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Weibull-generalized inverted exponential {log-logistic} distribution : Features and practicalities

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Abstract

Here, we have introduced a novel probability distribution called the Weibull-Generalized inverted exponential{Log-logistic} distribution (WGIELLD) adopting the Transformed-Transformer method. The density curves of the new distribution show that it can model data sets that are approximately normal, left and right skewed; while the hazard function graphs indicate that the distribution can model components with various shapes of failure rate. We have derived various statistical features of this distribution, such as the reliability function (rf), hazard function (hf), mode, quantile function (qf), median, Renyi entropy (Re), moment generating function (mgf), mean absolute deviations (MAD) from the mean and median, and order statistics. We have used maximum likelihood estimation (MLE) approach to estimate the parameters of this distribution and also run a simulation study to verify the suitability of the approach. Finally, two sets of data are employed to demonstrate the versatility of the distribution compared to the baseline distribution and other related distributions.

Subject Classification: *60E05, 62F10, 62N05, 62P30.*

Keywords: *T-R{Y} framework, Weibull distribution, Generalized inverse exponential distribution, Log-logistic distribution, Maximum likelihood estimate.*

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1. Introduction

In search for a distribution suitable to model life-time data with inverted bathtub-shaped hazard rate, Keller, Kamath, and Perera [17] transformed the exponential distribution to obtain inverse exponential distribution (IED).

In order to achieve more flexibility with respect to modeling failure rates, Aboudmmoh, and Alshingiti [1] generalized the IED to obtain the generalized inverted exponential distribution (GIED) with the cdf and pdf given respectively as

$$G_{GIE}(z) = 1 - (1 - e^{-\varphi z^{-1}})^{\vartheta} \tag{1}$$

$$g_{GIE}(z) = \frac{\vartheta\varphi}{z^2} e^{-\varphi z^{-1}} (1 - e^{-\varphi z^{-1}})^{\vartheta - 1}; \; z > 0, \; \varphi, \vartheta > 0 \tag{2}$$

Alzaatreh, Lee, and Famoye [5] introduced a method of generating families of continuous distributions called T-X family with the cdf and pdf respectively given as

$$F(z) = \int_a^{J(G(z))} dR(t) = R(J(G(z))) \tag{3}$$

$$f(z) = r\{J(G(z))\}\{J'(G(z))\} \tag{4}$$

where $J'(G(z))$ is the derivative of $J(G(z))$, J is an increasing function defined on (0,1) having the support of T as its range, R is the cdf of T, and G is the cdf of Z. Jamal, Chesneau, and Elgarhy [16] applied the T-X transformation method in obtaining Type II general inverse exponential family of distributions.

Aljarrah, Lee, and Famoye [3] replaced the J function in (3) with the quantile function to generate a family of continuous distributions referred to as T-X{Y} family of distributions.

Alzaatreh, Lee, and Famoye [6] gave a unified notation by replacing the notation X in T-X{Y} family with R, to have T-R{Y} family. With the unified notation, they defined the cdf, pdf and qf of T-R{Y} category of distributions respectively as

$$F_Z(z) = \int_a^{G_Y^{-1}(G_R(z))} dG_T(t) = G_T(G_Y^{-1}(G_R(z))) \tag{5}$$

$$f_Z(z) = g_R(z) \frac{g_T(G_Y^{-1}(G_R(z)))}{g_Y(G_Y^{-1}(G_R(z)))} \tag{6}$$

$$Q_Z(p) = G_R^{-1}(G_Y(G_T^{-1}(p))) \tag{7}$$

This study aims at using the T-R{Y} framework under transformed-transformer method of generating new distributions to generate a new distribution, WGIELLD which generalizes the distribution proposed by [1], to explore the statistical features of the new distribution and demonstrate the flexibility gain in the new distribution in fitting various datasets over its baseline and other related distributions as a result of the generalization.

Many authors had adopted the concept of T-R{Y} in generalizing existing distributions to develop new and more flexible ones. Such authors, among others, include:

Alzaatreh, Lee, and Famoye [6], Almheidat, Famoye, and Lee [4], Alzaatreh et al. [7], Tahir et al. [23], Alzaatreh, Lee, and Ghosh [8], Nasir et al. [21], Aldeni, Lee, and Famoye [2], Jamal et al. [15], Hamed, Famoye, and Lee [13], Alzaghal, and Hamed [9], Ekum, Adamu, and Akwarawak [10], Osatohanmwen et al. [22] Ibeh et al. [14]

The subsequent sections of this research work is set up in the following way: section 2 explains the derivation of WGIELLD, which is a special case of a broader range of distributions. This section also covers some statistical features of WGIELLD and the estimation of its parameters utilizing likelihood function maximization method. Section 3 includes a simulation study to determine whether the MLE approach is acceptable and the demonstration of the gain in flexibility as a result of the generalization, using real life data sets. Section 4 brings the article to a close by presenting the conclusion.

