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The half-logistic (Type-I)-skew-t model : Comparison of different estimation approaches using Monte Carlo simulations

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Abstract

In the paper, we looked at the problem of employing several non-Bayesian estimation approaches to estimate the parameters of the Type-I half-logistic skew-t (TIHLST) model. The maximum likelihood, least squares, weighted least squares, Anderson-Darling, maximum product of spacing, and Cramer-von Mises are the six methods for TIHLST parameter estimation that we examined. The Monte Carlo simulations were used to assess the TIHLST parameter estimations' finite sample performance. The partial and overall ranks of the estimating techniques for every combination of parameters were used to determine the ordering performance of the six criteria. Based on the overall ranks, the results indicate that the criteria performance is ranked from best to worst using the following estimators: maximum likelihood, weighted least squares, maximum product of spacing, least squares, and Cramer-von Mises for all parameter combinations. Based on an overall rank of 34,

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we infer that the maximum-likelihood criterion fared better than all other criteria. For the TIHLST model, we thereby affirm the supremacy of the maximum likelihood criterion with an overall rank of 34 and the Anderson-Darling criterion with an overall rank of 57.5.

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Keywords: *Simulation, Maximum likelihood, Cramer-von mises, Skew-t distribution, Type-I half-logistic-G.*

1. Introduction

The skew-t (ST) model having a complex structure, absence of moment generating function, and heavy tail feature, has prompted numerous statisticians to suggest more adaptable and hybridized variants. These variants can simulate real-world datasets with unimodal, increasing-decreasing, bathtub, inverted bathtub, and increasing-decreasing failure rates, which are frequently seen in time series, reliability, finance, econometrics, and medicine. Some notable variants of the skew-t model include the exponentiated generalized-ST [1], the Balakrishnan-ST [2], the generalized hyperbolic-ST [3], the Kumaraswamy-ST [4], the Beta-ST distribution [5], and the new Beta-ST [6], among others.

[7] used the family of distributions established by [9] to suggest a hybridized form of the ST model of [8]. They investigated the two-parameter Type-I half logistic-ST ($TIHL_{ST}$) model, which has bathtub hazard rate forms that might be increasing, decreasing, or inverted. It is possible for the $TIHL_{ST}$ density to be symmetric, left- or right-skewed. A few of its significant statistical characteristics, including the quantile function, ordinary and incomplete moments, probability weighted moments, entropies, and order statistics, were also studied. Two real datasets from the disciplines of engineering and medicine were fitted using the $TIHL_{ST}$ model, which yielded superior results than the half logistic-ST, basic pareto, exponentiated generalized Pareto, Fréchet, and ST models. Comparing several classical estimate approaches for parameter estimation of various distributions has attracted significant attention recently. Examples include the following: the odd exponential half-logistic exponential [15], the Type-I half-logistic Top-Leone [16], the extended exponential geometric [10], the Poisson exponential [11], the Binomial exponential 2 [12], the polynomial exponential [13], the Fréchet [14], and the maximum likelihood estimation for the proportion difference of two-sample Binomial data [17], among others.

Based on the half logistic (Type-I)-G (TIHL-G) family described by [9], the $TIHL_{ST}$ is developed. Given is the cumulative distribution function (CDF):

$$F(y;\varphi,\kappa)=\frac{(1-[1-H(y;\kappa)]^{\varphi})}{(1+[1-H(y;\kappa)]^{\varphi})},\quad y\in\Re \tag{1}$$

and corresponding probability density function (PDF) of:

$$f(y;\varphi,\kappa)=\frac{2\varphi h(y;\kappa)\langle[1-H(y;\kappa)]^{\varphi-1}\rangle}{\{1+[1-H(y;\kappa)]^{\varphi}\}^{2}},\quad y\in\Re \tag{2}$$

where $\varphi>0$ is the shape parameter, $H(y;\kappa)$ and $h(y;\kappa)$ are the baseline CDF and PDF depending on the parameter vector (κ) enabling the resultant model greater flexibility to handle all significant hazard rate function (HRF) forms. The $TIHL_{ST}$ model's CDF and PDF developed by [7] are expressed as

$$F(y,\upsilon)=\left\langle\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}}\right\rangle,\quad y\in(-\infty,\infty),\varphi,\eta>0 \tag{3}$$

and

$$f(y,\upsilon)=\frac{2\varphi\left(\frac{\eta}{2(\eta+y^{2})^{3/2}}\right)\left\langle\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi-1}\right\rangle}{\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}\right\}^{2}},\quad y\in(-\infty,\infty),\varphi,\eta>0 \tag{4}$$

