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Two classes of two-phase sampling estimators for the ratio of two population means using stratified random sampling in the presence of non-response

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Abstract

This paper introduce two distinct classes of estimators to calculate the ratio between two population means by utilizing the knowledge of two-phase sampling with in the stratified population, particularly addressing the situation of non-response in variables under the study. The expressions related to the description of bias and mean square error (MSE) of the given classes of two-phase method of sampling estimators for ratio between two population means (R) have been derived. Members of the suggested classes have been identified. An empirical analysis using a data set (census-2011) has been conducted to

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validate the effectiveness of the suggested estimators alongside the pertinent estimator. For the fixed cost, the optimized sample size of 1st-phase, 2nd-phase and sub-sampling fraction i.e. (n_i'), (n_i) and $(1/g_i)$ respectively for i^{th} stratum have been obtained.

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Keywords: *Auxiliary variable, Ratio of means, Stratified population, Two-phase, Non-response.*

1. Introduction

The utilization of auxiliary information(s) in sample surveys typically involves combining it with survey data through statistical methods such as regression modeling, ratio estimation, or post-stratification. In nearly all the surveys, the auxiliary information(s) are often collected or obtained from sources other than the surveys, and external studies. The auxiliary information can be made available in various forms such as the value of one or more parameters e.g. mean and variance of population, coefficient of variation, skewness and kurtosis, etc. of the related variables, depending on the source and nature of the data.

The challenges of estimation to determining the ratio between two population means holds significant importance in both Scientific and Social research surveys. In a medical research study, one method involves deriving the growth index through the estimation of the ratio of weight to height, while taking hip/chest/arm circumferences as an additional characteristic.

Singh [20], Tripathi [24], Singh [21], khare [6] and Upadhayaya et al. [25] have all explored the estimation challenge of ratio of two population means by incorporating supplementary variable.

In a Sample survey, the encountering of non-response phenomenon is a frequent issue in practice. Hansen and Hurwitz [3] have suggested a technique to minimize the effect of non-response by picking a small sub-sample from the non-responding part. Further modifications to make the less impact of non-response have been done by El-Badry [2]. Rao [17, 18] and Khare and Srivastava [7, 9] have suggested some estimators for the population mean in the scenario of non-response. Khare [5] has proposed an estimator for the population mean by taking optimal allocation despite the existence of non-response for the stratified population. Khare and Panday [8], Khare and Sinha [12] and Khare et al. [13] have all examined the challenge of estimating the ratio between two population means by make use of the population mean (Known) of auxiliary characteristic in the context of non-response.

When the mean of supplementary variable within the population isn't known, we employ a two-phase sampling method to estimate this average by taking a larger sample from the population. For finite populations, Khare and Sinha [10, 11, 14] have explored the estimators for ratio between two population means for unknown population mean $(\bar{X})$ of additional information. This study is particularly focused on scenarios where there exist non-response on study variables $(\bar{Y}_m(m=1,2))$.

In the case of stratified population, Cekim & Kadilar [4] have suggested the unbiased estimators and Zaman & Kadilar [23] have defined estimators for population mean utilizing supplementary information and Chaudhary et al. [1] have suggested an estimator for the population mean in stratified random sampling in the scenario of non-response with an unknown population mean of additional variable. Khare and Jha [15] and Singh et al. [19] have proposed several classes of estimators aiming to determine the population mean within a stratified population, considering the scenarios for known and unknown population mean $\bar{X}$. And also consider the scenario of non-response on study character.

In present study, we have proposed two classes of estimators for the ratio between two population means in existence of non-response with an unknown population mean of the auxiliary information in stratified random sampling. The expressions for the bias (u_j) and MSE (u_j) of proposed classes of estimators have derived. The identification of the Members of the suggested classes are also done. The precision of the proposed classes of estimators with estimator $(\hat{R}_{st})$ have been studied for the fixed sample sizes (n_i', n_i) and sub-sampling fraction $(1/g_i)$ for each stratum. The expressions of optimum sizes of (n_i', n_i) and sub-sampling fraction $(1/g_i)$ are also determined.

2. Notations and Sampling methodology:

A population $\phi:(\phi_1,\phi_2,\phi_3.......\phi_N)$ has been considered for the study variable y_m $(m=1,2.)$ and auxiliary variable x with respective population means $\bar{Y}_m(m=1,2.)$ and (unknown) $\bar{X}$. Let assume that the size of population is N to be stratified into L homogeneous strata. The size of i^{th} stratum is denoted by $N_i(i=1,2,3....L)$.

For estimating the unknown value of $\bar{X}$, we employ the methodology of two-phase sampling. In sampling strategy of two-phase, we choose a large sample of size n_i' from i^{th} stratum of the population of size N_i by using the sampling method i.e. simple random sampling without replacement (SRSWOR). Let the sample mean for the variable (x) for n_i'

units is denoted by $\bar{x}_i'$. Further, from each n_i' units, comparatively small sample of size $n_i (< n_i')$ by using the SRSWOR has been drawn and it has been also remarked that n_{i1} responding units and $n_{i(2)}$ units do not respond from n_i sub-sampled units. Further, from $n_{i(2)}$ units a sub-sample of size $r_i \left(= \frac{n_{i(2)}}{g_i}, g_i > 1\right)$ is taken from each stratum and on the basis of $(n_{i1} + r_i)$ units, we observe the study characters y_1, y_2 and the auxiliary character x. The weights of stratum for response group and non-response groups are denoted by $W_{i1} = \frac{N_{i1}}{N}$ and $W_{i2} = \frac{N_{i(2)}}{N}$ in the i^{th} stratum and their estimates are given by $w_{i1} = \frac{n_{i1}}{n}$ and $w_{i2} = \frac{n_{i(2)}}{n}$ respectively for each stratum.

Now, the estimator for $\bar{Y}_m (m = 1, 2)$ on support of information available on $(n_{i1} + r_i)$ units is given by using the Hansen and Hurwitz [3] technique as follows:

$$\bar{y}_{mi}^* = \frac{n_{i1}}{ni} \bar{y}_{mi(1)} + \frac{n_{i(2)}}{ni} \bar{y}'_{mi(2)}, (m = 1, 2) \tag{2.1}$$

Where, $\bar{y}_{mi(1)}$ and $\bar{y}'_{mi(2)}$ represent mean of sample for the variable $y_m (m = 1, 2)$ based on n_{i1} units and r_i units from the i^{th} stratum, $i = 1, 2, 3 ... L$.

