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The Marshall-Olkin-exponentiated half logistic-G family of distributions : Model, properties and applications

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Abstract

A new generalization of the exponentiated half-logistic-G distribution is developed to produce a new family of distributions referred to as the Marshall-Olkin-exponentiated half logistic-G distribution. The new family of distributions is an infinite linear combination of the exponentiated-G family of distributions. We considered some special cases of the new distribution and we conclude that the distribution can handle heavy tailed data and non-monotonic hazard rate functions. A simulation study was conducted to assess the consistency of the maximum likelihood estimates. Real data examples are provided to demonstrate the usefulness of the proposed model in comparison with other competing models.

Subject Classification: *62E99, 60E05, 62E15.*

Keywords: *Marshall-Olkin-G distribution, Exponentiated half-logistic distribution, Maximum likelihood estimation.*

1. Introduction

There is an increasing demand for generalized distributions in lifetime data analysis. Marshall and Olkin [24] added an extra parameter to a family of distributions in order to improve flexibility and goodness-of-fit. Gupta et al. [16] developed the exponentiated-G (Exp-G) distribution

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based on the Lehmann-type alternatives [22] by adding one parameter to the baseline cumulative distribution function (cdf). Gupta and Kundu [17] also developed a generalized-exponential (GE), also known as the exponentiated-exponential distribution. The GE distribution is an extension of the exponential distribution. These generalized distributions received much attention from applied and theoretical statisticians. Of interest was the flexibility enjoyed in data fitting as the models fit data that has monotonic or non-monotonic hazard rates. In addition, Eugene et al. [15], introduced a method of generalizing classical distributions by adding two shape parameters. Jones [19] also developed the beta-G distribution. Some well known generators include the Kumaraswamy-G by Cordeiro et al. [11], gamma-G (type I and type II) by ([29], [27]), Transformed-Transformer by Alzaatreh et al. [3], Weibull-G by Bourguignon et al. [5], Type I half-logistic-G by Cordeiro et al. [12], Garhy-G by Elgarhy et al. [14], Topp-Leone-G by Al-Shomrani et al. [1], beta-odd Lindley-G, Kumaraswamy-odd Lindley-G and Topp-Leone Marshall-Olkin-G by [9], [10] and [8], respectively, to mention a few.

Makubate et al. [23] developed the Marshall-Olkin-Half-logistic-G distribution with cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO-TL-G}(x;b,\delta,\xi)=1-\frac{\delta[1-[1-\bar{G}^2(x;\xi)]^b]}{1-\bar{\delta}[1-[1-\bar{G}^2(x;\xi)]^b]} \tag{1.1}$$

and

$$f_{MO-TL-G}(x;b,\delta,\xi)=\frac{2\delta b g(x;\xi)\bar{G}(x;\xi)[1-\bar{G}^2(x;\xi)]^{b-1}}{(1-\bar{\delta}[1-[1-\bar{G}^2(x;\xi)]^b])^2}, \tag{1.2}$$

respectively, for $b,\ \delta>0,\ \bar{\delta}=1-\delta$ and ξ is a vector of parameters from the baseline distribution function G(.).

The new proposed distribution is versatile and flexible, since it can be applied to data sets of varying skewness and kurtosis. Also, the new proposed model applies to data sets with monotonic or non-monotonic hazard rate functions. Furthermore, the proposed distribution is a linear combination of the exponentiated-G (Exp-G) densities, which makes the derivation of other important statistical properties easier.

The rest of the paper is organized as follows: We present the new proposed model and statistical properties in Section 2. Estimation of model parameters are given in Section 3. Some special cases are presented in Section 4. Section 5, contain results from Monte Carlo simulation study.

Real data examples are presented in Section 6, followed by concluding remarks.

2. The Model and Properties

We introduced an extra shape parameter to the half logistic-G distribution in the model proposed by Makubate et al. [23] (see equation (1.1)) to come up with the new family of distributions called the Marshall-Olkin-Exponentiated Half Logistic-G (MO-EHL-G) distribution with cdf and pdf respectively given by

$$F_{MO-EHL-G}(x;\alpha,\delta,\lambda,\xi)=1-\left(\frac{\delta(1-H(x;\alpha,\lambda,\xi))}{1-\overline{\delta}(1-H(x;\alpha,\lambda,\xi))}\right), \tag{2.1}$$

$$f_{MO-EHL-G}(x;\alpha,\delta,\lambda,\xi)=\frac{2\alpha\delta\lambda g(x;\xi)\overline{G}^{\lambda-1}(x;\xi)[1-\overline{G}^{\lambda}(x;\xi)]^{\alpha-1}}{(1+\overline{G}^{\lambda}(x;\xi))^{\alpha+1}[1-\overline{\delta}(1-H(x;\alpha,\lambda,\xi))]^{2}}, \tag{2.2}$$

where $H(x;\alpha,\lambda,\xi)=\left[\frac{1-\overline{G}^{\lambda}(x;\xi)}{1+\overline{G}^{\lambda}(x;\xi)}\right]^{\alpha}$, for $\alpha,\ \delta,\ \lambda>0,\ \overline{\delta}=1-\delta,$ where ξ is a vector of parameters from the baseline distribution function $G(.)$. The quantiles for the MO-EHL-G family of distributions may be determined by solving the non-linear equation

$$\mathrm{x}(u)=G^{-1}\left(1-\left[\frac{1-(1-z)^{1/\alpha}}{1+(1-z)^{1/\alpha}}\right]^{1/\lambda}\right). \tag{2.3}$$

