tau^\gamma u(\xi,\tau) = u(\xi,\tau) - \sum_{k=0}^{m-1} \frac{\tau^k}{k!} u_\tau^{(k)}(\xi,0), \\ \mathcal{D}_\tau^\gamma \mathcal{I}_\tau^\gamma u(\xi,\tau) = u(\xi,\tau) \end{cases} \tag{2.4}$$

Definition 2.4: The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$, valid in the whole complex plane defined as,

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \tag{2.5}$$

Definition 2.5: Let ϕ be function of τ, then Laplace transform of $\phi(\tau)$ is

$$F(s) = L[\phi(\tau)] = \int_0^\infty e^{-s\tau} \phi(\tau) \mathrm{d}\tau. \tag{2.6}$$

Definition 2.6: The Laplace transform $L[\phi(\tau)]$ of the Riemann-Liouville fractional integral is defined as

$$L\{\mathcal{I}^{\alpha}\phi(\tau)\} = s^{-\alpha}\Phi(s). \tag{2.7}$$

Definition 2.7: The Laplace transform $L[\phi(\tau)]$ of the Caputo fractional derivative is defined as

$$L\{\mathcal{D}^{\alpha}\phi(\tau)\} = s^{\alpha}\Phi(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)}\phi^{(k)}(0), \quad n-1 < \alpha \le n. \tag{2.8}$$

2.1 *Basic Ideas of the RDTM*

This section reviews the RDTM. Consider $u(\xi,\tau)$ be an analytical and k-times continuously differentiable with respect to space and time. Now, $u(\xi,\tau)$ as the product of two single-variable functions, for example, $u(\xi,\tau)$ $= \phi(\xi).\psi(\tau).$ The function may so be expressed as follows:

$$u(\xi,\tau) = \left(\sum_{p=0}^{\infty}\Phi(p)\xi^{p}\right)\left(\sum_{q=0}^{\infty}\Psi(q)\tau^{q}\right) = \sum_{k=0}^{\infty}U_k(\xi)\tau^{k} \tag{2.1.1}$$

where $U_k(\xi)$ is transformed funtion of the $u(\xi,\tau)$.

Definition 2.8: If $u(\xi,\tau)$ is analytic and continuously differentiable with respect to space ξ and time τ in the domain, then the reduced transformed function of $u(\xi,\tau)$ is defined as

$$U_k(\xi) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^{k}}{\partial\tau^{k}}u(\xi,\tau)\right]_{\tau=\tau_0} \tag{2.1.2}$$

Throughout in this article, $U_k(\xi)$ is reduced differential transformed function of $u(\xi,\tau)$.

Definition 2.9: The reduced differential inverse transform of $U_k(\xi)$ is given by

$$u(\xi,\tau) = \sum_{k=0}^{\infty}U_k(\xi)(\tau-\tau_0)^{k} \tag{2.1.3}$$

From Eqs. (2.1.2) and (2.1.3) one can deduce

$$u(\xi,\tau) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^{k}}{\partial\tau^{k}}u(\xi,\tau)\right]_{\tau=\tau_0}(\tau-\tau_0)^{k}, \tag{2.1.4}$$

Table 1

Basic properties of the RDTM [27]

Functional form	Reduced Differential Transformed form
$u(\xi,\tau)$	$\frac{1}{k!}\left[\frac{\partial^k}{\partial\tau^k}u(\xi,\tau)\right]_{\tau=\tau_0}$
$\gamma u(\xi,\tau)\pm\beta v(\xi,\tau)$	$\gamma U_k(\xi)\pm\beta v_k(\xi)$, where γ and β are constants
$u(\xi,\tau)\cdot v(\xi,\tau)$	$\sum_{i=0}^{k}U_j(\xi)V_{k-i}(\xi)$
$u(\xi,\tau)\cdot v(\xi,\tau)\cdot w(\xi,\tau)$	$\sum_{i=0}^{k}\sum_{j=0}^{i}U_j(\xi)V_{i-j}(\xi)W_{k-i}(\xi)$
$\frac{\partial^n}{\partial\tau^n}u(\xi,\tau)$	$\frac{(k+n)!}{k!}U_{k+n}(\xi),\quad n=1,2,\ldots$
$\frac{\partial^n}{\partial\xi^m}u(\xi,\tau)$	$\frac{\partial^n}{\partial\xi^n}U_k(\xi),\quad n=1,2,\ldots$
$\xi^m\tau^n u(\xi,\tau)$	$\xi^m U_{k-n}(\xi)$
$\xi^m\tau^n$	$\xi^m\delta(k-n)$, where $\delta(k-n)=\begin{cases}1, & \text{if } k=n,\\ 0, & \text{if } k\neq n\end{cases}$

when $\tau_0=0$, Eq. (2.1.4) it becomes

$$u(\xi,\tau)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial\xi^k}u(\xi,\tau)\right]_{\tau=0}\tau^k \tag{2.15}$$