2. Derivation of wgielld and its statistical features

2.1 *Wgielld*

Suppose the continuous random variable, T defined in the T-R{Y} family is distributed as Weibull distribution [24] with parameters, δ and ϖ, then the cdf, pdf, and qf are obtained, correspondingly as:

$$G_T(z) = 1 - \exp\left(-\left\{z/\delta\right\}^{\varpi}\right) \tag{8}$$

$$g_T(z) = \frac{d(G_T(z))}{dz} = \frac{\varpi}{\delta}\left(\frac{z}{\delta}\right)^{\varpi-1} \exp\left(-\left\{z/\delta\right\}^{\varpi}\right);\ \delta,\ \varpi > 0,\ z > 0 \tag{9}$$

$$G_T^{-1}(p) = Q_T(p) = \delta(-\log(1-p))^{1/\varpi} \tag{10}$$

Similarly, if R follows GIED with the cdf and pdf respectively expressed as in (1) and (2), then qf is obtained as

$$G_R^{-1}(p) = Q_R(p) = -\varphi(\log(1-(1-p)^{1/\vartheta}))^{-1} \tag{11}$$

If the continuous random variable denoted with Y is assumed to be distributed as standard logarithmic logistic distribution [11], then the cdf, pdf and qf are respectively given as

$$G_Y(z) = (1+z^{-1})^{-1} \tag{12}$$

$$g_Y(z) = \frac{1}{(1+z)^2}; z > 0 \tag{13}$$

$$G_Y^{-1}(p) = Q_Y(p) = p(1-p)^{-1} \tag{14}$$

Using (1), (8) and (14) in (5), the cdf of WGIELLD is obtained as

$$F_Z(z) = 1 - \exp(-\{\delta^{-1}[1-e^{-\varphi z^{-1}}]^{-\vartheta} - 1\}^{\varpi}) \tag{15}$$

Using (2), (9) and (13) in (6), the pdf of WGIELLD is derived as

$$f_Z(z) = \frac{\vartheta\varpi\varphi}{\delta^{\varpi} z^2}(1-e^{-\varphi z^{-1}})^{-\vartheta-1}[(1-e^{-\varphi z^{-1}})^{-\vartheta} - 1]^{\varpi-1} e^{-\varphi z^{-1}} \exp(-\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta} - 1)\}^{\varpi}) \tag{16}$$

The scale parameters of WGIELLD are φ and δ, whereas ϖ and ϑ are the shape parameters.

2.2 *Useful Expansions of Distribution and Density Functions of WGIELLD*

Utilizing Taylor's series expansion for the exponential function and extended binomial theorem on the term that appears second on right side of (15), we have

$$\exp(-\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta} - 1)\}^{\varpi}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\delta^{\varpi m}}\{[1-e^{-\varphi z^{-1}}]^{-\vartheta} - 1\}^{\varpi m}$$

$$\{[1-e^{-\varphi z^{-1}}]^{-\vartheta} - 1\}^{\varpi m} = \sum_{s=0}^{\infty} (-1)^s \binom{\varpi m}{s}[1-e^{-\varphi z^{-1}}]^{-\vartheta s}; [1-e^{-\varphi z^{-1}}]^{-\vartheta s}$$

$$= \sum_{w=0}^{\infty} \binom{\vartheta s + w - 1}{w} e^{-\varphi w z^{-1}}$$

Thus, $\exp(-\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta} - 1)\}^{\varpi}) = \sum_{m,s,w=0}^{\infty} \frac{(-1)^{m(1+\varpi)+s}}{m!\delta^{\varpi m}} e^{-\varphi w z^{-1}}$

$$\therefore, F_Z(z) = 1 - \sum_{m,s,w=0}^{\infty} \frac{(-1)^{m(1+\varpi)+s}}{m!\delta^{\varpi m}} \binom{\varpi m}{s}\binom{\vartheta s + w - 1}{w} e^{-\varphi w z^{-1}} \tag{17}$$

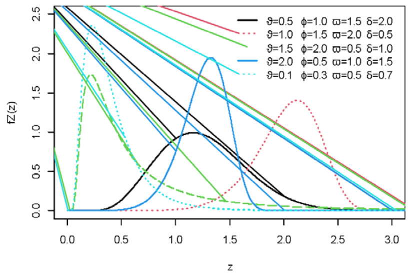

Figure 1

Plots of WGIELLD pdf

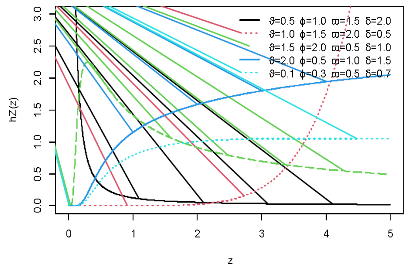

Figure 2

Plots of WGIELLD hf

Again, with use of Taylor's series expansion for the exponential function and extended binomial theorem in (16), we obtain

$$\exp(-\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)\}^{\varpi}) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\delta^{-\varpi m}\{[1-e^{-\varphi z^{-1}}]^{-\vartheta}-1\}^{\varpi m}$$

$$([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)^{\varpi(1+m)-1} = \sum_{s=0}^{\infty}(-1)^s\binom{\varpi(1+m)-1}{s}[1-e^{-\varphi z^{-1}}]^{-\vartheta s}$$

$$[1-e^{-\varphi z^{-1}}]^{-\vartheta(1+s)-1} = \sum_{w=0}^{\infty}\binom{\vartheta(1+s)+w}{w}e^{-w\varphi z^{-1}}; e^{-\varphi z^{-1}}e^{-\varphi w z^{-1}} = e^{-\varphi(1+w)z^{-1}}$$