2.1 *Expansion of Density Function*

In this section, we derive the statistical properties of the MO-EHL-G family of distributions using general results for the Marshall and Olkin's family of distributions by Barreto-Souza et al. [4]. The details are given in the web appendix. The MO-EHL-G family of distributions can be expressed as an infinite linear combination of the Exponentiated-G (Exp-G) distribution with power parameter $(q+1)$ as follows:

For $\delta\in(0,1)$,

$$\begin{aligned} f_{MO-EHL-G}(x;\alpha,\delta,\lambda,\xi)&=\sum_{j,m,p,q=0}^{\infty}\sum_{i=0}^{j}2\alpha\lambda(-1)^{m+p+q}w_{j,i}\binom{\alpha(j-i+1)+1}{m}\\ &\quad\times\binom{\alpha(j-i+1)-1}{p}\binom{\lambda(m+p+1)-1}{q}\\ &\quad\times g(x;\xi)[G(x;\xi)]^{q}\\ &=\sum_{q=0}^{\infty}w^{*}_{q+1}g_{q+1}(x;\xi), \end{aligned} \tag{2.4}$$

where

$$w^{*}_{q+1} = \sum_{j,m,p=0}^{\infty} \sum_{i=0}^{j} \frac{2\alpha\lambda(-1)^{m+p+q}}{q+1} w_{j,i} \binom{\alpha(j-i+1)+1}{m} \binom{\alpha(j-i+1)-1}{p} \times \binom{\lambda(m+p+1)-1}{q}. \tag{2.5}$$

Furthermore, for $\delta > 1$,

$$\begin{aligned} f_{MO-EHL-G}(x;\alpha,\delta,\lambda,\xi) &= \sum_{j,m,p,q=0}^{\infty} 2\alpha\lambda(-1)^{m+p+q} v_j \binom{-(\alpha(j+1)+1)}{m} \\ &\times \binom{\alpha(j+1)-1}{p} \binom{\lambda(m+p+1)-1}{q} \\ &\times g(x;\xi)[G(x;\xi)]^{q} \\ &= \sum_{q=0}^{\infty} v^{*}_{q+1} g_{q+1}(x;\xi), \end{aligned} \tag{2.6}$$

where

$$v^{*}_{q+1} = \sum_{j,m,p=0}^{\infty} 2\alpha\lambda(-1)^{m+p+q} v_j \binom{-(\alpha(j+1)+1)}{m} \binom{\alpha(j+1)-1}{p} \times \binom{\lambda(m+p+1)-1}{q}. \tag{2.7}$$

2.2 *Entropy and Order Statistics*

In this section, we derive Rényi entropy [28] and distribution of order statistics for the MO-EHL-G family of distributions. See appendix for details.

2.2.1 Rényi Entropy

Thus, for $\delta \in (0,1)$ we get

$$I_R(v) = (1-v)^{-1} \log\left[\sum_{q=0}^{\infty} \eta_{q+1} e^{(1-v) I_{REG}} \right], \tag{2.8}$$

where

$$\eta_{q+1} = \sum_{i,j,m,p=0}^{\infty} \frac{(2\alpha\delta\lambda)^{v}(1-\delta)^{j}(2v+j)}{\Gamma(2v)j!} (-1)^{i+m+p+q} \binom{j}{i} \binom{-(\alpha(i+v)+v)}{m} \times \binom{\alpha(i+v)-v}{p} \binom{\lambda(m+p+v)-v}{q} \frac{1}{\left[1+\frac{q}{v}\right]^{v}}.$$

Furthermore, for $\delta > 1$, the Rényi of the MO-EHL-G family of distributions is given by

$$I_R(\nu) = (1-\nu)^{-1} \log\left[\sum_{q=0}^{\infty} \eta_{q+1}^{*} e^{(1-\nu)I_{REG}}\right], \tag{2.9}$$

where

$$\eta_{q+1}^{*} = \sum_{j,m,p=0}^{\infty} \frac{(2\alpha\lambda)^{\nu}(\delta-1)^{j}\Gamma(2\nu+j)}{\delta^{\infty}\Gamma(2\nu)j!}(-1)^{m+p+q}\binom{-(\alpha(j+\nu)+\nu)}{m}$$
$$\times\binom{\alpha(j+\nu)-\nu}{p}\binom{\lambda(m+p+\nu)-\nu}{q}\frac{1}{\left[1+\frac{q}{v}\right]^{v}}.$$

for $v>0,\quad v\neq 1,$ where $I_{REG} = \frac{1}{1-v}\log\left[\int_0^{\infty}\left(\left[\frac{q}{v}+1\right](G(x;\xi))^{\frac{q}{v}} g(x;\xi)\right)^{v} dx\right]$ is the Rényi entropy of Exp-G distribution with power parameter $\left(\frac{q}{v}+1\right)$.