The table 1 is shows that Basic properties of the RDTM [27]

3. Methodology

Consider the system of FPDE:

$$\mathcal{D}^\beta u_j(\xi,\tau)+\mathcal{L}u_j(\xi,\tau)+\mathcal{N}u_j(\xi,\tau)=r_j(\xi,\tau),$$
$$\xi,\tau\geq 0,\ \beta\in(m-1,m],\quad j=1,2,\ldots,m. \tag{3.1}$$

The Caputo sense is used in the expression of equation (3.1). $\mathcal{L}$ and $\mathcal{N}$ stand for the linear & nonlinear terms, respectively and $r_j(\xi,\tau)$ is the remaining terms with the initial condition.

$$u_{j,t}^{(k)}(\xi,0)=f_{j,k}(\xi),\quad k=0,1,2,\ldots,m-1. \tag{3.2}$$

Taking the Laplace transform of equation (3.1), we get

$$\begin{aligned} s^{\beta}L[u_j(\xi,\tau)]-s^{\beta-1}u_j(\xi,0)&=L[r_j(\xi,\tau)]-L[\mathcal{L}u_j(\xi,\tau)+\mathcal{N}u_j(\xi,\tau)],\\ L[u_j(\xi,\tau)]&=\frac{f_k(\xi)}{s}-\frac{1}{s^{\beta}}L[\mathcal{L}u_j(\xi,\tau)+\mathcal{N}u_j(\xi,\tau)+r_j(\xi,\tau)]. \end{aligned} \tag{3.3}$$

Using the inverse Laplace transform to solve equation (3.3) on both sides, we obtain

$$u_j(\xi,\tau)=F_j(\xi,\tau)-L^{-1}\left(\frac{1}{s^{\beta}}L[\mathcal{L}u_j(\xi,\tau)+\mathcal{N}u_j(\xi,\tau)]\right), \tag{3.4}$$

where $F_j(\xi,\tau)$ represents the prescribed initial conditions.

By representing the solution as an infinite series given by in the second phase of the reduced differential decomposition approach, we can:

$$u_j(\xi,\tau)=\sum_{k=0}^{\infty}U_{j,k}(\xi). \tag{3.5}$$

In the problem, the nonlinear term is given as

$$\mathcal{N}u_j(\xi,\tau)=\sum_{k=0}^{\infty}A_{j,k}, \tag{3.6}$$

where

$$A_{j,k}=\frac{1}{k!}\left[\frac{d^k}{d\lambda}\left[\mathcal{N}\sum_{k=0}^{\infty}(\lambda^k U_{j,k})\right]\right]_{\lambda=0}, \quad j=1,2,\dots,n \tag{3.7}$$

known as Adomian polynomials. By replacing equation (3.4) with equations (3.5) and (3.6), we obtain

$$\sum_{k=0}^{\infty}U_{j,k}(\xi)=F_j(\xi,\tau)-L^{-1}\left(\frac{1}{s^{\beta}}L\left[\mathcal{L}\sum_{k=0}^{\infty}U_{j,k}(\xi)+\sum_{k=0}^{\infty}A_{j,k}\right]\right).$$

Reduced differential decomposition method, we get

$$\begin{aligned} U_{j,0}(\xi)&=F_j(\xi,\tau),\\ U_{j,k+1}(\xi)&=-L^{-1}\left(\frac{1}{s^{\beta}}L\left[\mathcal{L}\sum_{k=0}^{\infty}U_{j,k}(\xi)+\sum_{k=0}^{\infty}A_{j,k}\right]\right), \quad k\ge 0, \end{aligned} \tag{3.8}$$

Then, the approximate analytical solution of equations (3.1)-(3.2) is given as

$$u_j(\xi,\tau)=\sum_{k=0}^{\infty}U_{j,k}(\xi).$$

3.1 *Analysis of Convergence and Error Estimate*

Theorem 3.1: *If* $\mathcal{B}$ *be a Banach space, then the series solution of the system* (3.1) *converges to, for* $j = 1\ldots, n$ *if* $\exists\, \sigma \in [0,1)$ *such that,* $\| U_{jm} \| \leq \sigma \| U_{j(m-1)} \|$ $\forall\, m \in \mathbb{N}$.