For the estimator $\bar{y}_{mi}^*$ the expression of variance up to the term of order $\frac{1}{n}$ is given by:

$$V(\bar{y}_{mi}^*) = \lambda_i S_{y_{mi}}^2 + \frac{W_{i2}(g_i - 1)}{n_i} S_{y_{mi}(2)}^2, \tag{2.2}$$

where, $S_{y_{mi}}^2$ is mean square error of population for the variable y_m $(m = 1, 2)$ in the i^{th} stratum and $S_{y_{mi}(2)}^2$ is the mean square error of population based on r_i sub-sampled units from the non-responding units for the variable y_m $(m = 1, 2)$ in the i^{th} stratum and $\lambda_i = \left(\frac{1}{n_i} - \frac{1}{N_I}\right)$.

The sample means in stratified population for $\bar{Y}_m$ $(m = 1, 2)$ and $\bar{X}$ in scenario of the non-response are given by:

$$\bar{y}_{mst}^* = \sum_{i=1}^{L} W_i \bar{y}_{mi}^*, \ \bar{x}_{st}^* = \sum_{i=1}^{L} W_i \bar{x}_i^*, \text{ for } (m = 1, 2) \tag{2.3}$$

where, $\bar{x}_i^* = \frac{n_{i1}}{ni} \bar{x}_{i(1)} + \frac{n_{i(2)}}{ni} \bar{x}'_{i(2)}$ and known $W_i = \frac{N_i}{N}$.

The mean $\bar{x}'_i$ of the sample based on first-phase sample (n'_i) selected from N_i units of the stratum i^{th} for x_i is given as below:

$$\bar{x}'_i = \frac{1}{n'_i}\sum_{i=1}^{ni} x_{ij} \tag{2.4}$$

For stratified population, the mean of sample for estimating $\bar{X}$ on the basis of n'_i units is given as follows:

$$\bar{x}'_{st} = \sum_{i=1}^{L} W_i \bar{x}'_i \ , \tag{2.5}$$

Under the stratified population, in the stratum i^{th}, the sample mean for n_i units is given as follows:

$$\bar{x}_{st} = \sum_{i=1}^{L} W_i \bar{x}_i . \tag{2.6}$$

3. Classes of estimators [u_j for j = 1,2] under the study:

Let for the study variables y_1 and y_2, $R\left(=\bar{Y}_1 / \bar{Y}_2\right)$ represents the ratio for two population means. Here, we propose two different classes of estimators to estimate R with incorporation of auxiliary information with an unknown mean of population $(\bar{X})$ in scenario of stratified population in the existence of non-response.

Case (i): Non-response on $y_{mi}\,(m=1,2)$ and corresponding units of auxiliary variable x for unknown $\bar{\boldsymbol{X}}$.

$$u_1 = t_{(1)}(\hat{R}_{st}, s_1) \tag{3.1}$$

such that, $t_{(1)}(R,1) = R$, $t_{1(1)}(R,1) = 1$ and $t_{1(1)}(R,1) = \frac{\partial}{\partial \hat{R}_{st}} t_{(1)}(\hat{R}_{st}, s_1)$. (3.2)

Case (ii): Non-response on $y_{mi}\,(m=1,2)$ and complete response in auxiliary variable x for unknown $\bar{\boldsymbol{X}}$.

$$u_2 = t_{(2)}(\hat{R}_{st}, s_2), \tag{3.3}$$

such that, $t_{(2)}(R,1) = R$, $t_{1(2)}(R,1) = 1$ and $t_{1(2)}(R,1) = \frac{\partial}{\partial \hat{R}_{st}} t_{(2)}(\hat{R}_{st}, s_2)$, (3.4) where, $\hat{R}_{st} = \frac{\bar{y}_{1st}^{*}}{\bar{y}_{2st}^{*}}$, $s_1 = \frac{\bar{x}_{st}^{*}}{\bar{x}'_{st}}$ and $s_2 = \frac{\bar{x}_{st}}{\bar{x}'_{st}}$.

The function $t_{(j)}(\hat{R}_{st}, s_j)$, $\forall\ j = 1, 2$ holds the regularity conditions which are given bellow:

- For any strategy of sampling Q whatever be the selected sample, the function $t_{(1)}(\hat{R}_{st}, s_1)$ and $t_{(2)}(\hat{R}_{st}, s_2)$ takes values within a bounded, closed convex subset Q of two-dimensional real spaces having the point $(\theta = R, 1)$. .
- The partial derivatives for 1st and 2nd-order for the function $t_{(1)}(\hat{R}_{st}, s_1)$ and $t_{(2)}(\hat{R}_{st}, s_2)$ exist, and the function $t_{(1)}(\hat{R}_{st}, s_1)$ and $t_{(2)}(\hat{R}_{st}, s_2)$ are bounded and continuous in Q.

(3.5)

Expanding the functions $t_{(1)}(\hat{R}_{st}, s_1)$ and $t_{(2)}(\hat{R}_{st}, s_2)$ about the point $\theta(= R, 1)$ upto the partial derivatives of the 2nd order by using Taylor's series and using the conditions of regularity given in equation (3.5), we have

$$u_j = t_{(j)}(\theta) + (\hat{R}_{st} - R)t_{1(j)}(\theta) + (s_j - 1)t_{2(j)}(\theta)$$
$$+\frac{1}{2}\{(\hat{R}_{st} - R)^2 t_{11(j)}(\theta^*) + (s_j - 1)^2 t_{22(j)}(\theta^*) + 2(\hat{R}_{st} - R)(s_j - 1)t_{12(j)}(\theta^*)\} \qquad (3.6)$$

for $j = 1, 2$.

Using the condition given in equations (3.2) and (3.4) and likely to be neglected, substituting $t_{11(j)}(\theta^*) = 0$ and neglecting $t_{22(j)}(\theta^*)$ considering that it is very small in (3.6). The expression for u_j, $j = 1, 2$ is given as follows:

$$u_j = \hat{R}_{st} + (s_j - 1)t_{2(j)}(\theta) + R^{-1}(\hat{R}_{st} - R)(s_j - 1)t_{2(j)}(\theta^*), \qquad (3.7)$$

where, $[\hat{R}_{st}^{*} = R + \beta_1(\hat{R}_{st} - R)]$, $\theta = (R, 1)$, $\theta^* = (\hat{R}_{st}^{*}, s_j^*)$ and $s_j^* = 1 + \beta_2(s_j - 1)$, $0 < \beta_j < 1$, $\forall\ j = 1, 2$.