2.2.2. Distribution of Order Statistics

Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d) random variables distributed according to equation (2.2),

- for $\delta\in(0,1)$, the pdf of the i^{th} order statistic is

$$f_{i:n}(x) = \sum_{j,m,p,q=0}^{\infty}(2\alpha\lambda)(-1)^{m+p+n}U_{j,l,k}\binom{-(\alpha(j+l-k+i)+1)}{m}$$
$$\times\binom{\alpha(j+l-k+i)-1}{p}\binom{\lambda(m+p+1)-1}{q}g(x;\xi)G^{q}(x;\xi)$$
$$=\sum_{q=0}^{\infty}h_{q+1}g_{q+1}(x;\xi),$$

  where

$$h_{q+1} = \sum_{j,m,p=0}^{\infty}\frac{(2\alpha\lambda)(-1)^{m+p+n}}{q+1}U_{j,l,k}\binom{-(\alpha(j+l-k+i)+1)}{m}$$
$$\times\binom{\alpha(j+l-k+i)-1}{p}\binom{\lambda(m+p+1)-1}{q}$$

  and $g_{q+1}(x;\xi)$ is an Exp-G distribution with power parameter $(q+1)$.

- For $\delta>1$, the pdf of the i^{th} order statistic is

$$f_{i:n}(x) = \sum_{j,m,p,q=0}^{\infty}\sum_{l=0}^{n-i}2\alpha\lambda(-1)^{m+p+q}C_{j,l}\binom{-(\alpha(j+l+i)+1)}{m}\binom{\alpha(j+l+i)+1}{p}$$
$$\times\binom{\lambda(m+p+1)-1}{q}g(x;\xi)[G(x;\xi)]^{q}$$
$$=\sum_{q=0}^{\infty}C_{q+1}^{*}g_{q+1}(x;\xi),$$

where

$$C^*_{q+1} = \sum_{j,m,p=0}^{\infty} \sum_{i=0}^{n-i} \frac{2\alpha\lambda(-1)^{m+p+q}}{q+1} C_{j,l} \binom{-(\alpha(j+l+i)+1)}{m} \binom{\alpha(j+l+i)+1}{p} \times \binom{\lambda(m+p+1)-1}{q}.$$

Therefore, the distribution of the i^{th} order statistics from the MO-EHL-G family of distributions can be obtained from those of the Exp-G distribution since the pdf of the i^{th} order statistic is an infinite linear combination of Exp-G densities.

3. Maximum Likelihood Estimation

Let $X_i \sim MO-EHL-G(\alpha,\delta,\lambda,\xi)$ with the parameter vector Ψ = (α, δ, λ, $\xi)^T$. The log-likelihood function $\ell = \ell(\Psi)$ from a random sample of size n is given by

$$\ell(\Psi) = n\log(2\alpha\delta\lambda) + \sum_{i=1}^{n} \log[g(x_i;\xi)] + (\lambda-1)\sum_{i=1}^{n} \log[\overline{G}(x_i;\xi)] + (\alpha-1)\sum_{i=1}^{n} \log[1-\overline{G}^{\lambda}(x_i;\xi)] - (\alpha+1)\sum_{i=1}^{n} \log[1+\overline{G}^{\lambda}(x_i;\xi)] - 2\sum_{i=1}^{n} \log\left[1-\overline{\delta}\left(1-\left[\frac{1-\overline{G}^{\lambda}(x_i;\xi)}{1+\overline{G}^{\lambda}(x_i;\xi)}\right]^{\alpha}\right)\right].$$

The elements of the score vector $U(\Psi)$ are given in the appendix section.

4. Some Special Cases

Two special cases are considered in this section by taking baseline distribution to be Weibull and log-logistic distributions.

4.1 *Marshall-Olkin-Exponentiated Half Logistic-Weibull (MO-EHL-W) Distribution*

Suppose Weibull distribution is the baseline distribution with pdf and cdf given by $g(x;\theta) = \theta x^{\theta-1} e^{-x^{\theta}}$ and $G(x;\theta) = 1-e^{-x^{\theta}}$, respectively, for $\theta > 0$, then cdf and pdf of the MO-EHL-W distribution are given by

$$F_{MO-EHL-W}(x;\alpha,\delta,\lambda,\theta) = 1-\left(\frac{\delta(1-H(x;\theta,\alpha,\lambda))}{1-\overline{\delta}(1-H(x;\theta,\alpha,\lambda))}\right),$$