Proof: Let the sequences S_{jm}, be a partial sums of the series given by the system (3.8) as

$$\begin{cases} S_{j0} = U_{j0}(\xi) \\ S_{j1} = U_{j0}(\xi) + U_{j1}(\xi) \\ S_{j2} = U_{j0}(\xi) + U_{j1}(\xi) + U_{j2}(\xi) \\ \vdots \\ S_{jm} = U_{j0}(\xi) + U_{j1}(\xi) + U_{j2}(\xi) + \cdots + U_{jm}(\xi), \end{cases} \tag{3.1.1}$$

then we must prove that in Banach space $\mathcal{B}$, $\{S_{jm}\}_{m=0}^{\infty}$ are Cauchy sequences. We examine the following factors in this regard:

$$\begin{aligned} \| S_{j(m+1)} - S_{jm} \| &= \| U_{j(m+1)}(\xi) \| \leq \sigma \| U_{jm}(\xi) \| \leq \sigma^2 \| U_{j(m-1)}(\xi) \| \leq \cdots \\ &\leq \sigma^{m+1} \| U_{j0}(\xi) \|. \end{aligned} \tag{3.1.2}$$

$\forall\, m, r \in \mathbb{N}, m \geq r,$ by using the (3.1.1) and triangle inequality successively, we have,

$$\begin{aligned} \| S_{jm} - S_{jr} \| &= \| S_{jm} - S_{j(m-1)} + S_{j(m-1)} - S_{j(m-2)} + \cdots + S_{j(r+1)} - S_{jr} \| \\ &\leq \| S_{jm} - S_{j(m-1)} \| + \| S_{j(m-1)} - S_{j(m-2)} \| + \cdots + \| S_{j(r+1)} - S_{jr} \| \\ &\leq \sigma^{m} \| U_{j0}(\xi) \| + \sigma^{m-1} \| U_{j0}(\xi) \| + \cdots + \sigma^{r+1} \| U_{j0}(\xi) \| \\ &= \sigma^{r+1}(1 + \sigma + \cdots + \sigma^{m-r-1}) \| U_{j0}(\xi) \| \\ &\leq \sigma^{r+1} \left(\frac{1-\sigma^{m-r}}{1-\sigma} \right) \| U_{j0}(\xi) \|. \end{aligned}$$

As $0 < \sigma < 1,$ so $1 - \sigma^{m-r} \leq 1$ then

$$\| S_{jm} - S_{jr} \| \leq \frac{\sigma^{r+1}}{1-\sigma} \| U_{j0}(\xi) \|.$$

for $j = 1, 2, \ldots, n$ Since $U_{j0}(\xi)$ is bounded, then

$$\lim_{m,r\to\infty} \| S_{jm} - S_{jr} \| = 0, \quad j = 1, 2, \ldots, n.$$

As a result, the sequences $\{S_{jm}\}_{m=0}^{\infty}$ in the Banach space $\mathcal{B}$ are Cauchy sequences, and the series solution specified in system (3.5) converges.

Theorem 3.2: *The series solution* (3.5) *of the nonlinear fractional differential system* (3.1) *is determined to have a maximum absolute truncation error of* (3.5).

$$\sup_{(\xi,\tau)\in\Theta}\left|u_j(\xi,\tau)-\sum_{k=0}^{r}U_{jk}(\xi)\right|\le\frac{\sigma^{r+1}}{1-\sigma}\sup_{\xi\in\Theta}|U_{j0}(\xi)|,\quad j=1,2,\dots,n.$$

where the region $\Theta\subset\mathbb{R}^{n+1}$.

Proof: We deduce the following from Theorem 3.1:

$$\|S_{jm}-S_{jr}\|\le\frac{\sigma^{r+1}}{1-\sigma}\sup_{(\xi)\in\Theta}|U_{j0}(\xi)|, \tag{3.1.3}$$

However, we suppose that $S_{jm}=\sum_{k=0}^{n}U_{jk}(\xi)$ and since $m\to\infty$, we obtain $S_{jm}\to u_j(\xi,\tau)$, so the system (3.1.3) can be rephrased as

$$\begin{aligned}\|u_j(\xi,\tau)-S_{jr}\| &= \left\|u_j(\xi,\tau)-\sum_{k=0}^{r}U_{jk}(\xi)\right\|\\ &\le\frac{\sigma^{r+1}}{1-\sigma}\sup_{\xi\in\Theta}|U_{j0}(\xi)|, j=1,2,\dots,n.\end{aligned}$$

As a result, in the Θ region, the maximum absolute truncation error is

$$\sup_{(\xi,\tau)\in\Theta}\left|u_j(\xi,\tau)-\sum_{k=0}^{r}U_{jk}(\xi)\right|\le\frac{\sigma^{r+1}}{1-\sigma}\sup_{(\xi)\in\Theta}|U_{j0}(\xi)|, j=1,2,\dots,n$$

and this completes the proof.