The $TIHL_{ST}$ distribution quantile function takes the form

$$x_{q}=\eta^{\frac{1}{2}}\frac{\left[1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right]}{\left\{\left[1-\left(1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right)^{2}\right]^{\frac{1}{2}}\right\}},\qquad 0<q<1. \tag{5}$$

[7] examined a number of explicit statistical characteristics of the $TIHL_{ST}$ model, such as the order statistics, quantile function, probability weighted moment, ordinary and incomplete moments, and entropies. Additionally, they examined two real datasets from the engineering and medical domains to show its efficacy. They only employed the most popular estimating technique, the maximum likelihood method, to estimate the $TIHL_{ST}$ parameters. Figures 1 and 2 display some of the $TIHL_{ST}$ density and hazard rate function shapes. The purpose of this article is estimation of the $TIHL_{ST}$ parameters using various non-Bayesian estimation methods: maximum-likelihood-estimation ('MLE'), least-squares-estimation ('LSE'), weighted-least-squares-estimation ('WLSE'), Anderson-Darling-estimation ('ADE'), maximum-product of spacing-estimation ('MPSE'), and Cramer-Von-Mises-estimation ('CVME'). These estimating approaches are evaluated using extensive-simulations based on Monte-Carlo study to determine their performance.

The article is organized as follows: Section 2 describes the six non-Bayesian estimators produced for the $TIHL_{ST}$ model parameters. Section 3: comprehensive Monte Carlo simulation to compare the estimating algorithms' performance. The conclusion is found in Section 4.

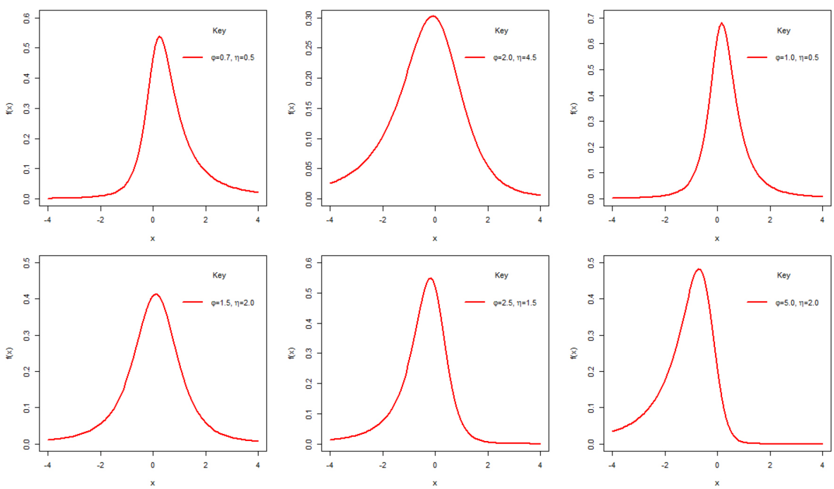

Figure 1

The $TIHL_{ST}$ model PDF Plots with selected value of the parameters.

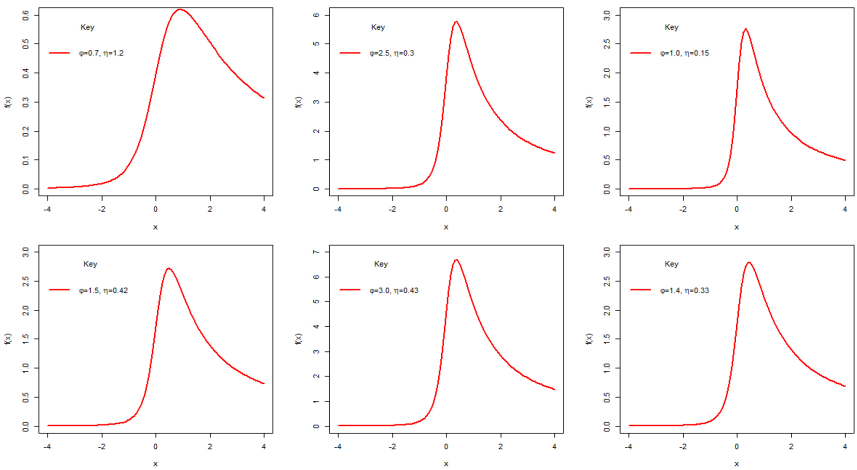

Figure 2

The TIHL_{ST} model HRF Plots with selected value of the parameters

2. Estimation Methods

The functional forms of the estimation techniques for the TIHL_{ST} are discussed here.