Here, the derivatives (partial) of function $t_{(j)}(\theta)$ at $\theta(= R, 1)$ point are denoted by $t_{1(j)}(\theta)$, $t_{2(j)}(\theta)$, $t_{11(j)}(\theta)$, $t_{12(j)}(\theta)$ and $t_{22(j)}(\theta)$ are given as follows:

$$t_{1(j)}(\theta) = \left(\frac{\partial}{\partial \hat{R}_{st}} t_{(j)}(\hat{\theta}_j)\right)_{(\theta)}, \quad t_{2(j)}(\theta) = \left(\frac{\partial}{\partial s_j} t_{(j)}(\hat{\theta}_j)\right)_{(\theta)},$$

$$t_{11(j)}\left(\theta\right)=\left(\frac{\partial^2}{\partial\hat{R}_{st}^{\,2}}t_{(j)}(\hat{\theta}_j)\right)_{(\theta)},\ t_{12(j)}\left(\theta\right)=\left(\frac{\partial^2}{\partial\hat{R}_{st}\partial s_j}t_{(j)}(\hat{\theta}_j)\right)_{(\theta)},\ t_{22(j)}\left(\theta\right)=\left(\frac{\partial^2}{\partial s_j^{\,2}}t_{(j)}(\hat{\theta}_j)\right)_{(\theta)},$$

where $\hat{\theta}_j=(\hat{R}_{st},s_j)\ \forall\ j=1,2.$

4. Bias and Mean square error

Using the large sample approximation, we assume that

$\bar{y}^*_{1st}=\bar{Y}_1(1+\varepsilon_0),\quad \bar{y}^*_{2st}=\bar{Y}_2(1+\varepsilon_1),\quad \bar{x}_{st}=\bar{X}(1+\varepsilon_2),\quad \bar{x}'_{st}=\bar{X}(1+\varepsilon_3)$ and $\bar{x}^*_{st}=\bar{X}(1+\varepsilon_4)$,

Such that, $|\varepsilon_i|<1$, and $E(\varepsilon_i)=0\ \forall\ i=0, 1, 2, 3, 4$.

We have,

$$E(\varepsilon_0^{\,2})=\frac{V(\bar{y}^*_{1st})}{\bar{Y}_1^{\,2}}=\frac{1}{\bar{Y}_1^{\,2}}\sum_{i=1}^{L}W_i^2\left\{\lambda_i S^2_{y_{1i}}+\frac{(g_i-1)}{n_i}W_{i2}S^2_{y_{1i}(2)}\right\}$$

$$E(\varepsilon_1^{\,2})=\frac{V(\bar{y}^*_{2st})}{\bar{Y}_2^{\,2}}=\frac{1}{\bar{Y}_2^{\,2}}\sum_{i=1}^{L}W_i^2\left\{\lambda_i S^2_{y_{2i}}+\frac{(g_i-1)}{n_i}W_{i2}S^2_{y_{2i}(2)}\right\}$$

$$E(\varepsilon_2^{\,2})=E(\varepsilon_2\varepsilon_4)=\frac{V(\bar{x}_{st})}{\bar{X}^2}=\frac{1}{\bar{X}^2}\sum_{i=1}^{L}W_i^2\{\lambda_i S^2_{x_i}\}$$

$$E(\varepsilon_3^{\,2})=E(\varepsilon_2\varepsilon_3)=E(\varepsilon_3\varepsilon_4)=\frac{V(\bar{x}'_{st})}{\bar{X}^2}=\frac{1}{\bar{X}^2}\sum_{i=1}^{L}W_i^2\{\lambda'_i S^2_{x_i}\}$$

$$E(\varepsilon_4^{\,2})=\frac{V(\bar{x}^*_{st})}{\bar{X}^2}=\frac{1}{\bar{X}^2}\sum_{i=1}^{L}W_i^2\left\{\lambda_i S^2_{x_i}+\frac{(g_i-1)}{n_i}W_{i2}S^2_{x_i(2)}\right\}$$

$$E(\varepsilon_0\varepsilon_1)=\frac{\mathrm{Cov}(\bar{y}^*_{1st},\bar{y}^*_{2st})}{\bar{Y}_1\bar{Y}_2}=\frac{1}{\bar{Y}_1\bar{Y}_2}\sum_{i=1}^{L}W_i^2\left\{\lambda_i S_{y_{1i}y_{2i}}+\frac{(g_i-1)}{n_i}W_{i2}S_{y_{1i}y_{2i}(2)}\right\}$$

$$E(\varepsilon_0\varepsilon_2)=\frac{\mathrm{Cov}(\bar{y}^*_{1st},\bar{x}_{st})}{\bar{Y}_1\bar{X}}=\frac{1}{\bar{Y}_1\bar{X}}\sum_{i=1}^{L}W_i^2\{\lambda_i S_{y_{1i}x_i}\}$$

$$E(\varepsilon_0\varepsilon_3)=\frac{\mathrm{Cov}(\bar{y}^*_{1st},\bar{x}'_{st})}{\bar{Y}_1\bar{X}}=\frac{1}{\bar{Y}_1\bar{X}}\sum_{i=1}^{L}W_i^2\{\lambda'_i S_{y_{1i}x_i}\}$$

$$E(\varepsilon_0\varepsilon_4)=\frac{Cov(\bar{y}^*_{1st},\bar{x}^*_{st})}{\bar{Y}_1\bar{X}}=\frac{1}{\bar{Y}_1\bar{X}}\sum_{i=1}^{L}W_i^2\left\{\lambda_i S_{y_{1i}x_i}+\frac{(g_i-1)}{n_i}W_{i2}S_{y_{1i}x_i(2)}\right\}$$

$$E(\varepsilon_1\varepsilon_2)=\frac{Cov(\bar{y}^*_{2st},\bar{x}_{st})}{\bar{Y}_2\bar{X}}=\frac{1}{\bar{Y}_2\bar{X}}\sum_{i=1}^{L}W_i^2\{\lambda_i S_{y_{12}x_i}\}$$

$$E(\varepsilon_1\varepsilon_3)=\frac{Cov(\bar{y}^*_{2st},\bar{x}'_{st})}{\bar{Y}_2\bar{X}}=\frac{1}{\bar{Y}_2\bar{X}}\sum_{i=1}^{L}W_i^2\{\lambda'_i S_{y_{12}x_i}\} \text{ and}$$