4. Solving system of FPDEs

Example 4.1: Consider the system of non-linear FPDEs.

$$\begin{aligned}&\mathcal{D}_\tau^\alpha u-u_{\xi\xi}-2uu_\xi+(uv)_\xi=0,\\ &\mathcal{D}_\tau^\alpha v-v_{\xi\xi}-2vv_\xi+(uv)_\xi=0,\quad 0<\alpha\leqslant1,\end{aligned} \tag{4.1}$$

with conditions

$$u(\xi,0)=\sin\xi,\quad v(\xi,0)=\sin\xi. \tag{4.2}$$

Equations (4.1) and (4.2) are solved using the same methods as in Section 3, and the resulting iteration formula is as follows.

$$U_0(\xi)=\sin\xi\quad\&\quad V_0(\xi)=\sin\xi \tag{4.3}$$

$$
\begin{aligned}
U_{k+1}(\xi) &= \mathcal{L}^{-1}\left(\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^2 U_k}{\partial \xi^2} + A_k - B_k - C_k\right]\right) \\
V_{k+1}(\xi) &= \mathcal{L}^{-1}\left(\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^2 U_k}{\partial \xi^2} + D_k - B_k - C_k\right]\right)
\end{aligned} \tag{4.4}
$$

The transform functions are $U_k(\xi), V_k(\xi)$ and the nonlinear terms have been changed into $A_k(\xi)$, $B_k(\xi)$, $C_k(\xi)$, & $D_k(\xi)$. Here are the first few nonlinear terms for the reader's convenience:

$A_0(u,u) = 2U_0 \frac{\partial U_0}{\partial \xi}$,	$B_0(u,v) = U_0 \frac{\partial U_0}{\partial \xi}$,
$A_1(u,u) = 2\left(U_0 \frac{\partial U_1}{\partial \xi} + U_1 \frac{\partial U_0}{\partial \xi}\right)$,	$B_1(u,v) = U_0 \frac{\partial V_1}{\partial \xi} + U_1 \frac{\partial V_0}{\partial \xi}$,
$A_2(u,u) = 2\left(U_0 \frac{\partial U_2}{\partial \xi} + U_1 \frac{\partial U_1}{\partial \xi} + U_2 \frac{\partial U_0}{\partial \xi}\right)$,	$B_2(u,v) = U_0 \frac{\partial V_2}{\partial \xi} + U_1 \frac{\partial V_1}{\partial \xi} + U_2 \frac{\partial V_0}{\partial \xi}$,
$A_3(u,u) = 2\left(U_0 \frac{\partial U_3}{\partial \xi} + U_1 \frac{\partial U_2}{\partial \xi} + U_2 \frac{\partial U_1}{\partial \xi} + U_3 \frac{\partial U_0}{\partial \xi}\right)$,	$B_3(u,v) = U_0 \frac{\partial U_3}{\partial \xi} + U_1 \frac{\partial U_2}{\partial \xi} + U_2 \frac{\partial U_1}{\partial \xi} + U_3 \frac{\partial U_0}{\partial \xi}$,
$\vdots$	$\vdots$

.

$C_0(u,v) = \frac{\partial U_0}{\partial \xi} V_0$,	$D_0(v,v) = 2V_0 \frac{\partial U_0}{\partial \xi}$,
$C_1(u,v) = \frac{\partial U_0}{\partial \xi} V_1 + \frac{\partial U_1}{\partial \xi} V_0$,	$D_1(v,v) = 2\left(U_0 \frac{\partial U_1}{\partial \xi} + U_1 \frac{\partial u_0}{\partial \xi}\right)$,
$C_2(u,v) = \frac{\partial U_0}{\partial \xi} V_2 + \frac{\partial U_1}{\partial \xi} V_1 + \frac{\partial U_2}{\partial \xi} V_0$,	$D_2(v,v) = 2\left(V_0 \frac{\partial V_2}{\partial \xi} + V_1 \frac{\partial V_1}{\partial \xi} + V_2 \frac{\partial V_0}{\partial \xi}\right)$,
$C_3(u,v) = \frac{\partial U_0}{\partial \xi} V_3 + \frac{\partial U_1}{\partial \xi} v_2 + \frac{\partial U_2}{\partial \xi} V_1 + \frac{\partial U_3}{\partial \xi} V_0$,	$D_3(v,v) = 2\left(V_0 \frac{\partial V_3}{\partial \xi} + V_1 \frac{\partial V_2}{\partial \xi} + V_2 \frac{\partial V_1}{\partial \xi} + V_3 \frac{\partial V_0}{\partial \xi}\right)$,
$\vdots$	$\vdots$

.