Thus, $$f_Z(z) = \vartheta\varpi\varphi A z^{-2}\,\mathrm{e}^{-\varphi(1+w)z^{-1}} \tag{18}$$

where $$A = \sum_{m,s,w=0}^{\infty}\frac{(-1)^{m+\varpi(1+m)+s-1}}{m!\,\delta^{\varpi(1+m)}}\binom{\varpi(1+m)-1}{s}\binom{\vartheta(1+s)+w}{w}$$

Figure 1 reveals that WGIELLD can model datasets that are approximately normal, right and left skewed.

2.3 *Statistical Features of WGIELLD*

2.3.1 Reliability Function

Subtracting (15) from one, the rf $(R_Z(z))$ of the WGIELLD is expressed as

$$R_Z(z) = \exp(-\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)\}^{\varpi}) \tag{19}$$

2.3.2 Hazard Function

Dividing (16) by (19), the hf $(h_Z(z))$ of WGIELLD is

$$h_Z(z) = \vartheta\varpi\varphi z^{-2}\delta^{-\varpi}(1-e^{-\varphi z^{-1}})^{-(\vartheta+1)}[(1-e^{-\varphi z^{-1}})^{-\vartheta}-1]^{\varpi-1}e^{-\varphi z^{-1}} \tag{20}$$

The shapes of the graphs of the hf of WGIELLD for certain parameter values depicted in Figure 2 show that the distribution can model events with various shapes of failure rate.

2.3.3 Shapes

Remember, the mode of a distribution is the value of x, say m_0 for which $\frac{df_Z(z)}{dz}=0$ and $\frac{d^2 f_Z(z)}{dz^2}<0$ or $\frac{d\log(f_Z(z))}{dz}=0$ and $\frac{d^2\log(f_Z(z))}{dz^2}<0$

From (16), $\log(f_Z(z)) = \log(\vartheta\varpi\varphi) - \varpi\log(\delta) - 2\log(z) - (1+\vartheta)$

$$\log(1-e^{-\varphi z^{-1}}) + (\varpi-1)\log[(1-e^{-\varphi z^{-1}})^{-\vartheta}-1] - \frac{\varphi}{z} - \{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)\}^{\varpi}$$

$$\frac{d\log(f_Z(z))}{dz} = -\frac{2}{z} + \frac{\varphi(\vartheta+1)e^{-\varphi z^{-1}}}{z^2(1-e^{-\varphi z^{-1}})} + \frac{\varphi\vartheta(\varpi-1)(1-e^{-\varphi z^{-1}})^{-(\vartheta+1)}e^{-\varphi z^{-1}}}{z^2[(1-e^{-\varphi z^{-1}})^{-\vartheta}-1]} + \frac{\varphi}{z^2} -$$
$$\frac{\vartheta\varphi\varpi}{\delta^{\varpi}z^2}[(1-e^{-\varphi z^{-1}})^{-\vartheta}-1]^{\varpi-1}(1-e^{-\varphi z^{-1}})^{-(\vartheta+1)}e^{-\varphi z^{-1}} \tag{21}$$

$$\frac{d^2\log(f_Z(z))}{dz^2} = \frac{2}{z^2} - \frac{2\varphi}{z^3} + \varphi(\vartheta+1)\left(\frac{\varphi e^{\varphi z^{-1}}}{z^4(e^{\varphi z^{-1}}-1)^2} - \frac{2}{x^3(e^{\varphi z^{-1}}-1)}\right) +$$
$$\frac{\varphi\vartheta(\varpi-1)(1-e^{-\varphi z^{-1}})^{-(\vartheta+1)}e^{-\varphi z^{-1}}}{z^2((1-e^{-\varphi z^{-1}})^{-\vartheta}-1)} \times \left[\frac{\varphi(\vartheta+1)e^{-\varphi z^{-1}}}{z^2(1-e^{-\varphi z^{-1}})} + \frac{\varphi}{z^2} - \frac{2}{z} - \frac{\vartheta\varphi e^{-\varphi z^{-1}}}{z^2(1-e^{-\varphi z^{-1}})^{\vartheta+1}}\right] -$$
$$\frac{\vartheta\varphi\varpi}{\delta^{\varpi}z^2}((1-e^{-\varphi z^{-1}})^{-\vartheta}-1)^{\varpi-1}(1-e^{-\varphi z^{-1}})^{-(\vartheta+1)} \times$$
$$e^{-\varphi z^{-1}}\left[\frac{\varphi}{z^2} - \frac{2}{z} + \frac{\vartheta\varphi(\varpi-1)(1-e^{-\varphi z^{-1}})^{-(\vartheta+1)}e^{-\varphi z^{-1}}}{z^2((1-e^{-\varphi z^{-1}})^{-\vartheta}-1)} + \frac{\varphi(\vartheta+1)}{z^2(e^{\varphi z^{-1}}-1)}\right] \tag{22}$$