2.1 *MLE-Method*

Given that $y_i,\ i=1,\ldots,n$ is a random sample from the TIHL_{ST} model with unknown vector of parameter $\upsilon=(\varphi,\eta)^T$, then the log-likelihood-function, say l, is specified as:

$$l=\log L(\upsilon)=n\log 2\varphi+n\log\eta-n\log 2-3/2\sum_{i=1}^{n}\log(\eta+y^2)$$
$$+(\varphi-1)\sum_{i=1}^{n}\log\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)-2\sum_{i=1}^{n}\log\left(1+\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\varphi}\right) \tag{6}$$

Differentiating Eq. (6) partially with respect, we have the following non-linear equations:

$$\frac{\partial l}{\partial\varphi}=\frac{n}{\varphi}+\sum_{i=1}^{n}\ln\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)-2\sum_{i=1}^{n}\frac{\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\varphi}\ln\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)}{\left\{1+\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\varphi+y^2}}\right)\right)^{\alpha}\right\}}=0 \tag{7}$$

$$\frac{\partial l}{\partial \eta}=\frac{n}{\eta}-\frac{3}{2}\sum_{i=1}^{n}\frac{1}{(\eta+y^2)}+(\varphi-1)\sum_{i=1}^{n}\frac{y}{4(\eta+y^2)^{3/2}\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)}$$

$$-\varphi\sum_{i=1}^{n}\frac{x\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\varphi}}{2(\eta+y^2)^{3/2}\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)\left\{1+\left(1-\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^2}}\right)\right)^{\alpha}\right\}}=0 \tag{8}$$

The nonlinear system of equations can be solved numerically utilizing R', MATLAB', SAS and Mathematica owing to its complexity. In this work, R-software was used.

2.2 *MPSE-Method*

The MPSE-method proposed by [18,19] then developed by [20], is a good substitute to the MLE-method for unknown parameters estimation. Given that $y_{(i,j)}i,\ j=1,\ n,\ldots,n,n$ are 'ordered sample' from TIHL_{ST} model, then the random sample uniform spacing function is specified as

$$D_i(\varphi,\eta)=F(y_{(i)}\,|\,\varphi,\eta)-F(y_{(i-1)}\,|\,\varphi,\eta),\ \text{for}\ i=1,2,\ldots,n+1, \tag{9}$$

where $F(y_{(0)}\,|\,\varphi,\eta)=0,\ \ F(y_{(n+1)}\,|\,\varphi,\eta)=1,$ such that $\sum_{i=1}^{n+1}D_i(\varphi,\eta)=1.$ Then,

$$F(y_{(i)}\,|\,\varphi,\eta)=\left\{\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i)}}{\sqrt{\eta+y_{(i)}^2}}\right)\right)\right]^{\varphi}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i)}}{\sqrt{\eta+y_{(i)}^2}}\right)\right)\right]^{\varphi}}\right\}$$

and

$$F(y_{(i-1)}\,|\,\varphi,\eta)=\left\{\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i-1)}}{\sqrt{\eta+y_{(i-1)}^2}}\right)\right)\right]^{\varphi}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i-1)}}{\sqrt{\eta+y_{(i-1)}^2}}\right)\right)\right]^{\varphi}}\right\}.$$

The parameter estimates $\hat{\varphi}_{MPS}$ and $\hat{\eta}_{MPS}$ are found by maximising the geometric mean specified as

$$G(\varphi,\eta)=\left\langle\prod_{i=1}^{n+1}D_i(\varphi,\eta)\right\rangle^{\frac{1}{n+1}} \tag{10}$$