Now, substituting (4.3) with (4.4) respectively, we obtain

$$U_1(\xi) = \frac{-\sin \xi \tau^{\alpha}}{\Gamma(\alpha+1)}, U_2(\xi) = \frac{\sin \xi \tau^{2\alpha}}{\Gamma(2\alpha+1)}, U_3(\xi) = \frac{-\sin \xi \tau^{3\alpha}}{\Gamma(3\alpha+1)}, \cdots$$

$$V_1(\xi) = \frac{-\sin \xi \tau^{\alpha}}{\Gamma(\alpha+1)}, V_2(\xi) = \frac{\sin \xi \tau^{2\alpha}}{\Gamma(2\alpha+1)}, V_3(\xi) = \frac{-\sin \xi \tau^{3\alpha}}{\Gamma(3\alpha+1)}, \cdots$$

Taking the inverse transformation of the set of values $\{U_k^{\alpha}(\xi)\}_{k=0}^{4}$ and $\{U_k^{\alpha}(\xi)\}_{k=0}^{4}$ gives 4-terms approximation solutions as follows

$$
\begin{aligned}
\tilde{u}_4(\xi,\tau) &= \sum_{k=0}^{4} U_k(\xi) = U_0(\xi) + U_1(\xi) + U_2(\xi) + U_3(\xi) \\
&= \sin \xi - \frac{\sin \xi \tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{\sin \xi \tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\sin \xi \tau^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\sin \xi \tau^{4\alpha}}{\Gamma(4\alpha+1)}
\end{aligned}
$$

Table 2

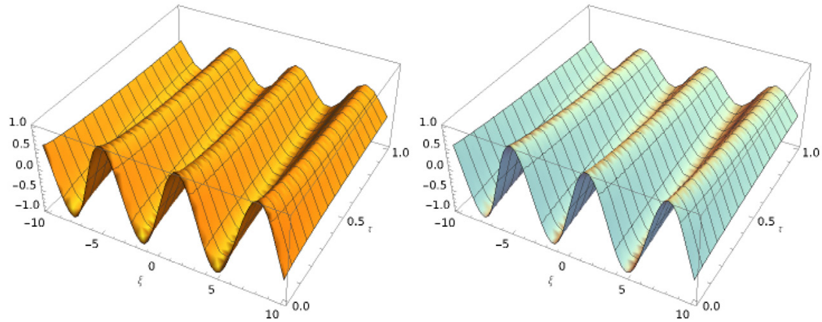
For $\alpha = 1$ and $\tau = 0.01, 0.05, 0.1 - 10 \leq \xi \leq 10$, the error between the exact and fourth order approximation solution u and v of system by LRDTM for $\alpha = 1$.

ξ	τ	Exact	Approximate		Absolute Error u and v for $\alpha = 1$
			$\alpha = 1$	$\alpha = 0.9$	$\lvert u_{exact} - u_{appr} \rvert$
-10	0.01	0.53860801	0.53860801	0.53513757	4.52526905e-13
-5		0.94938282	0.94938282	0.94326562	7.97695243e-13
0		0.	0.	0.	0.00000000e+00
5		-0.94938282	-0.94938282	-0.94326562	7.97695243e-13
10		-0.53860801	-0.53860801	-0.53513757	4.52526905e-13
-10	0.05	0.51748889	0.51748889	0.50729786	1.40499934e-09
-5		0.91215699	0.91215699	0.89419367	2.47653631e-09
0		0.	0.	0.	0.00000000e+00
5		-0.91215699	-0.91215699	-0.89419367	2.47653631e-09
10		-0.51748889	-0.51748889	-0.50729786	1.40499934e-09
-10	0.1	0.49225066	0.4922507	0.477697	4.45901684e-08
-5		0.86767056	0.86767064	0.84201742	7.85973081e-08
0		0.	0.	0.	0.00000000e+00
5		-0.86767056	-0.86767064	-0.84201742	7.85973081e-08
10		-0.49225066	-0.4922507	-0.477697	4.45901684e-08

$$\tilde{v}_4(\xi,\tau) = \sum_{k=0}^{4} V_k(\xi)\tau^{k\alpha} = V_0(\xi) + V_1(\xi)\tau^{\alpha} + V_2(\xi)\tau^{2\alpha} + V_3(\xi)\tau^{3\alpha}$$
$$= \sin\xi - \frac{\sin\xi\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{\sin\xi\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\sin\xi\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\sin\xi\tau^{4\alpha}}{\Gamma(4\alpha+1)}$$

In Table2, the numerical values of the approximate solutions for $\alpha =$ 0.9, 1 and exact solutions for Example 4.1, show the accuracy and efficiency of our technique at different values of ξ, τ. The absolute error between an exact and a fourth-order approximate solution. In Figure 1, we plot the graph of fourth-term approximate solutions for Example 4.1, when $\alpha = 0.7$, 0.9 and Figure 2, we plot the exact solution of Example 4.1.