$$E(\varepsilon_1\varepsilon_4)=\frac{Cov(\bar{y}^*_{2st},\bar{x}^*_{st})}{\bar{Y}_2\bar{X}}=\frac{1}{\bar{Y}_2\bar{X}}\sum_{i=1}^{L}W_i^2\left\{\lambda_i S_{y_{2i}x_i}+\frac{(g_i-1)}{n_i}W_{i2}S_{y_{2i}x_i(2)}\right\}, \quad (4.1)$$

where, $(S^2_{y_{1i}},S^2_{y_{2i}})$ and $(S^2_{y_{1i}(2)},S^2_{y_{2i}(2)})$ show the mean square error of the population for the variable (y_1,y_2) and non-response part for the i^{th} stratum. In the same manner, $S^2_{x_i}$ and $S^2_{x_i(2)}$ are the population mean square error of x for i^{th} stratum and for the non-responding part of the i^{th} stratum. $(S_{y_{1i}y_{2i}},S_{y_{1i}x_i},S_{y_{2i}x_i})$ and $(S_{y_{1i}y_{2i}(2)},S_{y_{1i}x_i(2)},S_{y_{2i}x_i(2)})$ are the population covariance of (y_1,y_2), (y_1,x) and (y_2,x) for the i^{th} stratum and for the units which are not respond of the i^{th} stratum and $\lambda_i=\left(\frac{1}{n_i}-\frac{1}{N_i}\right)$, $\lambda'_i=\left(\frac{1}{n'_i}-\frac{1}{N_i}\right)$.

By using the equation (3.7) and (4.1) upto the (n^{-1}) terms of order, the expression for the bias and MSE of u_j for $j=1,2$ are given as follows:

$$Bias(u_j)=E(u_j-R)=E[t_{(j)}(\theta)+(\hat{R}_{st}-R)t_{1(j)}(\theta)+(s_j-1)t_{2(j)}(\theta)$$
$$+R^{-1}(\hat{R}_{st}-R)(s_j-1)t_{2(j)}(\theta^*)-R]$$

$$=E(\hat{R}_{st}-R)+R^{-1}E(\hat{R}_{st}-R)(s_j-1)t_{2(j)}(\theta^*) \quad (4.2)$$

$$MSE(u_j)=E(u_j-R)^2=E(\hat{R}_{st}-R)^2+E(s_j-1)^2 t_{2(j)}{}^2(\theta)$$
$$+2E(\hat{R}_{st}-R)(s_j-1)t_{2(j)}(\theta^*), \quad (4.3)$$

where, $E(\hat{R}_{st}-R)^2=\frac{\bar{Y}_1^2}{\bar{Y}_2^2}\left[\frac{V(\bar{y}^*_{1st})}{\bar{Y}_1^2}+\frac{V(\bar{y}^*_{1st})}{\bar{Y}_2^2}-2\frac{cov(\bar{y}^*_{1st},\bar{y}^*_{2st})}{\bar{Y}_1\bar{Y}_2}\right]$,

$$E(z_1-1)^2=\frac{1}{\overline{X}^2}(V(\overline{x}^*_{st})-V(\overline{x}'_{st})),\ E(z_2-1)^2=\frac{1}{\overline{X}^2}(V(\overline{x}_{st})-V(\overline{x}'_{st})),$$

$$E(\hat{R}_{st}-R)(z_1-1)$$
$$=\frac{\overline{Y}_1}{\overline{Y}_2}\left(\frac{Cov(\overline{y}^*_{1st},\overline{x}^*_{st})}{\overline{Y}_1\overline{X}}-\frac{Cov(\overline{y}^*_{2st},\overline{x}^*_{st})}{\overline{Y}_2\overline{X}}-\frac{Cov(\overline{y}^*_{1st},\overline{x}'_{st})}{\overline{Y}_1\overline{X}}+\frac{Cov(\overline{y}^*_{2st},\overline{x}'_{st})}{\overline{Y}_2\overline{X}}\right),$$

$$E(\hat{R}_{st}-R)(z_2-1)$$
$$=\frac{\overline{Y}_1}{\overline{Y}_2}\left(\frac{Cov(\overline{y}^*_{1st},\overline{x}_{st})}{\overline{Y}_1\overline{X}}-\frac{Cov(\overline{y}^*_{2st},\overline{x}_{st})}{\overline{Y}_2\overline{X}}-\frac{Cov(\overline{y}^*_{1st},\overline{x}'_{st})}{\overline{Y}_1\overline{X}}+\frac{Cov(\overline{y}^*_{2st},\overline{x}'_{st})}{\overline{Y}_2\overline{X}}\right).$$

Now, differentiating the expression (4.3) w.r.t. $t_{2(j)}(\theta)$ for $j=1,2$ and making equal to zero and assuming that the derivatives of the second order are positive, we have

$$t_{2(1)}(\theta)_{opt}=-\frac{R\overline{X}}{(V(\overline{x}^*_{st})-V(\overline{x}'_{st}))}\left(\frac{Cov(\overline{y}^*_{1st},\overline{x}^*_{st})}{\overline{Y}_1}-\frac{Cov(\overline{y}^*_{2st},\overline{x}^*_{st})}{\overline{Y}_2}-\frac{Cov(\overline{y}^*_{1st},\overline{x}'_{st})}{\overline{Y}_1}+\frac{Cov(\overline{y}^*_{2st},\overline{x}'_{st})}{\overline{Y}_2}\right) \tag{4.4}$$

$$t_{2(2)}(\theta)_{opt}=-\frac{R\overline{X}}{(V(\overline{x}_{st})-V(\overline{x}'_{st}))}\left(\frac{Cov(\overline{y}^*_{1st},\overline{x}_{st})}{\overline{Y}_1}-\frac{Cov(\overline{y}^*_{2st},\overline{x}_{st})}{\overline{Y}_2}-\frac{Cov(\overline{y}^*_{1st},\overline{x}'_{st})}{\overline{Y}_1}+\frac{Cov(\overline{y}^*_{2st},\overline{x}'_{st})}{\overline{Y}_2}\right) \tag{4.5}$$