Let $\zeta(z) = \frac{d^2\log(f_Z(z))}{dz^2}$

If $z=m_0$ satisfies (21), then it shows a point of local maxima if $\zeta(z)>0$ for every $z<m_0$ and $\zeta(z)<0$ for every $z>m_0$. It shows a point of local maxima if $\zeta(z)<0$ for every $z>m_0$ and $\zeta(z)>0$ for every $z>m_0$. It represents a vertex of curvature change if either $\zeta(z)>0$ for every $z\neq m_0$ or $\zeta(z)<0$ for every $x\neq m_0$

2.3.4 Quantile Function

Theorem 1: *The pth quantile, z_p of WGIELL random variable is defined as*

$$z_p = Q_Z(p) = -\varphi(\log[1-\{1+\delta[\log(1-p)^{-1}]^{1/\varpi}\}^{-1/\vartheta}])^{-1},\ \ 0<p<1 \tag{23}$$

Proof: Using (15), we have

$$p = F(z_p) = 1 - \exp(-\{\delta^{-1}([1 - e^{-\varphi z_p^{-1}}]^{-\vartheta} - 1)\}^{\varpi})$$
$$\exp(-\{\delta^{-1}([1 - e^{-\varphi z_p^{-1}}]^{-\vartheta} - 1)\}^{\varpi}) = 1 - p$$
$$z_p = Q_Z(p) = -\varphi[\log(1 - \{1 + \delta(\log(1-p)^{-1})^{1/\varpi}\}^{-1/\vartheta})]^{-1}$$

Substituting p=0.5 in (23), the median is

$$Q_Z(0.5) = -\varphi(\log(1 - (1 + \delta(\log(2))^{1/\varpi})^{-1/\vartheta}))^{-1} \quad (24)$$

2.3.5 Rényi Entropy

Theorem 2: *The R_e of the WGIELLD is expressed as*

$$\eta_R = \frac{1}{1-\rho}\log\left[\sum_{m,s,w=0}^{\infty}\frac{(\vartheta\varpi\varphi)^{\rho}\rho^{m}}{m!\delta^{\varpi(\rho+m)}}\binom{\varpi(\rho+m)-\rho}{s}\binom{\vartheta(1+s)+\rho+w-1}{w}((\rho+w)\varphi)^{1-2\rho}\Gamma(2\rho-1)\right] \quad (25)$$

Proof:

$$\eta_R = \frac{1}{1-\rho}\log\left\{\int_0^{\infty}[f_Z(z)]^{\rho}dz\right\} \quad (26)$$

$$\int_0^{\infty}\left[f_Z(z)\right]^{\rho}dz = \int_0^{\infty}\left(\frac{\vartheta\varpi\varphi}{z^2\delta^{\varpi}e^{\varphi z^{-1}}}\right)^{\rho}(1-e^{-\varphi z^{-1}})^{-\rho(\vartheta+1)}[(1-e^{-\varphi z^{-1}})^{-\vartheta}-1]^{\rho(\varpi-1)}$$
$$\exp(-\rho\{\delta^{-1}[(1-e^{-\varphi z^{-1}})^{-\vartheta}-1]\}^{\rho})dz$$

$$= \left(\frac{\vartheta\varpi\varphi}{\delta^{\varpi}}\right)^{\rho}\sum_{m,s,w=0}^{\infty}\frac{(\delta^{-\varpi}\rho)^{m}(-1)^{m+\varpi(1+m)+s-1}}{m!}\binom{\varpi(\rho+m)-\rho}{s}\binom{\vartheta(1+s)+\rho+w-1}{w}\int_0^{\infty}z^{-2\rho}e^{-\varphi(\rho+w)z^{-1}}dz \quad (27)$$

$$\int_0^{\infty}z^{-2\rho}e^{-\varphi(\rho+w)z^{-1}}dx = ((\rho+w)\varphi)^{1-2\rho}\int_0^{\infty}u^{2(\rho-1)}e^{-u}du = ((\rho+w)\varphi)^{1-2\rho}\Gamma(2\rho-1) \quad (28)$$

Putting (28) into (27) results to

$$\left(\frac{\vartheta\varpi\varphi}{\delta^{\varpi}}\right)^{\rho}\sum_{m,s,w=0}^{\infty}\frac{(\delta^{-\varpi}\rho)^{m}(-1)^{m+\varpi(1+m)+s-1}}{m!}\binom{\varpi(\rho+m)-\rho}{s}\binom{\vartheta(1+s)+\rho+w-1}{w}((\rho+w)\varphi)^{1-2\rho}\Gamma(2\rho-1) \quad (29)$$

putting (29) into (26), we have (25)

(Also see, Kumar [18] for more on Renyi's entropy)

2.3.6 Moment

Theorem 3: *The mgf* $(M_Z(t))$ *of the WGIELLD is*

$$M_Z(t) = \vartheta\varpi A \sum_{k=0}^{\infty} \frac{(\varphi t)^k (-1)^k}{k!(1+w)^{1-k}} \Gamma(1-k) \quad (30)$$