Example 4.2: Consider the system of fractional PDEs



(a) Graph of $u(\xi,\tau)$ & $v(\xi,\tau)$ of (4.1) for $\alpha = 0.7$ by new approach.

(b) Graph of $u(\xi,\tau)$ & $v(\xi,\tau)$ of (4.1) for $\alpha = 0.9$ by using new approach.

Figure 1

4th-order of approximation solution $u(\xi,\tau)$ and $v(\xi,\tau)$ of example 4.1 for $\alpha = 0.7, 0.9$

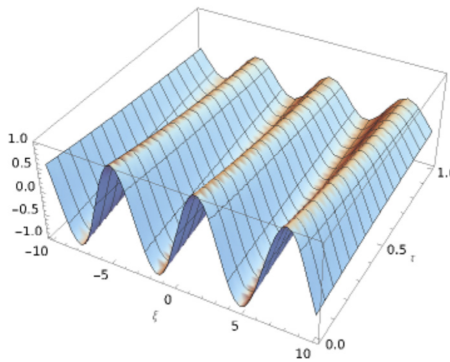


Figure 2

Exact solution $u(\xi,\tau)$ and $v(\xi,\tau)$ of example 4.1

$$\begin{aligned} &\frac{\partial^\beta u}{\partial \tau^\beta} + v\frac{\partial u}{\partial \xi} + u = 1, \\ &\frac{\partial^\beta v}{\partial \tau^\beta} - u\frac{\partial v}{\partial \xi} - v = 1, \quad 0 < \beta \le 1, \end{aligned} \tag{4.5}$$

with the initial conditions

$$u(\xi,0) = e^{\xi}, \quad v(\xi,0) = e^{-\xi}. \tag{4.6}$$

Equations (4.5) and (4.6) are solved using the same methods as in Section 3, and the resulting iteration formula is as follows.

$$U_0^{\alpha}(\xi) = e^{\xi} \quad \& \quad V_0^{\alpha}(x) = e^{-\xi}$$

$$\begin{aligned} U_{k+1}^{\alpha}(\xi) &= \mathcal{L}^{-1}\left(\frac{1}{s^{\beta}}\mathcal{L}[\delta(k) + A_k + U_k^{\alpha}]\right) \\ V_{k+1}^{\alpha}(\xi) &= \mathcal{L}^{-1}\left(\frac{1}{s^{\beta}}\mathcal{L}[\delta(k) - B_k - V_k^{\alpha}]\right) \end{aligned} \tag{4.7}$$

The transform functions are $U_k(\xi), V_k(\xi)$ and the nonlinear terms have been changed into $A_k(\xi), B_k(\xi)$. Here are the first few nonlinear terms for the reader's convenience:

$A_0(v,u) = V_0 \frac{\partial U_0}{\partial \xi},$	$B_0(u,v) = U_0 \frac{\partial V_0}{\partial \xi},$
$A_1(v,u) = V_0 \frac{\partial U_1}{\partial \xi} + V_1 \frac{\partial U_0}{\partial \xi},$	$B_1(u,v) = U_0 \frac{\partial V_1}{\partial \xi} + U_1 \frac{\partial V_0}{\partial \xi},$
$A_2(v,u) = V_0 \frac{\partial U_2}{\partial \xi} + V_1 \frac{\partial U_1}{\partial \xi} + V_2 \frac{\partial U_0}{\partial \xi},$	$B_2(u,v) = U_0 \frac{\partial V_2}{\partial \xi} + U_1 \frac{\partial V_1}{\partial \xi} + U_2 \frac{\partial V_0}{\partial \xi},$
$\vdots$	$\vdots$

Now, substituting (4.6) with (4.7) respectively, we obtain

$$U_1(\xi) = \frac{-e^{\xi}\tau^{\alpha}}{\Gamma(\alpha+1)}, U_2(\xi) = \frac{e^{\xi}\tau^{2\alpha}}{\Gamma(2\alpha+1)}, U_3(\xi) = \frac{-e^{\xi}\tau^{3\alpha}}{\Gamma(3\alpha+1)}, \cdots$$

$$V_1(\xi) = \frac{e^{-\xi}\tau^{\alpha}}{\Gamma(\alpha+1)}, V_2(\xi) = \frac{e^{-\xi}\tau^{2\alpha}}{\Gamma(2\alpha+1)}, V_3(\xi) = \frac{e^{-\xi}\tau^{3\alpha}}{\Gamma(3\alpha+1)}, \cdots$$