Now, putting the optimized values of $t_{2(1)}(\theta)_{opt}$ and $t_{2(2)}(\theta)_{opt}$ from equations (4.4) and (4.5) in the expression of $MSE(u_j)$ for $j=1,2$ respectively, we obtain the minimum values of the MSE which are given as:

$$MSE(u_1)_{\min}=MSE(\widehat{R}_{st})-\frac{R^2}{(V(\overline{x}^*_{st})-V(\overline{x}'_{st}))}\left(\frac{Cov(\overline{y}^*_{1st},\overline{x}^*_{st})}{\overline{Y}_1}-\frac{Cov(\overline{y}^*_{2st},\overline{x}^*_{st})}{\overline{Y}_2}-\frac{Cov(\overline{y}^*_{1st},\overline{x}'_{st})}{\overline{Y}_1}+\frac{Cov(\overline{y}^*_{2st},\overline{x}'_{st})}{\overline{Y}_2}\right)^2 \tag{4.6}$$

$$MSE(u_2)_{\min} = MSE(\hat{R}_{st}) - \frac{R^2}{(V(\bar{x}_{st}) - V(\bar{x}'_{st}))}$$

$$\left(\frac{Cov(\bar{y}^*_{1st}, \bar{x}_{st})}{\bar{Y}_1} - \frac{Cov(\bar{y}^*_{2st}, \bar{x}_{st})}{\bar{Y}_2} - \frac{Cov(\bar{y}^*_{1st}, \bar{x}'_{st})}{\bar{Y}_1} + \frac{Cov(\bar{y}^*_{2st}, \bar{x}'_{st})}{\bar{Y}_2} \right)^2 \quad (4.7)$$

Members of the proposed classes u_j for $j = 1,2$:

Any parametric function $t_{(1)}(\hat{R}_{st}, s_1)$ and $t_{(2)}(\hat{R}_{st}, s_2)$ satisfying the condition given in equation (3.5) can produce the class of estimators. Such types of estimators have a wider class. Some members of the proposed classes u_j, $j = 1,2$ are given as follows:

Table 1

Members of the proposed classes:

Members of class u_1	**Members of class u_2**
$u_{11} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}}(\alpha_1 + (1-\alpha_1)s_1)$	$u_{21} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}}(\alpha_1 + (1-\alpha_1)s_2)$
$u_{12} = \left(\frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}} + \alpha_2(s_1 - 1) \right) s_1^{\beta}$	$u_{22} = \left(\frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}} + \alpha_2(s_2 - 1) \right) s_2^{\beta}$
$u_{13} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}}(s_1)^{\alpha_3}$	$u_{23} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}}(s_2)^{\alpha_3}$
$u_{14} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}}(2 - s_1^{\alpha_4})$	$u_{24} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}}(2 - s_2^{\alpha_4})$
$u_{15} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}} \exp\left(\frac{s_1 - 1}{s_1 + 1} \right)$	$u_{25} = \frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}} \exp\left(\frac{s_2 - 1}{s_2 + 1} \right)$

Where, α_i, $i = 1,2,3,4$ and β are constant.

For the optimized value of $t_{2(j)}(\theta)$ for $(j = 1,2)$ given in equations (4.4) and (4.5), the MSE of the member of the proposed classes (u_j) of the estimators given in **Table 1** will also attain the same minimum value of the MSE given in equations (4.6) and (4.7) for the optimum values of $t_{2(j)}(\theta)$ for $j = 1,2.$ One may have to estimate the bias of the estimator by making appropriate adjustments in the bias which does not affect the minimum MSE upto the terms of order (n^{-1}). In case, when the optimized values of the constant occasionally are in the form of unknown parameters then one may estimate it by using past data or utilize the sample values. Reddy [16] has demonstrated that such values are not altered over time and location and don't affect the MSE of the estimator upto the term of order n^{-1} (Srivastava and Jhajj [22]). Therefore, the estimators u_1 and u_2 are suggested to use in case of a large sample survey.

5. Comparison of the proposed classes $[u_j$ for $j=1,2]$ of the estimators with the usual estimator:

The usual estimator for stratified population under the existence of non-response is defined as: $\left[\hat{R}_{st}=\frac{\bar{y}^*_{1st}}{\bar{y}^*_{2st}}\right]$

$$MSE(\hat{R}_{st})=R^2\left[\frac{V(\bar{y}^*_{1st})}{\bar{Y}_1^2}+\frac{V(\bar{y}^*_{2st})}{\bar{Y}_2^2}-2\frac{Cov(\bar{y}^*_{1st},\bar{y}^*_{2st})}{\bar{Y}_1\bar{Y}_2}\right] \tag{5.1}$$

Here, we compare the estimators u_1 and u_2 given in equations (2.3) and (2.4) with the relevant estimator $\hat{R}_{st}$ on the basis of MSE with the help of a set of real data.

6. Survey Cost and Determination of the Optimum size of n'_i, n_i and g_i (for fixed cost $Z \le Z_0$):

Let us suppose that, excluding the overhead cost, the total cost i.e. overall survey cost is denoted by Z_0 (fixed). Here, n'_i is the size of the first-phase sample and n_i the size of the second-phase sample. The expected cost per stratum of the survey excluding the overhead cost is given as-

$$E(Z_i)=Z'_i n'_i+n_i\left(Z_{i0}+Z_{i1}W_{i1}+Z_{i2}\frac{W_{i2}}{g_i}\right) \tag{6.1}$$

So, we can write the total cost over all the strata is given by-

$$Z=\sum_{i=1}^{L}E(Z_i)=\sum_{i=1}^{L}\left[Z'_i n'_i+n_i\left(Z_{i0}+Z_{i1}W_{i1}+Z_{i2}\frac{W_{i2}}{g_i}\right)\right] \tag{6.2}$$

Where,

Z'_i = Cost per unit for mailing the questionnaire and observing the auxiliary character in the 1st-phase.

Z_{i0} = Cost per unit for mailing the questionnaire in the second phase from the first- phase units.

Z_{i1} = Cost per unit on the responding units in the 2nd-phase sample on y_1 and y_2.

Z_{i2} = Cost per unit on observing the units from non-response units at the second phase on y_1 and y_2.

We can also rewrite the expression of MSE in the form of:

$$MSE(u_1) = R^2 \sum_{i=1}^{L} W_i^2 \left\{ \lambda_i \left(\frac{S_{y_{1i}}^2}{\bar{Y}_1^2} + \frac{S_{y_{2i}}^2}{\bar{Y}_2^2} - \frac{2S_{y_{1i}y_{2i}}}{\bar{Y}_1\bar{Y}_2} \right) + \frac{(g_i-1)W_{i2}}{n_i} \left(\frac{S_{y_{1i}(2)}^2}{\bar{Y}_1^2} + \frac{S_{y_{2i}(2)}^2}{\bar{Y}_2^2} - \frac{2S_{y_{1i}y_{2i}(2)}}{\bar{Y}_1\bar{Y}_2} \right) \right\}$$