Likewise, the parameter estimates $\hat{\varphi}_{MPS}$ and $\hat{\eta}_{MPS}$ are found by evaluating the succeeding function

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\varphi,\eta)}\langle\Delta_t(y_{(i)}\mid\varphi,\eta)-\Delta_t(y_{(i-1)}\mid\varphi,\eta)\rangle=0,\text{ for } t=1,2 \tag{11}$$

where

$$\begin{aligned}\Delta_1(y_{(i)}\mid\varphi,\eta)&=\frac{\partial}{\partial\varphi}F(y_{(i)}\mid\varphi,\eta)\\ \Delta_2(y_{(i)}\mid\varphi,\eta)&=\frac{\partial}{\partial\eta}F(y_{(i)}\mid\varphi,\eta)\end{aligned} \tag{12}$$

2.3 *ADE-Method*

The ADE-method is a minimum distance estimator used in estimating model parameters [21]. The ADE estimates $\hat{\varphi}_{ADE}$ and $\hat{\eta}_{ADE}$ can be derived by minimizing

$$\mathrm{A}(\varphi,\eta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\langle\log[F(y_{(i:n)}\mid\varphi,\eta)]+\log[\bar{F}(y_{(n+1-i:n)}\mid\varphi,\eta)]\rangle, \tag{13}$$

with respect to φ and η. Also, utilizing Eq (14), the estimates $\hat{\varphi}_{ADE}$ and $\hat{\eta}_{ADE}$ can be found.

$$\sum_{i=1}^{n}(2i-1)\left\langle\frac{\Delta_t(y_{(i:n)}|\varphi,\eta)}{F(y_{(i:n)}|\varphi,\eta)}-\frac{\Delta_k(y_{(n+1-i:n)}|\varphi,\eta)}{1-F(y_{(n+1-i:n)}|\varphi,\eta)}\right\rangle=0,\text{ for } t,k=1,2 \tag{14}$$

where $\Delta_1(\cdot\mid\varphi,\eta)$ and $\Delta_2(\cdot\mid\varphi,\eta)$ are given in (12).

2.4 *CVME-Method*

The CVME method is another minimum distance estimator with a smaller bias [21,22]. The CVME estimates $\hat{\varphi}_{CVME}$ and $\hat{\eta}_{CVME}$ can be found by minimizing

$$C(\varphi,\eta)=\frac{1}{12n}+\sum_{i=1}^{n}\left\langle\left\{\frac{2i-1}{2n}\right\}-F(y_{(i:n)}\mid\varphi,\eta)\right\rangle^2, \tag{15}$$

relative to φ and η. Utilizing Eq. (16), the estimates $\hat{\varphi}_{CVME}$ and $\hat{\eta}_{CVME}$ can be found.

$$\sum_{i=1}^{n}\left\langle F(y_{(i:n)}\mid\varphi,\eta)-\left\{\frac{2i-1}{2n}\right\}\right\rangle\Delta_t(y_{(i:n)}\mid\varphi,\eta)=0,\text{ for } t=1,2 \tag{16}$$

where $\Delta_1(\cdot\mid\varphi,\eta)$ and $\Delta_2(\cdot\mid\varphi,\eta)$ are given in (12).

2.5. *LSE and WLSE-Methods*

The LSE and WLSE methods are based on ordered sample from a particular model [23]. Hence, The LSE estimates $\hat{\varphi}_{LSE}$ and $\hat{\eta}_{LSE}$ can be obtained by minimizing

$$L(\varphi,\eta)=\sum_{i=1}^{n}\left\langle F(y_{(i:n)}\,|\,\varphi,\eta)-\left\{\frac{i}{n+1}\right\}\right\rangle^{2} \tag{17}$$

relative to φ and η. Likewise, the estimates $\hat{\varphi}_{LSE}$ and $\hat{\eta}_{LSE}$ can be determined using

$$\sum_{i=1}^{n}\left\langle F(y_{(i:n)}\,|\,\varphi,\eta)-\left\{\frac{i}{n+1}\right\}\right\rangle \Delta_{t}(y_{(i:n)}\,|\,\varphi,\eta)=0,\ \text{for } t=1,2 \tag{18}$$

where $\Delta_1(\cdot\,|\,\varphi,\eta)$ and $\Delta_2(\cdot\,|\,\varphi,\eta)$ are given in (12).