Taking the inverse transformation of the set of values $\{U_k(\xi)\}_{k=0}^{4}$ and $\{U_k(\xi)\}_{k=0}^{4}$ gives 4-terms approximation solutions as follows

$$\begin{aligned} \tilde{u}_4(\xi,\tau) &= \sum_{k=0}^{4} U_k(\xi)\tau^{k\alpha} = U_0(\xi) + U_1(\xi)\tau^{\alpha} +_t U_2(\xi)\tau^{2\alpha} + U_3(\xi)\tau^{3\alpha} \\ &= e^{\xi} - \frac{e^{\xi}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{\xi}\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{e^{\xi}\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{e^{\xi}\tau^{4\alpha}}{\Gamma(4\alpha+1)} \end{aligned}$$

$$\begin{aligned} \tilde{v}_4(\xi,\tau) &= \sum_{k=0}^{4} V_k(\xi)\tau^{k\alpha} = V_0(\xi) + V_1(\xi)\tau^{\alpha} + V_2(\xi)\tau^{2\alpha} + V_3(\xi)\tau^{3\alpha} \\ &= e^{-\xi} + \frac{e^{-\xi}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{-\xi}\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{e^{-\xi}\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{e^{-\xi}\tau^{4\alpha}}{\Gamma(4\alpha+1)} \end{aligned}$$

In Table 3 and Table 4, the numerical values of the approximate solutions for α = 0.9, 1 and exact solutions for Example 4.2, show the accuracy and efficiency of our technique at different values of ξ, τ. The absolute error between an exact and a fourth-order approximate solution. In Figure 3 and Figure 4, we plot the graph of fourth-term approximate solutions for Example 4.2, when $\alpha = 0.7$, 0.9 respectively and Figure 5, we plot the exact solution of Example 4.2

Table 3

For $\alpha = 1$ and $\tau = 0.01, 0.05, -2 \le \xi \le 2$, the error between the exact and fourth order approximation solution u of system by LRDTM .

ξ	τ	Exact	Approximate		Absolute Error for $\alpha = 1$
			$\alpha = 1$	$\alpha = 0.9$	$\lvert u_{exact} - u_{appr} \rvert$
-2	0.01	0.13398867	0.13398867	0.13312524	1.12548859e-13
-1.5		0.22090998	0.22090998	0.21948642	1.85629290e-13
-1		0.36421898	0.36421898	0.36187193	3.06032977e-13
0.5		0.60049558	0.60049558	0.59662595	5.04485342e-13
0		0.99004983	0.99004983	0.98366989	8.31890112e-13
0.5		1.63231622	1.63231622	1.62179747	1.37179157e-12
1		2.69123447	2.69123447	2.67389198	2.26174635e-12
1.5		4.43709552	4.43709552	4.40850258	3.72857301e-12
2		7.31553376	7.31553376	7.26839198	6.14708284e-12
-2	0.05	0.1287349	0.1287349	0.1261997	3.49519469e-10
-1.5		0.21224797	0.21224797	0.20806813	5.76260262e-10
-1		0.34993775	0.34993775	0.34304635	9.50092505e-10
0.5		0.57694981	0.57694981	0.56558781	1.56643776e-09
0		0.95122942	0.95122943	0.93249665	2.58261923e-09
0.5		1.56831219	1.56831219	1.53742706	4.25801949e-09
1		2.58570966	2.58570967	2.53478869	7.02028702e-09
1.5		4.26311452	4.26311453	4.17916003	1.15744969e-08
2		7.02868758	7.0286876	6.89027004	1.90831191e-08

Table 4

For $\alpha = 1$ and $\tau = 0.01, 0.05, -2 \le \xi \le 2$, the error between the exact and fourth order approximation solution v of system by LRDTM .

ξ	τ	Exact	Approximate		Absolute Error for $\alpha = 1$
			$\alpha = 1$	$\alpha = 0.9$	$\lvert v_{exact} - v_{appr} \rvert$
-2	0.01	7.46331735	7.46331735	7.51193449	6.16484641e-12
-1.5		4.52673079	4.52673079	4.55621858	3.74100750e-12
-1		2.74560102	2.74560102	2.76348626	2.26840768e-12
0.5		1.66529119	1.66529119	1.67613915	1.37578837e-12
0		1.01005017	1.01005017	1.01662978	8.34443625e-13

Contd...