Proof:

$$M_Z(t) = E[\exp(tZ)] = \int_0^{\infty} \exp(tz) dF_Z(z)$$

$$M_Z(t) = \int_0^{\infty} e^{tz} \left(\sum_{m,s,w=0}^{\infty} \frac{\vartheta\varpi\varphi(-1)^{m+\varpi(m+1)+s-1}}{m!\delta^{\varpi(m+1)}} \binom{\varpi(m+1)-1}{s} \binom{\vartheta(1+s)+w}{w} z^{-2} e^{-\varphi(1+w)z^{-1}} \right) dz \quad (31)$$

$$e^{tz} = \sum_{k=0}^{\infty} \frac{t^k z^k}{k!}$$

Substituting the above series representation into (31) and simplifying, give

$$M_Z(t) = \sum_{m,s,w,k=0}^{\infty} \frac{\vartheta\varpi\varphi t^k (-1)^{m+\varpi(1+m)+s-1}}{k!m!\delta^{\varpi(1+m)}} \binom{\varpi(m+1)-1}{s} \binom{\vartheta(s+1)+w}{w} \int_0^{\infty} z^{k-2} e^{-\varphi(1+w)z^{-1}} dx \quad (32)$$

Recall inverse gamma distribution, $\int_0^{\infty} z^{-\vartheta-1} e^{-\delta z^{-1}} dz = \frac{\Gamma(\vartheta)}{\delta^{\vartheta}}$

Thus, $\int_0^{\infty} z^{k-2} e^{-\varphi(1+w)z^{-1}} dx = \frac{\Gamma(1-k)}{((1+w)\varphi)^{1-k}}$ [12] (33)

Putting (33) into (32) and simplifying result to (30)

2.3.7 Mean Absolute Deviations (MAD)

Theorem 4: *The MAD from the mean* $(D(\mu))$ *and median* $(D(\tilde{Z}))$ *for the WGIELLD are respectively obtained as*

$$D(\mu) = 2\mu F_X(\mu) - 2 \sum_{k,m,s,w=0}^{\infty} \frac{\vartheta\varpi(1+w)^k \varphi^{1+k} (-1)^{m+\varpi(m+1)+s+k-1}}{kk!m!\mu^k \delta^{\varpi(m+1)}} \binom{\varpi(m+1)-1}{s} \binom{\vartheta(s+1)+w}{w} \quad (34)$$

$$D(\tilde{X}) = \mu - 2 \sum_{k,m,s,w=0}^{\infty} \frac{\vartheta\varpi(1+w)^k \varphi^{1+k} (-1)^{m+\varpi(m+1)+s+k-1}}{kk!m!\tilde{x}^k \delta^{\varpi(m+1)}} \binom{\varpi(m+1)-1}{s} \binom{\vartheta(s+1)+w}{w} \quad (35)$$

Proof:

$$D(\mu) = E[\,|Z-\mu|\,] = 2\mu F_Z(\mu) - 2\int_{-\infty}^{\mu} z dF_Z(z) \quad (36)$$

$$D(\tilde{Z}) = E[\,|Z-\tilde{Z}|\,] = \mu - 2\int_{-\infty}^{\tilde{Z}} z dF_Z(z) \tag{37}$$

$$\int_{-\infty}^{c} z dF_Z(z) = \int_{-\infty}^{c} z f_Z(z) dz \;= \int_{-\infty}^{c} z(A\vartheta\varpi\varphi z^{-2} e^{-\varphi(1+w)z^{-1}}) dz \tag{38}$$

$$= A\vartheta\varpi\varphi \int_{-\infty}^{c} z^{-1} e^{-(1+w)\varphi z^{-1}} dz \;\; \text{Let } u = \frac{(1+w)\varphi}{z}$$

$$\int_{-\infty}^{c} z f_Z(z) dz \;= -A\vartheta\varpi\varphi \int_{0}^{\varphi(1+w)c^{-1}} u^{-1} e^{-u} du \tag{39}$$

$$\text{Recall } \int_0^z t^{a-1} e^{-t} dt = \gamma(a,z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{k+a}}{k!(k+a)}$$

$$\int_0^{\varphi(1+w)c^{-1}} u^{-1} e^{-u} du \;= \sum_{k=0}^{\infty} \frac{(-1)^k (\varphi(1+w))^k}{c^k k k!} \tag{40}$$

Substituting equation (40) into equation (39), gives

$$\int_{-\infty}^{c} z dF_Z(z) = \sum_{k=0}^{\infty} \frac{A\vartheta\varpi\varphi^{1+k}(1+w)^k}{k c^k k!} \tag{41}$$

Replacing c in (41) with μ and $\tilde{Z}$, respectively and substituting in (36) and (37) give (34) and (35).