$$-R^2 \frac{\left[\sum_{i=1}^{L} W_i^2 \left\{ \lambda_i' \left(\frac{S_{y_{1i}x_i}}{\bar{Y}_1^2} + \frac{S_{y_{2i}x_i}}{\bar{Y}_2^2} \right) + \frac{(g_i-1)W_{i2}}{n_i} \left(\frac{2S_{y_{1i}x_i(2)}}{\bar{Y}_1\bar{Y}_2} - \frac{2S_{y_{2i}x_i(2)}}{\bar{Y}_1\bar{Y}_2} \right) \right\} \right]^2}{\sum_{i=1}^{L} \left[W_i^2 \left\{ \lambda_i' S_{x_i}^2 + \frac{(g_i-1)}{n_i} W_{i2} S_{x_i(2)}^2 \right\} \right]} \tag{6.3}$$

$$MSE(u_2) = R^2 \sum_{i=1}^{L} W_i^2 \left\{ \lambda_i \left(\frac{S_{y_{1i}}^2}{\bar{Y}_1^2} + \frac{S_{y_{2i}}^2}{\bar{Y}_2^2} - \frac{2S_{y_{1i}y_{2i}}}{\bar{Y}_1\bar{Y}_2} \right) + \frac{(g_i-1)W_{i2}}{n_i} \left(\frac{S_{y_{1i}(2)}^2}{\bar{Y}_1^2} + \frac{S_{y_{2i}(2)}^2}{\bar{Y}_2^2} - \frac{2S_{y_{1i}y_{2i}(2)}}{\bar{Y}_1\bar{Y}_2} \right) \right\}$$

$$-R^2 \frac{\left[\sum_{i=1}^{L} W_i^2 \left\{ (\lambda_i - \lambda_i') \left(\frac{S_{y_{1i}x_i}}{\bar{Y}_1} - \frac{S_{y_{2i}x_i}}{\bar{Y}_2} \right) \right\} \right]^2}{\sum_{i=1}^{L} W_i^2 [(\lambda_i - \lambda_i') S_{X_i}^2]} \tag{6.4}$$

For the convenience purpose, we assume the following approximation for solving the optimum values of n_i, n_i' and g_i.

$$\begin{aligned}
S_{x_1}^2 &= S_{x_2}^2 = S_{x_3}^2 \ldots\ldots\ldots\ldots\ldots\ldots S_{x_L}^2 = V_1 \\
S_{y_{11}}^2 &= S_{y_{12}}^2 = S_{y_{13}}^2 \ldots\ldots\ldots\ldots\ldots\ldots S_{y_{1L}}^2 = V_{01} \\
S_{y_{21}}^2 &= S_{y_{22}}^2 = S_{y_{23}}^2 \ldots\ldots\ldots\ldots\ldots\ldots S_{y_{2L}}^2 = V_{02} \\
S_{y_{11}(2)}^2 &= S_{y_{12}(2)}^2 = S_{y_{13}(2)}^2 \ldots\ldots\ldots\ldots\ldots\ldots S_{y_{1L}(2)}^2 = V_{01(2)} \\
S_{y_{21}(2)}^2 &= S_{y_{22}(2)}^2 = S_{y_{23}(2)}^2 \ldots\ldots\ldots\ldots\ldots\ldots S_{y_{2L}(2)}^2 = V_{02(2)} \\
S_{x_1(2)}^2 &= S_{x_2(2)}^2 = S_{x_3(2)}^2 \ldots\ldots S_{x_L(2)}^2 = V_{1(2)} \\
S_{y_{11}x_1} &= S_{y_{12}x_2} = S_{y_{13}x_3} \ldots\ldots S_{y_{1L}x_L} = V_{11} \\
S_{y_{21}x_1} &= S_{y_{22}x_2} = S_{y_{23}x_3} \ldots\ldots S_{y_{2L}x_L} = V_{21}
\end{aligned} \tag{6.5}$$

$$S_{y_{11}x_1(2)} = S_{y_{12}x_2(2)} = S_{y_{13}x_3(2)} \ldots\ldots\ldots S_{y_{1L}x_L(2)} = V_{11(2)}$$
$$S_{y_{21}x_1(2)} = S_{y_{22}x_2(2)} = S_{y_{23}x_3(2)} \ldots\ldots S_{y_{2L}x_L(2)} = V_{21(2)}$$
$$S_{y_{11}y_{21}} = S_{y_{12}y_{22}} = S_{y_{13}y_{23}} \ldots\ldots. S_{y_{1L}y_{2L}} = V_{03}$$
$$S_{y_{11}y_{21}(2)} = S_{y_{12}y_{22}(2)} = S_{y_{13}y_{23}(2)} \ldots\ldots. S_{y_{1L}y_{2L}(2)} = V_{03(2)}$$

Theorem: *If* $\dfrac{\left[\frac{V_{11}}{\bar{Y}_1} - \frac{V_{21}}{\bar{Y}_2}\right]}{V_1} = \dfrac{\left[\frac{V_{11(2)}}{\bar{Y}_1} - \frac{V_{21(2)}}{\bar{Y}_2}\right]}{V_{1(2)}}$ *then* $MSE(u_1)$ *and* $MSE(u_2)$ *can be given by:*

$$MSE(u_1) = R^2\left[D\sum_{i=1}^{L} W_i^2\left(\frac{1}{n_i} - \frac{1}{N_i}\right) + E\sum_{i=1}^{L} W_i^2 \frac{(g_i-1)}{n_i} W_{i2} - \frac{A^2}{C}\sum_{i=1}^{L} W_i^2\left(\frac{1}{n_i} - \frac{1}{n_i'}\right) - \frac{AB}{C}\sum_{i=1}^{L} W_i^2 \frac{(g_i-1)}{n_i} W_{i2}\right] \tag{6.6}$$

$$MSE(u_2) = R^2\left[D\sum_{i=1}^{L} W_i^2\left(\frac{1}{n_i} - \frac{1}{N_i}\right) + E\sum_{i=1}^{L} W_i^2 \frac{(g_i-1)}{n_i} W_{i2} - \frac{A^2}{C}\sum_{i=1}^{L} W_i^2\left(\frac{1}{n_i} - \frac{1}{n_i'}\right)\right] \tag{6.7}$$

Where,

$$A = \left(\frac{V_{11}}{\bar{Y}_1} - \frac{V_{21}}{\bar{Y}_2}\right), B = \left(\frac{V_{11(2)}}{\bar{Y}_1} - \frac{V_{21(2)}}{\bar{Y}_2}\right), C = V_1, D = \left(\frac{V_{01}}{\bar{Y}_1^2} + \frac{V_{02}}{\bar{Y}_2^2} - \frac{V_{03}}{\bar{Y}_1\bar{Y}_2}\right) \text{ and}$$
$$E = \left(\frac{V_{01(2)}}{\bar{Y}_1^2} + \frac{V_{02(2)}}{\bar{Y}_2^2} - \frac{V_{03(2)}}{\bar{Y}_1\bar{Y}_2}\right)$$

Let us define the Lagrange function $\Psi_m^*(m=1,2)$ for minimizing the $MSE(u_1)$ and $MSE(u_2)$ and to obtain the optimized values of n_i', n_i and g_i for both expressions (6.6) and (6.7). Which are given bellow as :

$$\Psi_1^* = MSE(u_1) + \eta_1(Z - Z_0) \tag{6.8}$$

$$\Psi_2^* = MSE(u_2) + \eta_2(Z - Z_0) \tag{6.9}$$

Where, η_1 and η_2 are the Lagrange multipliers.