and

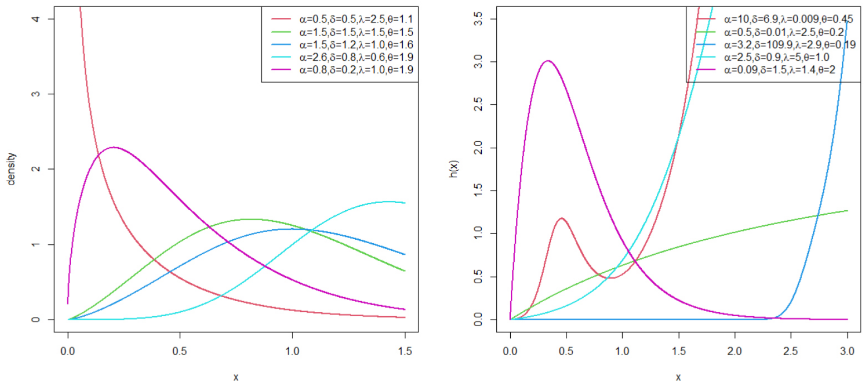

Figure 4.1

Plots of the pdf and hrf for the EHL-MO-W distribution

$$f_{MO-EHL-W}(x;\alpha,\delta,\lambda,\theta)=\frac{2\alpha\delta\lambda\theta x^{\theta-1}e^{-\lambda x^{\theta}}[1-e^{-\lambda x^{\theta}}]^{\alpha-1}}{(1+e^{-\lambda x^{\theta}})^{\alpha+1}[1-\bar{\delta}(1-H(x;\theta,\alpha,\lambda))]^{2}},$$

where $H(x;\theta,\alpha)=\left[\frac{1-e^{-\lambda x^{\theta}}}{1+e^{-\lambda x^{\theta}}}\right]^{\alpha}$, for α, δ, λ and $\theta>0$.

Plots of the pdfs and hrfs for the MO-EHL-W distribution are shown in Figure 4.1. The pdf depicts reverse-J, left or right-skewed shapes. The hrf takes both monotonic and non-monotonic shapes.

4.2 *Marshall-Olkin-Exponentiated Half Logistic-Log-Logistic (MO-EHL-LLoG) Distribution*

By taking the log-logistic distribution as the baseline distribution with pdf and cdf given by $g(x;\theta)=\theta x^{\theta-1}(1+x^{\theta})^{-2}$ and $G(x;\theta)=1-(1+x^{\theta})^{-1}$, respectively, for $\theta>0$, we obtain the MO-EHL-LLoG distribution with cdf and pdf given by

$$F_{MO-EHL-LLoG}(x;\alpha,\delta,\lambda,\theta)=1-\left(\frac{\delta(1-H(x;\theta,\alpha,\lambda))}{1-\bar{\delta}(1-H(x;\theta,\alpha,\lambda))}\right),$$

and

$$f_{MO-EHL-LLoG}(x;\alpha,\delta,\lambda,\theta)=\frac{2\alpha\delta\lambda\theta x^{\theta-1}(1+x^{\theta})^{\lambda-3}[1-(1+x^{\theta})^{-\lambda}]^{\alpha-1}}{(1+(1+x^{\theta})^{-\lambda})^{\alpha+1}[1-\bar{\delta}(1-H(x;\theta,\alpha,\lambda))]^{2}},$$

where $H(x;\theta,\alpha,\lambda)=\left[\frac{1-(1+x^{\theta})^{-\lambda}}{1+(1+x^{\theta})^{-\lambda}}\right]^{\alpha}$, for α, δ, λ and $\theta>0$.

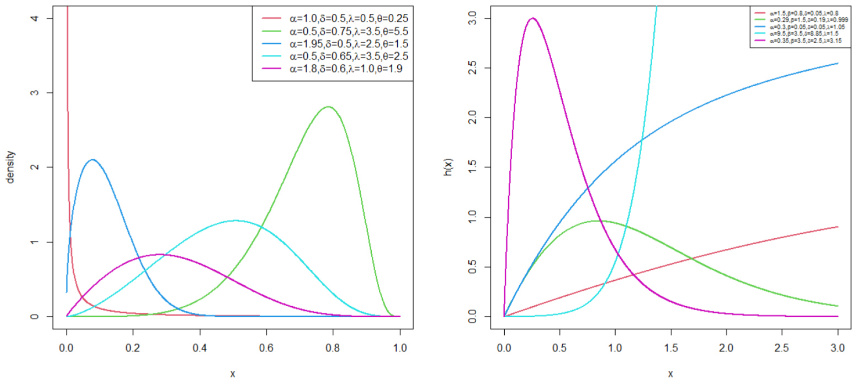

Figure 4.2

Plots of the pdf and hrf for the EHL-MO-LLoG distribution

Plots of the pdfs and hrfs for the MO-EHL-LLoG distribution are shown in Figure 4.2. The pdf can take various shapes that include almost symmetric, reverse-J, left or right-skewed. The hrf exhibit increasing, reverse-J and uni-modal shapes.

5. Simulation Study

In this section, we conducted a simulation study to assess the performance of the maximum likelihood estimates of the MO-EHL-LLoG distribution. We considered N=1000 times for samples n = 50, 100, 200, 400, 800 and 1000. Simulation results are shown in Table 5.1. We estimate the mean, average bias and root mean square error (RMSE). The bias and RMSE for the estimated parameter, say, $\hat{\Delta}$, are given by:

$$Bias(\hat{\Delta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta), \quad \text{and} \quad RMSE(\hat{\Delta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta)^2}{N}},$$

respectively. We fixed the parameter $\lambda = 1$ in the following sets of initial parameter values (I: $\alpha = 0.5$, $\delta = 1.0$, $\theta = 1.5$), (II: $\alpha = 1.0$, $\delta = 1.1$, $\theta = 0.5$), (III: $\alpha = 1.0$, $\delta = 0.5$, $\theta = 1.50$ and (IV: $\alpha = 1.5$, $\delta = 0.5$, $\theta = 0.5$). Our model produces consistent results as evidence from the simulation study results. As the sample size increase, the mean approximate the true parameter value, the bias and RMSE decay as well for large n.