0.5		0.61262639	0.61262639	0.61661713	5.06261699e-13
1		0.37157669	0.37157669	0.3739972	3.07032177e-13
1.5		0.22537266	0.22537266	0.22684077	1.86212157e-13
2		0.13669543	0.13669543	0.13758588	1.12965193e-13
-2	0.05	7.76790111	7.76790109	0.1261997	1.94038376e-08
-1.5		4.71147018	4.71147017	0.20806813	1.17690231e-08
-1		2.85765112	2.85765111	0.34304635	7.13827353e-09
0.5		1.73325302	1.73325301	0.56558781	4.32958158e-09
0		1.0512711	1.05127109	0.93249665	2.62602406e-09
0.5		0.63762815	0.63762815	1.53742706	1.59276414e-09
1		0.38674102	0.38674102	2.53478869	9.66060232e-10
1.5		0.23457029	0.23457029	4.17916003	5.85945154e-10
2		0.14227407	0.14227407	6.89027004	3.55393714e-10

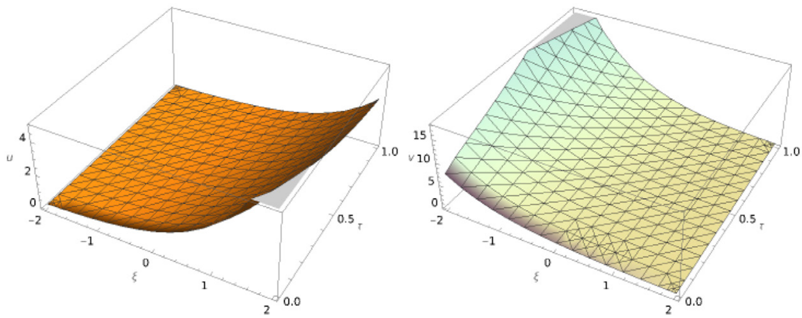


Figure 3

4th-order of approximation solution $u(\xi,\tau)$ and $v(\xi,\tau)$ of example 4.2 for $\alpha = 0.7$

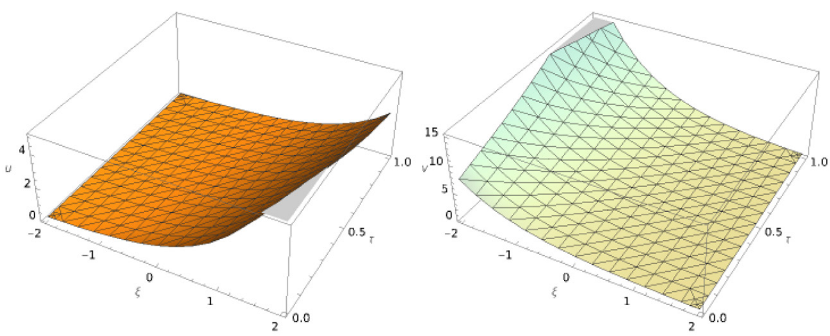


Figure 4

4th-order of approximation solution $u(\xi,\tau)$ and $v(\xi,\tau)$ of example 4.2 for $\alpha = 0.9$

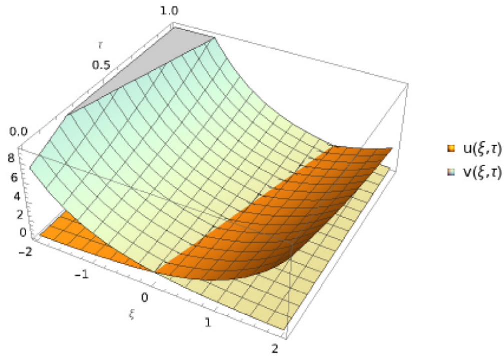


Figure 5

Exact solution $u(\xi,\tau)$ and $v(\xi,\tau)$ of example 4.2

5. Conclusion

In this work, we discussed how to solve systems of NFPDEs by combining the Laplace transform and reduced differential transform method with Adomian decomposition. It is generally considered how to deal with these concerns, we approximate solutions to the system of nonlinear equations using this innovative method and discussed this suggested technique of convergence analysis and Absolute error. Some numerical illustrates are presented to support the results & the numerical approximation's performance.

List of Abbriviation

- FDEs Fractional Differential Equations.
- NFPDEs Nonlinear Fractional Partial Differential Equations.
- FPDEs Fractional Partial Differential Equations.
- DT Differential Transform.
- FDTM Fractional Differential Transform Method.
- HAM Homotopy Analysis Method.
- VIM Variational Iterational Method.
- ADM Adomian Decomposition Method.
- RDTM Reduced Differential Transform Method.
- FRDTM Fractional Reduced Differential Transform Method.
- LADM Laplace Adomian Decomposition Method.

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