Now, to get the optimize values of n_i', n_i and g_i for case **(i)** and case **(ii)**. We differentiate the equation (6.8) and (6.9) with respect to n_i', n_i and g_i respectively and make the derivatives equal to zero, we obtain

For Case (i):

$$\frac{\partial}{\partial n_i}\Psi_1^* = -\frac{R^2W_i^2}{n_i^2}\left(D - \frac{A^2}{C}\right) - \frac{R^2W_i^2(g_i-1)W_{i2}}{n_i^2}\left(E - \frac{AB}{C}\right) + \eta_1\left(Z_{i0} + Z_{i1}W_{i1} + Z_{i2}\frac{W_{i2}}{g_i}\right) \tag{6.10}$$

$$\frac{\partial}{\partial n_i'}\Psi_1^* = -\frac{R^2A^2W_i^2}{Cn_i^2} + \eta_1 Z_i' \tag{6.11}$$

$$\frac{\partial}{\partial g_i}\Psi_1^* = -\frac{R^2W_i^2\left(E-\frac{AB}{C}\right)W_{i2}}{n_i} - \frac{\eta_1 n_i Z_{i2}W_{i2}}{g_i^2} \tag{6.12}$$

From equation (6.10), (6.11) and (6.12), we respectively get

$$n_i = RW_i\frac{\sqrt{(DC-A^2)+(g_i-1)W_{i2}(EC-AB)}}{\sqrt{\eta_1 C\left(Z_{i0}+Z_{i1}W_{i1}+Z_{i2}\frac{W_{i2}}{g_i}\right)}} \tag{6.13}$$

$$n_i' = \frac{RAW_i}{\sqrt{\eta_1 CZ_i'}} \tag{6.14}$$

$$\sqrt{\eta_1} = \frac{RW_i g_i\sqrt{(EC-AB)}}{n_i\sqrt{Z_{i2}C}} \tag{6.15}$$

Substituting the value of $\sqrt{\eta_1}$ from equation (6.15) into (6.13), we get the optimized value of g_i.

$$g_{iopt} = \frac{T_i\sqrt{Z_{i2}}}{S_iU_i} \tag{6.16}$$

Where,

$S_i = \sqrt{Z_{i0} + Z_{i1}W_{i1}}, T_i = \sqrt{DC - A^2 - W_{i2}(EC - AB)}$ and $U_i = \sqrt{(EC - AB)}$.

From the Equations, (6.13) and (6.16), n_i can be expressed as:

$$n_i = RW_i\frac{\sqrt{T_i^2+\frac{T_iU_iW_{i2}\sqrt{Z_{i2}}}{S_i}}}{\sqrt{\eta_1}\sqrt{C\left(S_i^2+\frac{W_{i2}S_iU_i\sqrt{Z_{i2}}}{T_i}\right)}} \tag{6.17}$$

To get the value of $\sqrt{\eta_1}$ in terms of the total cost Z_0, we put the values of n_i', n_i and g_i from equations (6.14), (6.16) and (6.17) into equation (6.2).

$$\sqrt{\eta_1} = \frac{R}{D_0\sqrt{C}}\sum_{i=1}^{L}[AW_i\sqrt{d_i'} + W_i(S_iT_i + W_{i2}U_i\sqrt{d_{i2}})] \tag{6.18}$$

Now, to find the optimum size of n_i and n_i' we substitute the value of $\sqrt{\eta_1}$ from equation (6.18) into equations (6.14) and (6.17).

$$n_{iopt} = \frac{D_0 W_i T_i}{S_i \sum_{i=1}^{L} [AW_i \sqrt{Z_i'} + W_i (S_i T_i + W_{i2} U_i \sqrt{Z_{i2}})]} \tag{6.19}$$

$$n'_{i\,opt} = \frac{D_0 W_i A}{\sqrt{Z_i'} \sum_{i=1}^{L} [AW_i \sqrt{Z_i'} + W_i (S_i T_i + W_{i2} U_i \sqrt{Z_{i2}})]} \tag{6.20}$$

Similarly, the optimum values of n_i', n_i and g_i for **case (ii)** are obtained and given as follows:

$$n'_{i\,opt} = \frac{D_0 W_i A}{\sqrt{Z_i'} \sum_{i=1}^{L} [AW_i \sqrt{Z_i'} + W_i (S_i T_i^* + W_{i2} U_i^* \sqrt{Z_{i2}})]} \tag{6.21}$$

$$n_{iopt} = \frac{D_0 W_i T_i^*}{S_i \sum_{i=1}^{L} [AW_i \sqrt{Z_i'} + W_i (S_i T_i^* + W_{i2} U_i^* \sqrt{Z_{i2}})]} \tag{6.22}$$

$$g_{iopt} = \frac{T_i^* \sqrt{Z_{i2}}}{S_i U_i^*} \tag{6.23}$$

Where, $T_i^* = \sqrt{(DC - A^2) - W_{i2} EC}$ and $U_i^* = \sqrt{EC}$

7. An Empirical study:

Table 2, shows the census data (2011) published by the government of India, 595 districts from various Indian states have been taken into account. The study variables are considered as the number of cultivators (y_1) and the number of main workers (y_2). And as an auxiliary variable (x), we take the number of literate persons. Here, we have six strata: Strata (i) =Northern states, Strata (ii)= Southern states, Strata (iii) =Central states, Strata (iv) = Eastern states, Strata (iv) = Western states, and Strata (vi) = North-East states.

Table 2

Population parameters for each stratum:

Parameters	Northern States	Southern States	Central States	Western States	Eastern States	North-Eastern states
N_i	75	104	144	88	105	79
n_i'	44	61	85	52	62	46
n_i	24	33	46	28	33	25

Contd...