Table 5.1

Monte Carlo Simulation Results for MO-EHL-LLoG Distribution: Mean, RMSE and Average Bias

		I : $\alpha = 0.5, \delta = 1.0, \theta = 1.5$			II : $\alpha = 1.0, \delta = 1.1, \theta = 0.5$		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	50	0.5801	0.2805	0.0801	1.1460	0.5254	0.1460
	100	0.5452	0.1944	0.0452	1.1240	0.4261	0.1240
α	200	0.5218	0.1236	0.0218	1.0987	0.3194	0.0987
	400	0.5182	0.0844	0.0182	1.0922	0.2056	0.0922
	800	0.5085	0.0603	0.0085	1.0805	0.1575	0.0805
	1000	0.5084	0.0513	0.0084	1.0752	0.1264	0.0752
	50	1.1168	0.8224	0.1168	1.3533	1.2286	0.2533
	100	1.0520	0.5207	0.0520	1.1671	0.7242	0.0671
δ	200	1.0341	0.3588	0.0341	1.0906	0.4844	-0.0094
	400	0.9936	0.2400	-0.0065	1.0231	0.3342	-0.0769
	800	0.9969	0.1774	-0.0031	1.0012	0.2545	-0.0988
	1000	0.9918	0.1518	-0.0082	0.9986	0.2088	-0.1014
	50	1.5148	0.2856	0.0148	0.5076	0.0971	0.0076
	100	1.5120	0.2076	0.0120	0.4993	0.0716	-0.0007
θ	200	1.5056	0.1525	0.0056	0.4948	0.0545	-0.0052
	400	1.4965	0.1065	-0.0035	0.4918	0.0384	-0.0082
	800	1.4984	0.0759	-0.0016	0.4914	0.0275	-0.0086
	1000	1.4995	0.0668	-0.0005	0.4942	0.0228	-0.0058
		III : $\alpha = 1.0, \delta = 0.5, \theta = 1.5$			IV : $\alpha = 1.5, \delta = 0.5, \theta = 0.5$		
	50	1.1326	0.5814	0.1326	1.5989	0.7433	0.0989
	100	1.0820	0.4376	0.0820	1.5662	0.5862	0.0662
α	200	1.0444	0.3053	0.0444	1.5513	0.4446	0.0513
	400	1.0288	0.2042	0.0288	1.5530	0.2961	0.0530
	800	1.0081	0.1312	0.0081	1.5269	0.2154	0.0269
	1000	1.0067	0.1110	0.0067	1.5233	0.1789	0.0233
	50	0.6473	0.6147	0.1473	0.7517	0.7701	0.2517
	100	0.5665	0.3528	0.0665	0.6392	0.4818	0.1392
δ	200	0.5410	0.2491	0.0410	0.5714	0.3171	0.0714
	400	0.5127	0.1653	0.0127	0.5156	0.2028	0.0156
	800	0.5102	0.1197	0.0102	0.5085	0.1503	0.0085
	1000	0.5069	0.1030	0.0069	0.5043	0.1296	0.0043
	50	1.5306	0.3721	0.0306	0.5187	0.1255	0.0187
	100	1.5154	0.2674	0.0154	0.5115	0.0946	0.0115
θ	200	1.5055	0.2025	0.0055	0.5032	0.0718	0.0032
	400	1.4972	0.1390	-0.0028	0.4976	0.0503	-0.0024
	800	1.5022	0.0986	0.0022	0.4987	0.0366	-0.0013
	1000	1.5020	0.0862	0.0020	0.4994	0.0319	-0.0006

6. Applications

Two real data examples are presented to demonstrate the flexibility and usefulness of the MO-EHL-LLoG distribution. The new proposed model is compared to other competing models. Model performance was assessed using the goodness-of-fit statistics, namely, -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-von-Mises (W^*) and Andersen-Darling (A^*) as described by Chen and Balakrishnan [7]. The model with the smaller values of these statistics is regarded as the best model.

Data analysis results are shown in Tables 6.1 and 6.2. The maximum likelihood estimation method was used to estimate the moedl parameters and the technique was used by Rahardja et al. [26]. Graphical representations are also used (see Chambers et al. [6] for details), to demonstrate how our model fit the selected data sets (see Figures 6.1 and 6.2 for details).

We compared the MO-EHL-LLoG distribution to the following models: Kumaraswamy odd Lindley-Log logistic (KOL-LLoG) by Chipepa et al. [10], beta generalized Lindley (BGL) by Oluyede and Yang [25], Kumaraswamy-Weibull (KW) by Cordeiro et al. [13], odd exponentiated half logistic-Burr XII (OEHLBXII) by Aldahlan et al. [2], beta odd Lindley-uniform by Chipepa et al. [9], the Topp-Leone-Weibull-Lomax (TLWLx) distributions by Jamal et al. [18], and the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [20].