2.3.8 Order Statistics

Theorem 5: *The pdf of pth order statistics of the WGIELLD is*

$$f_{p:n}(z) = \frac{\vartheta\varpi\varphi n!}{(p-1)!(n-p)!} B \sum_{i=0}^{n-p} \binom{n-p}{i} (1+j)^l z^{-2} e^{-\varphi(1+w)z^{-1}} \tag{42}$$

where $B = \sum_{j,l,m,w=0}^{\infty} \binom{p+i-1}{j} \binom{\sigma(1+l)-1}{m} \binom{\vartheta(1+m)+w}{w} \frac{(-1)^{i+j+l+m+w+\varpi(1+l)-1}}{l!\delta^{\varpi(1+l)}}$

Proof:

$$f_{p:n}(z) = \frac{n!}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i [F_Z(z)]^{p-i+1} f_Z(z) \tag{43}$$

Substituting (15) and (16) into (43)

$$f_{p:n}(z) = \frac{n!}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i [1-\exp(-\{\delta^{-1}[(1-e^{-\varphi z^{-1}})^{-\vartheta} - 1]\}^{\varpi})]^{p-i+1} \times$$

$$\frac{\vartheta\varpi\varphi}{z^2} \delta^{-\varpi} (1-e^{-\varphi z^{-1}})^{-\vartheta-1} [(1-e^{-\varphi z^{-1}})^{-\vartheta} - 1]^{\varpi-1} e^{-\varphi z^{-1}} \exp(-\{\delta^{-1}[1-e^{-\varphi z^{-1}}]^{-\vartheta} - 1\}^{\varpi}) \tag{44}$$

$$[1-\exp(-\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)\}^{\varpi})]^{p+i-1}=\sum_{j=0}^{\infty}\binom{p+i-1}{j}(-1)^{j}$$

$$\exp(-j\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)\}^{\varpi})\exp(-(1+j)\{\delta^{-1}([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)\}^{\varpi})$$

$$=\sum_{l=0}^{\infty}\frac{(-1)^{l}(1+j)^{l}}{l!\delta^{\varpi l}}([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)^{\varpi l}$$

$$([1-e^{-\varphi z^{-1}}]^{-\vartheta}-1)^{\varpi(1+l)-1}=\sum_{m=0}^{\infty}\binom{\varpi(1+l)-1}{m}(-1)^{m}(1-e^{-\varphi z^{-1}})^{-\vartheta m}$$

$$(1-e^{-\varphi z^{-1}})^{-(\vartheta(1+m)+1)}=\sum_{w=0}^{\infty}\binom{\vartheta(1+m)+w}{w}(-1)^{w}e^{-\varphi w z^{-1}}$$

Thus

$$f_{p:n}(z)=\frac{\vartheta\varpi\varphi n!}{(p-1)!(n-p)!}B\sum_{i=0}^{n-p}(1+j)^{l}z^{-2}e^{-\varphi(1+w)z^{-1}}$$

2.4 *Estimation of Parameters*

Here, the MLE method will be adopted in determining the parameters of the WGIELLD. Suppose $z_1, z_2, ..., z_n$ constitute a representative sample of n elements from the WGIELLD. Then, the natural logarithm likelihood function, $\ell=\log\left(\prod_{i=1}^{n}f_Z(z)\right)$ of the WGIELLD can be represented as

$$\ell=n\log\vartheta+n\log\varpi+n\log\varphi-2\sum_{i=1}^{n}\log(z_i)-(\vartheta+1)\sum_{i=1}^{n}\log(1-e^{\varphi z_i^{-1}})$$
$$+(\varpi-1)\sum_{i=1}^{n}\log[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]-\frac{\varphi}{\sum_{i=1}^{n}z_i}-\sum_{i=1}^{n}\{\delta^{-1}([1-e^{-\varphi z_i^{-1}}]^{-\vartheta}-1)\}^{\varpi} \tag{45}$$

By computing the partial derivative of (45) with respect to each of the parameters, the elements of the score function $U(\Theta)=(\vartheta,\varphi,\varpi,\delta)^{T}$ are obtained as follows

$$\frac{\partial\ell}{\partial\vartheta}=\frac{n}{\vartheta}-\sum_{i=1}^{n}\log(1-e^{-\varphi z_i^{-1}})-(\varpi-1)\sum_{i=1}^{n}\frac{(1-e^{-\varphi z_i^{-1}})\log(1-e^{-\varphi z_i^{-1}})}{[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]}+$$
$$\frac{\varpi}{\delta^{\varpi}}\sum_{i=1}^{n}((1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1)^{\varpi-1}(1-e^{-\varphi z_i^{-1}})^{-\vartheta}\log(1-e^{-\varphi z_i^{-1}})$$

$$\frac{\partial\ell}{\partial\varphi}=\frac{n}{\varphi}-\sum_{i=1}^{n}\frac{(\vartheta+1)}{z_i(e^{\varphi/z_i}-1)}-\vartheta(\varpi-1)\sum_{i=1}^{n}\frac{(1-e^{-\varphi z_i^{-1}})^{-(\vartheta+1)}e^{-\varphi/z_i}}{z_i[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]}-\frac{1}{\sum_{i=1}^{n}z_i}$$
$$+\frac{\vartheta\varpi}{\delta^{\sigma}}\sum_{i=1}^{n}\frac{[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]^{\varpi-1}e^{-\varphi z_i^{-1}}}{z_i(1-e^{-\varphi z_i^{-1}})^{\vartheta+1}}$$

$$\frac{\partial \ell}{\partial \varpi}=\frac{n}{\varpi}-n\log(\delta)+\sum_{i=1}^{n}\log[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]-\frac{1}{\delta^{\varpi}}\sum_{i=1}^{n}[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]^{\varpi}$$
$$\log\{\delta^{-1}[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]\}$$

$$\frac{\partial \ell}{\partial \delta}=\frac{n\varpi}{\delta}-\frac{\varpi}{\delta^{\varpi+1}}\sum_{i=1}^{n}[(1-e^{-\varphi z_i^{-1}})^{-\vartheta}-1]^{\varpi}$$