$\rho_{x_i y_{1i}}$		0.486	0.209	0.565	0.475	0.746	0.724
$\rho_{x_i y_{2i}}$		0.975	0.837	0.888	0.967	0.960	0.982
$\rho_{y_{1i} y_{2i}}$		0.564	0.571	0.720	0.618	0.796	0.801
$\bar{Y}_{1i}$		70125.6	158587.9	187047.4	280337.7	135325.2	61273.5
$\bar{Y}_{2i}$		278585.6	873215.1	508445.6	817840.4	556079.9	158835.8
$\bar{X}_{1i}$		654860.6	1595972	1150503	152161	1373561	340604
$S_{y_{1i}}$		49478.36	126536.8	96788.4	174563.8	78901.1	48310.4
$S_{y_{2i}}$		202774	468645.5	251156.4	501108.8	434663.2	142244.5
S_{x_i}		464081.9	835015.6	643860.5	979540.8	1062407	328757
10% Non-response	$\rho_{x_i y_{1i}(2)}$	0.606	0.547	-0.012	0.531	0.049	0.682
	$\rho_{x_i y_{2i}(2)}$	0.939	0.856	0.972	0.933	0.983	0.954
	$\rho_{y_{1i} y_{2i}(2)}$	0.415	0.478	-0.062	0.749	0.057	0.822
	$S_{y_{1i}(2)}$	26767.3	97080.9	8555.4	161900.2	7598.2	19577.4
	$S_{y_{2i}(2)}$	112650.4	415843	176635.2	479532.4	161992.5	142243.1
	$S_{x_i(2)}$	214111.5	607765.8	441078	911233.7	518650.5	297983.7
20% Non-response	$\rho_{x_i y_{1i}(2)}$	0.020	0.341	0.128	0.730	-0.102	0.366
	$\rho_{x_i y_{2i}(2)}$	0.983	0.862	0.950	0.965	0.969	0.951
	$\rho_{y_{1i} y_{2i}(2)}$	0.0071	0.288	0.1216	0.839	-0.0437	0.522
	$S_{y_{1i}(2)}$	30228.5	95709.9	16264.8	161871.1	10578.41	28934.2
	$S_{y_{2i}(2)}$	247136.9	396357.8	168625.6	505686.5	139988.8	114953.5
	$S_{x_i(2)}$	579454	551955.6	440631	982979.5	441124.1	264703.8
30% Non-response	$\rho_{x_i y_{1i}(2)}$	0.185	0.358	0.113	0.715	0.066	0.151
	$\rho_{x_i y_{2i}(2)}$	0.981	0.885	0.927	0.968	0.974	0.946
	$\rho_{y_{1i} y_{2i}(2)}$	0.191	0.406	0.189	0.803	0.144	0.347
	$S_{y_{1i}(2)}$	33606.9	98776.0	19799.7	157169.5	14230.7	34719.2
	$S_{y_{2i}(2)}$	217117.4	411540.4	145526.4	508584	250673.4	118325.6
	$S_{x_i(2)}$	512181.7	624390.1	381432.2	956207.3	644953.7	295764.8

Table 3, shows the optimum value of $t_{2(j)}(\theta)$ and $t_{2(j)}(\theta)$ for different non-response rates and sub-sampling fraction as $g=2$, $g=3$ and $g=4$.

Table 3

Optimum values of $t_{2(j)}(\theta)$ for $j=1,2$.

Non-response rate	Constant	$g=2$	$g=3$	$g=4$
10% Non-response rate	$t_{2(1)}(\theta)_{opt}$	0.1214	0.1235	0.1252
	$t_{2(2)}(\theta)_{opt}$	0.1189	0.1189	0.1189
20% Non-response rate	$t_{2(1)}(\theta)_{opt}$	0.1231	0.1259	0.1279
	$t_{2(2)}(\theta)_{opt}$	0.1189	0.1189	0.1189
30% Non-response rate	$t_{2(1)}(\theta)_{opt}$	0.1291	0.1349	0.1387
	$t_{2(2)}(\theta)_{opt}$	0.1189	0.1189	0.1189

Table 4

PRE of u_j for $j=1,2$ with respect to $\hat{R}_{st}$.

Non-response rates	**Estimators**	(N=595)		
		$1/g$		
		$1/2$	$1/3$	$1/4$
10%	$\hat{R}_{st}$	**100**(103)	**100**(113)	**100**(121)
	u_1	**119.77**(86)	**122.82**(92)	**123.47**(98)
	u_2	**117.05**(88)	**116.50**(97)	**115.24**(105)
20%	$\hat{R}_{st}$	**100**(114)	**100**(134)	**100**(153)
	u_1	**121.30**(94)	**124.08**(108)	**125.40**(122)
	u_2	**116.33**(98)	**113.56**(118)	**111.68**(137)
30%	$\hat{R}_{st}$	**100**(124)	**100**(154)	**100**(184)
	u_1	**125.25**(99)	**129.41**(119)	**133.34**(138)
	u_2	**113.76**(109)	**111.60**(138)	**109.52**(168)

(# Figures shows the Percentage relative efficiency and in parenthesis() MSEs of the estimators in 10^{-6}.)

Results:

Table 4, depicts the PREs and MSEs of estimators $\hat{R}_{st}$, u_1 and u_2 for the various values of non-response rates and sub-sampling fractions.

In Table 4, we observe that the MSEs of $\hat{R}_{st}$, u_1 and u_2 increases as the rate of non-response increases and MSEs also increases as we increase the value of g.

Conclusion:

- Upon examination of Table 4., it is evident that across various sub-sampling fractions and rate of non-response, the MSEs of suggested classes of estimators (u_j for $j = 1, 2$) are lower than the usual estimator $(\hat{R}_{st})$ i.e. $MSE(u_1) < MSE(u_2) < MSE(\hat{R}_{st})$.
- The proposed class u_1 has less MSE than the proposed class u_2 for the various rates of non-response and various values of g due to high correlation coefficients and other factors related to parameters [see Rao [17] for more details].
- Hence, our conclusion is that the suggested classes of estimators u_1 and u_2 exhibit greater efficiency as compared to the usual estimator $(\hat{R}_{st})$ in scenarios of stratified population and existence of non-response. So, we can use it for application in the field of research in Medical science, Social science and Agriculture surveys.
- Further, the expressions for the optimum sizes of n_i', n_i and g_i have been carried out for the fixed cost for both cases which can be used in practice for fixing the optimum size of n_i', n_i and g_i prior to conducting the survey.

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