6.1 *Kevlar 49/Epoxy Strands Failure at 90% Data*

The first data set consist of 101 observations representing the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. The data set was also analyzed by Chipepa et al. [8].

Based on the values of the goodness-of-fit statistics A^*, W^*, K-S (and the p-value of the K-S statistic) as shown in Table 6.1, we conclude that the MO-EHL-LLoG model performs better than the other models considered on kevlar data set. Figure 6.1 shows the fitted densities and probability plots for the MO-EHL-LLoGP model and other competing models. We observe the flexibility enjoyed in data fitting from the MO-EHL-LLoG model compared to the other models considered using kevlar data set.

Table 6.1

Parameter estimates and goodness-of-fit statistics for various fitted models based on kevlar data set

Model	Estimates				Statistics							
	α	λ	δ	c	-2 log L	AIC	AICC	BIC	W^*	A^*	K-S	p-value
MO-EHL-LLoG	0.1955	2.3257	0.8963	2.9692	201.7	209.7	210.2	220.2	0.0878	0.5866	0.0758	0.6068
	(0.1331)	(1.06287)	(0.6708)	(1.6909)								
	a	b	λ	α								
BGL	2.4825	10.0965	0.1314	0.3958	204.9	212.9	213.4	223.4	0.1625	0.9395	0.0839	0.4745
	(4.6366)	(16.6595)	(0.2165)	(0.5156)								
	a	b	λ	θ								
BOL-U	0.87181	10.3230	2.3456×10^4	2.8294×10^5	205.7	213.7	214.1	224.1	0.1819	1.0332	0.0894	0.3940
	(0.1067)	(1.6271)	(7.1192×10^{-4})	(5.9016×10^{-5})								
	a	b	α	θ								
TL-WLx	0.6616	0.7096	1.3632	0.6078	205.4	213.4	213.8	223.9	0.0834	0.9447	0.0834	0.4825
	(0.6594)	(0.9639)	(0.9469)	(0.4455)								
	a	b	α	β								
KW	1.2800	2069.1000	2.6750×10^{-4}	0.7239	206.0	214.0	214.4	224.4	0.1992	1.1135	0.0911	0.3714
	(0.0245)	(7.6348×10^{-8})	(1.6886×10^{-4})	(0.0438)								
	β	λ	θ	γ								
ELOLLW	7.0692	0.1556	7.3936	0.8374	205.2	213.2	213.6	223.7	0.1665	0.9601	0.0815	0.5123
	(4.1286)	(0.1114)	(3.9446)	(0.1094)								
	α	λ	a	b								
OEHL-BXII	0.1395	0.7355	5.3708	0.1910	215.5	223.5	223.9	234.0	0.1689	0.9968	0.0993	0.2723
	(0.0313)	(0.4943)	(1.2409)	(0.0621)								
	a	b	λ	c								
KOL-LLoG	0.9437	3.1448	0.6028	0.8524	205.1	213.1	213.5	223.5	0.1481	0.8782	0.0787	0.5591
	(0.4861)	(12.7461)	(1.5014)	(0.4004)								

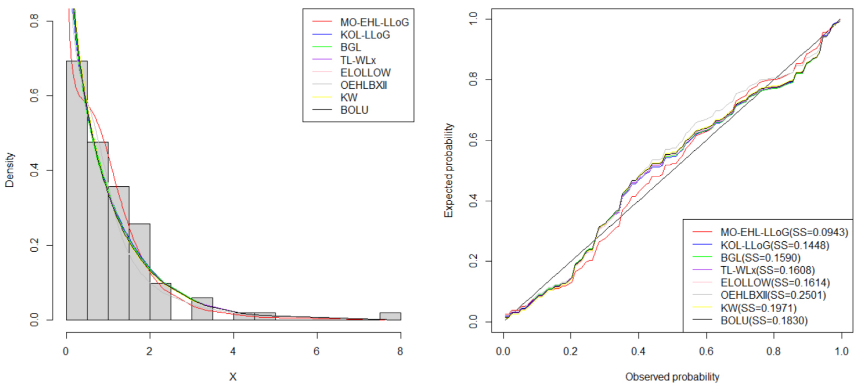

Figure 6.1

Fitted pdfs and probability plots for kevlar data set

6.2 *Cancer Patients*

The data set represents remission times of a random sample of 128 bladder cancer patients. The data set was analyzed by Lee and Wang [21].

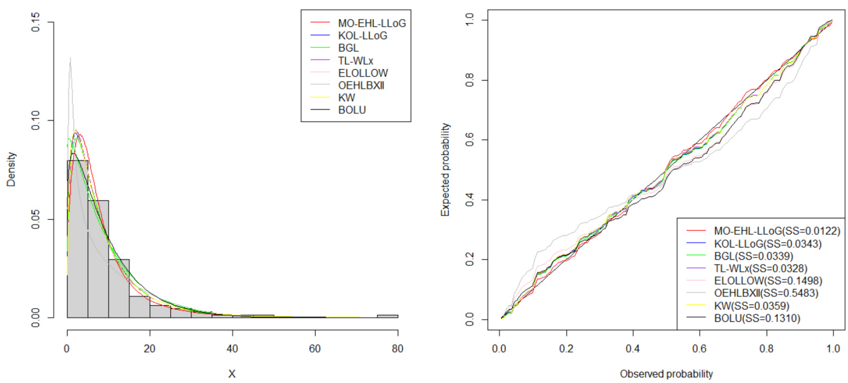

Figure 6.2

Fitted pdfs and probability plots for run off data set

Furthermore, based on the values of the goodness-of-fit statistics A^*, W^*, K-S (and the p-value of the K-S statistic) as shown in Table 6.2, we conclude that the MO-EHL-LLoGP model performs better than the other models on cancer patients data set. Figure 6.2 shows the fitted densities and probability plots for the fitted models. The second example affirms