By numerically solving the non-linear equations $U(\Theta)=0$ using Quasi-Newton method in R software, we will obtain the MLEs $((\hat{\vartheta},\hat{\varphi},\hat{\varpi},\hat{\delta})^T)$ of the parameter vector $U(\Theta)=(\vartheta,\varphi,\varpi,\delta)^T$.

3. Results and Discussion

3.1 *Simulation Study*

Here, the Monte-Carlo simulation of different sample sizes (n_i,i=1,2,…,5) and parameter values $(\vartheta,\varphi,\varpi,\delta)$ is undertaken to inspect the effectiveness of MLE method in estimating the parameter values of the WGIELLD. Using R, the simulation was repeated 3000 times with n_1=50, n_2=100, n_3=200, n_4= 500, n_5=1000 for a set of actual parameter values, $(\vartheta=0.5,\ \varphi=0.3,\ \varpi=0.3,\ \delta=0.2)$. For each of the four parameter values, the following three quantities are computed

(i) Mean Estimate (ME) $=\dfrac{\sum_{i=1}^{3000}\hat{\mu}_i}{3000}$

(ii) Average Bias (AB) $=\dfrac{\sum_{i=1}^{3000}(\hat{\mu}_i-\mu)}{3000}$

(iii) Root Mean Sqaure Error ($RMSE$) $=\sqrt{\dfrac{\sum_{i=1}^{3000}(\hat{\mu}_i-\mu)^2}{3000}}$

Table 1 displays the values of these quantities obtained. As shown in the table, AB and RMSE reduce with increase in sample volume. Therefore, the MLE method is a robust technique for parameter estimation of the proposed distribution.

3.2 *Practical Uses of WGIELLD*

In this segment, the WGIELLD is utilized to analyze two sets of life data in order to showcase its adaptability and effectiveness compared to other similar distributions in fitting certain data sets. In order to compare the distribution's fit to the data sets to other distributions, a number of

Table 1

Results of Simulation by Monte-Carlo Method

Sample size	parameter	ME	AB	RMSE
50	ϑ	0.4376	-0.0624	0.3256
	φ	0.2235	-0.0765	0.1249
	ϖ	0.4363	0.1363	0.2194
	δ	0.3422	0.1422	0.4045
100	ϑ	0.3881	-0.1119	0.2501
	φ	0.2238	-0.0762	0.1078
	ϖ	0.4181	0.1181	0.1790
	δ	0.2536	0.0536	0.1871
200	ϑ	0.3703	-0.1297	0.2096
	φ	0.2267	-0.0733	0.0952
	ϖ	0.3993	0.0993	0.1404
	δ	0.2247	0.0247	0.1080
500	ϑ	0.3817	-0.1183	0.1615
	φ	0.2394	-0.0606	0.0742
	ϖ	0.3727	0.0727	0.0944
	δ	0.2139	0.0139	0.0644
1000	ϑ	0.3965	-0.1035	0.1319
	φ	0.2473	-0.0527	0.0623
	ϖ	0.3583	0.0583	0.0721
	δ	0.2199	0.0140	0.0454

performance metrics, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Sirmnorv (K-S), and probability values, are used.

Data set I: The times (mins) at which the air conditioning system of an airplane stops working correctly [19]

23,261,87,7,120,14,62,47,225,71,246,21,42,20,5,12,120,11,3,14,71,11,14,1 1,16,90,1,16, 52,95

Data set II: Recorded times(hrs) to repair a particular kind of electronic system without regard to failure mode [20] (Values in parentheses are the frequencies)

1(3), 2(13), 3(10), 4(6), 5(6), 6(9), 7(8) 8(4) 9(8), 10(4), 11(5), 12(2), 13(3), 14(2),15, 16(4), 17(3), 18(3), 19(2), 20, 21, 22, 23(5), 24(2), 25, 26(2), 29, 30(2), 31(2),38, 39, 43, 53, 60

Table 2

Evaluation and Effectiveness of the Distributions in Fitting Data set I

Distributions $\Rightarrow$	WGIELLD	GIED	EGIED	IED
Parameter Estimate	$\hat{\vartheta}=0.7647$ (4.7378) $\hat{\varphi}=1.2376$ (13.8973) $\hat{\varpi}=2.6119$ (13.0885) $\hat{\delta}=4.8746$ (21.5497)	$\hat{\vartheta}=5.3567$ (1.7468) $\hat{\varphi}=19.3633$ (3.1956)	$\hat{\vartheta}=11.6689$ (4.5510) $\hat{\varphi}=41.8058$ (8.0847) $\hat{\delta}=0.3477$ (0.1075)	$\hat{\vartheta}=8.0875$ (1.3120)
LL	-112.4847	-116.1925	-114.3776	-129.2641
AIC	232.9694	236.3851	234.7552	260.5282
BIC	239.5198	239.6603	239.6680	262.1657
K-s	0.0677	0.1294	0.0985	0.3312
p-value	0.9902	0.5071	0.8199	0.0003