Table 6.2

Parameter estimates and goodness-of-fit statistics for various fitted models on kevlar data set

Model	Estimates				Statistics							
	α	λ	δ	c	-2log L	AIC	AICC	BIC	W^*	A^*	K-S	p-value
MO-EHL-LLoG	3.1465	104.8089	5.3491	0.4702	819.0	827.0	827.4	838.4	0.0130	0.0871	0.0299	0.9999
	(7.3793)	(0.1853)	(3.3208)	(0.2444)								
	a	b	λ	α								
BGL	32.3100	93.1320	7.5713×10^{-5}	8.6822×10^{-3}	820.4	828.4	828.7	839.8	0.0376	0.2389	0.0426	0.9746
	(9.2237×10^{-6})	(6.8577×10^{-7})	(3.6594×10^{-5})	(5.4131×10^{-3})								
	a	b	λ	c								
KOLLLoG	3.3468	1.8102	0.8226	0.4724	820.8	828.8	829.2	840.3	0.0393	0.2561	0.0429	0.9724
	(3.0986)	(3.3767)	(0.4034)	(0.3214)								
	a	b	λ	θ								
BOLU	1.1988	1.3360	6.9105×10^{5}	7.6215×10^{6}	826.4	834.4	834.7	845.8	0.1151	0.6908	0.0735	0.4940
	(1.3875×10^{-1})	(1.5755×10^{-1})	(3.0371×10^{-7})	(2.7537×10^{-8})								
	a	b	α	θ								
TL-WLx	0.5080	0.2070	1.0718	1.5290	820.8	828.8	829.1	840.2	0.0372	0.2469	0.0429	0.9728
	(0.4035)	(0.2299)	(0.6876)	(1.1201)								
	a	b	α	θ								
KW	4.1179	2.9414	0.2159	0.4589	821.1	829.1	829.5	840.5	0.0415	0.2732	0.0446	0.9602
	(5.8532)	(8.1732)	(0.2492)	(0.5169)								
	β	λ	θ	γ								
ELOLLW	6.1173×10^{-5}	3.7799×10^{-2}	2.9053	1.0478	828.2	836.2	836.5	847.6	0.131367	0.7864	0.0699	0.5573
	(0.4015)	(6.0762×10^{-3})	(8.4383×10^{-5})	(6.7577×10^{-2})								
	α	λ	a	b								
OEHL-BXII	0.3925	0.0274	3.6829	0.3660	881.8	889.8	890.1	901.2	0.3274	1.9125	0.1166	0.0616
	(0.0648)	(0.0187)	(1.0211)	(0.0976)								

that the MO-EHL-LLoG model performs better than the models considered in this paper.

7. Concluding Remarks

We developed a new family of distributions, by combining the Marshall-Olkin-G and the exponentiated half logistic-G distributions. The new distribution can handle heavy tailed data and also have monotonic as well as non-monotonic hazard rate shapes. The proposed distribution can be expressed as a linear combination of the Exp-G distribution. We applied the new distribution to two real data sets and our model performs better than several other competing equal parameter models as shown in Tables 6.1 and 6.2.

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Appendix

The following URL contains the appendix material:
https://drive.google.com/file/d/1RGO_lwxR13Vj-nwrt3F2_ifUPporODCV/view?usp=sharing

References

[1] A. N. Marshall and I. Olkin, "A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families," *Biometrika*, vol. 84, pp. 641-652 (1997).

[2] R. C. Gupta and R. D. Gupta, "Modeling failure time data by Lehmann alternatives," *Communications in Statistics: Theory and Methods*, pp. 887-904 (1998).

[3] E. L. Lehmann, "The power of rank tests," *Annals of Mathematics and Statistics*, vol. 24, pp. 23-43 (1953).

[4] R. D. Gupta and D. Kundu, "Generalized exponential distribution," *Austral and New Zealand Journal of Statistics*, vol. 41, pp. 173-188 (1999).

[5] N. Eugene, C. Lee, and F. Famoye, "The Beta-Normal distribution and its applications," *Communications in Statistics: Theory and Methods*, vol. 31, no. 4, pp. 497-512 (2002).

[6] M. C. Jones, "Families of distributions arising from the distribution of order statistics," Test, vol. 13, pp. 1-43 (2004).

[7] G. M. Cordeiro and M. A. de Castro, "A new family of generalized distribution," *Journal of Statistical Computations and Simulation*, vol. 81, no. 7, pp. 883-898 (2011).

[8] K. Zografos and N. Balakrishnan, "On families of beta- and generalized gamma-generated distributions and associated inference," *Statistical Methodology*, vol. 6, pp. 344-362 (2009). doi: 10.1016/j.stamet.2008.12.003.

[9] M. M. Ristic and N. Balakrishnan, "The gamma-exponentiated exponential distribution," *Journal of Statistical Computation and Simulation*, vol. 82, no. 8, pp. 1191-1206 (2012).

[10] A. Alzaatreh, C. Lee, and F. Famoye, "A new method for generating families of continuous distributions," *Metron*, vol. 71, no. 1, pp. 63-79 (2013).

[11] M. Bourguignon, R. B. Silva, and G. M. Cordeiro, "The Weibull-G family of probability distributions," *Journal of Data Science*, vol. 12, pp. 53-68 (2014).

[12] G. M. Cordeiro, M. Alizadeh, and P. R. D. Marinho, "The Type I half-logistic family of distributions," *Journal of Statistical Computation and Simulation*, vol. 86, no. 4, pp. 707-728 (2016).

[13] M. Elgarhy, A. S. Hassan, and M. Rashed, "Garhy-generated family of distributions with application," *Mathematical Theory and Modeling*, vol. 6, no. 2, pp. 1-15 (2016).

[14] A. Al-Shomrani, O. Arif, A. Shawky, S. Hanif, and M. Q. Shahbaz, "Topp-Leone family of distributions: Some properties and application," *Pakistan Journal of Statistics and Operation Research*, vol. 12, no. 3, pp. 443-451 (2016).

[15] F. Chipepa, B. Oluyede, and B. Makubate, "The Topp-Leone-Marshall-Olkin-G family of distributions with applications," *International*

Journal of Statistics and Probability, vol. 9, no. 4, pp. 15-32 (2020). doi: 10.5539/ijsp.v9n4p15.

[16] F. Chipepa, B. Oluyede, B. Makubate, and A. F. Fagbamigbe, "The Beta Odd Lindley-G family of distributions with applications," *Journal of Probability and Statistical Science*, vol. 17, no. 1, pp. 51-83 (2019). https://doi.org/10.5539/ijsp.v8n6p1.

[17] F. Chipepa, B. Oluyede, and B. Makubate, "A new generalized family of odd Lindley-G distributions with application," *International Journal of Statistics and Probability*, vol. 8, no. 6, pp. 1-22 (2019). doi: 10.5539/ijsp.v8n6p1.

[18] B. Makubate, F. Chipepa, B. Oluyede, and O. P. Peter, "The Marshall-Olkin Half Logistic-G family of distributions with applications," *International Journal of Statistics and Probability*, vol. 10, no. 2, pp. 120-136 (2021).

[19] W. Barreto-Souza, A. J. Lemonte, and G. M. Cordeiro, "General results for the Marshall and Olkin's family of distributions," *Annals of the Brazilian Academy of Sciences*, vol. 85, no. 1, pp. 3-21 (2013).

[20] A. Rényi, "On measures of entropy and information," in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, vol. 1, The Regents of the University of California (1961).

[21] G. Chen and N. Balakrishnan, "A general purpose approximate goodness-of-fit test," *Journal of Quality Technology,* vol. 27, no. 2, pp. 154-161 (1985).

[22] D. Rahardja, H. Wu, Z. Zhang, and A. D. Tiedt, "Maximum likelihood estimation for the proportion difference of two-sample binomial data subject to one type of misclassification," *Journal of Statistics and Management Systems*, vol. 22, no. 8, pp. 1365-1379 (2019). doi: 10.1080/09720510.2019.1606319.

[23] J. Chambers, W. Cleveland, B. Kleiner, and P. Tukey, Graphical Methods of Data Analysis, Chapman and Hall (1983).

[24] B. O. Oluyede and T. Yang, "A new class of generalized Lindley distributions with applications," *Journal of Statistical Computation and Simulation*, vol. 10, pp. 2072-2100 (2015). doi: 10.1080/00949655.2014.917308.

[25] G. M. Cordeiro, E. M. M. Ortega, and S. Nadarajaah, "The Kumaraswamy Weibull distribution with application to failure data," *Journal of the Franklin Institute*, vol. 347, pp. 1399-1429 (2010).

[26] M. Aldahlan and A. Z. Afify, "The odd exponentiated half-logistic Burr XII distribution," *Pakistan Journal of Statistics and Operation Research*, vol. 14, no. 2, pp. 305-317 (2018).

[27] F. Jamal, H. M. Reyad, M. A. Nasir, C. Chesneau, M. A. A. Shah, and S. O. Ahmed, "Topp-Leone Weibull-Lomax distribution: Properties, regression model and applications," hal-02270561 (2019).

[28] M. C. Korkmaz, H. M. Yousof, and G. G. Hamedani, "The exponential Lindley odd log-logistic-G family: Properties, characterizations and applications," *Journal of Statistical Theory and Applications*, vol. 17, no. 3, pp. 554-571 (2018).

[29] E. T. Lee and J. W. Wang, Statistical Methods for Survival Data Analysis, John Wiley and Sons, New York, ISBN: 9780471458555, 534 pp. (2003).

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