(EGIED=Exponentiated GIED)

Table 3

Evaluation and Effectiveness of the Distributions in Fitting Data set II

Distributions $\Rightarrow$	WGIELLD	GIED	EGIED	IED
Parameter Estimate	$\hat{\vartheta}=1.4103$ (1.2756) $\hat{\varphi}=4.4822$ (3.6008) $\hat{\varpi}=0.7698$ (0.4612) $\hat{\delta}=4.3025$ (5.8945)	$\hat{\vartheta}=1.3928$ (0.1830) $\hat{\varphi}=6.5039$ (0.7397)	$\hat{\vartheta}=3.8682$ (0.7655) $\hat{\varphi}=34.2111$ (0.0044) $\hat{\delta}=0.1898$ (0.0184)	$\hat{\vartheta}=5.2743$ (0.4835)
LL	-405.8595	-415.3119	-412.2988	-418.3714
AIC	819.7190	834.6238	830.5976	838.7428
BIC	830.8355	840.1821	838.9349	841.5219
K-s	0.0620	0.1180	0.1288	0.1144
p-value	0.7266	0.0670	0.0352	0.0818

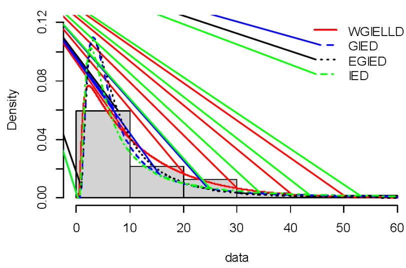

Figure 3

The distributions' fitted PDFs are shown for the first dataset.

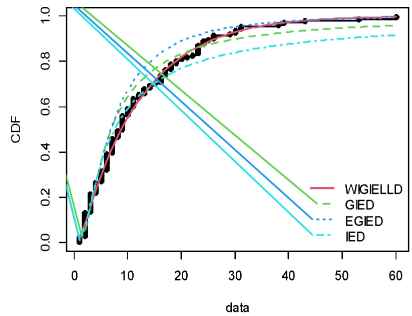

Figure 4

An illustration of the distributions' fitted CDFs to the first dataset

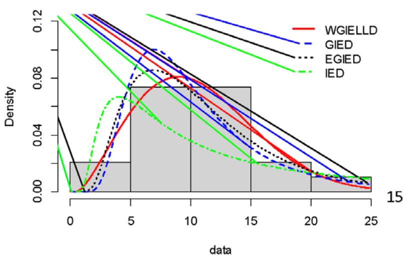

Figure 5

The distributions' fitted PDFs are shown for the second dataset.

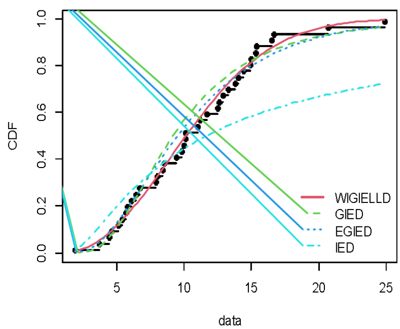

Figure 6

An illustration of the distributions' fitted CDFs to the second dataset

The standard errors of the calculated parameters for the distributions under consideration are shown in Tables 2 and 3 by the numbers in parenthesis. In this paper, the T-R{Y} framework is used to further generalize the generalized inverse exponential distribution (GIED). There are other methods of generalizing a distribution, such as exponentiation method. Here, we compared the performance of a distribution in fitting data sets when generalized using T-R{Y} (i.e WGIELLD) and exponentiation (EGIED) methods. The comparison is extended to their baseline distribution, the GIED as well as its parent distribution (IED). From Tables 2 and 3, it is observed that the values of log-likelihood (LL) and the probability values for the WGIELLD are higher than those of the competing models. Again, in comparison to the competing models, the WGIELLD has the least values of AIC, BIC and K-S. Consequently, among the models under consideration, the WGIELLD is the best distribution in fitting the two data sets. More so, in line with the numerical values obtained, the graphical representations of the data sets are better fitted by the WGIELLD as shown in Figures 3, 5, 4 and 6 compared to the other models.

4. Conclusion

A novel univariate continuous distribution called the Weibull-Generalized Inverted Exponential {Log-logistic} distribution (WGIELLD) is derived in this research employing the T-R{Y} framework proposed by [3] with the transformed-transformer method of generating new continuous distributions. The WGIELLD is a specific instance within the T-GIELLD family when the continuous random variable, T follows Weibull distribution. Certain statistical features of the distribution are derived and the WGIELLD is used to fit various datasets. Results show that the WGIELLD outperformed its parent distribution, (GIED) and other related distributions in modeling natural phenomena.

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