The WLSE estimates $\hat{\varphi}_{WLSE}$ and $\hat{\eta}_{WLSE}$ can be obtained by minimizing,

$$W(\varphi,\eta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\langle F(y_{(i:n)}\,|\,\varphi,\eta)-\left\{\frac{i}{n+1}\right\}\right\rangle^{2} \tag{19}$$

relative to φ and η Likewise, the estimates $\hat{\varphi}_{WLSE}$ and $\hat{\eta}_{WLSE}$ can be determined using

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\langle F(y_{(i:n)}\,|\,\varphi,\eta)-\left\{\frac{i}{n+1}\right\}\right\rangle \Delta_{t}(y_{(i:n)}\,|\,\varphi,\eta)=0,\text{for}\, t=1,2 \tag{20}$$

where $\Delta_1(\cdot\,|\,\varphi,\eta)$ and $\Delta_2(\cdot\,|\,\varphi,\eta)$ are given in (12).

3. Simulation

This section compares and evaluates the performance of the six frequentist estimators of the $TIHL_{ST}$ parameters using a Monte Carlo simulation analysis. $N=1000$ generated random sample of sizes n = 50, 100, 300, 1000 from the $TIHL_{ST}$ distribution is utilized. The parameter combinations (φ,η) are: $(\varphi=2,\ \eta=0.5)$, $(\varphi=2,\ \eta=1.5)$, $(\varphi=2,\ \eta=2)$, $(\varphi=3,\ \eta=0.5)$, $(\varphi=3,\ \eta=1.5)$ and $(\varphi=3,\ \eta=2)$. For each parameter combination, the MLE, MPSE, ANDE, CVME, LSE and WLSE estimates of φ and η were obtained for each sample, permitting the computation of the following measures: average estimates (AVEs), average absolute biases (AVABs) and mean square errors (MSEs) of these estimators. We also provide partial and overall rankings of the estimation methods for various parameter groupings to assist in choosing the optimal method for estimating the $TIHL_{ST}$ parameters.

Tables 1–6 contain the AVEs, AVABs, and MSEs for the six estimate methods. Furthermore, these tables display the rank of each estimator among all estimators in each row with superscripts as indicators, and $\Sigma\,ranks$ is the partial sum of the ranks for each column and sample size. Table 7 shows the partial and overall rankings of the six estimating methods for all possible $TIHL_{ST}$ parameter combinations.

Table 1

Simulation results of the six estimation methods for $\varphi = 2, \eta = 0.5$.

n	**Measures**	**Pa.**	**MLE**	**MPSE**	**ANDE**	**CVME**	**LSE**	**WLSE**
50	AVE	φ	$2.0374^{(5)}$	$1.9963^{(1)}$	$2.0306^{(4)}$	$2.0426^{(6)}$	$2.0285^{(2)}$	$2.0287^{(3)}$
		η	$0.5061^{(2)}$	$0.5564^{(6)}$	$0.5180^{(3)}$	$0.5036^{(1)}$	$0.5372^{(5)}$	$0.5245^{(4)}$
	AVAB	φ	$0.0374^{(5)}$	$0.0037^{(1)}$	$0.0306^{(4)}$	$0.0426^{(6)}$	$0.0285^{(2)}$	$0.0287^{(3)}$
		η	$0.0061^{(2)}$	$0.0564^{(6)}$	$0.0180^{(3)}$	$0.0036^{(1)}$	$0.0372^{(5)}$	$0.0245^{(4)}$
	MSE	φ	$0.0750^{(2)}$	$0.0676^{(1)}$	$0.0763^{(3)}$	$0.0892^{(6)}$	$0.0815^{(5)}$	$0.0775^{(4)}$
		η	$0.0265^{(1)}$	$0.0357^{(6)}$	$0.0281^{(3)}$	$0.0271^{(2)}$	$0.0320^{(5)}$	$0.0294^{(4)}$
	Σranks		$\mathbf{17^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{20^{(2)}}$	$\mathbf{22^{(4.5)}}$	$\mathbf{24^{(6)}}$	$\mathbf{22^{(4.5)}}$
100	AVE	φ	$2.0155^{(5)}$	$1.9907^{(1)}$	$2.0125^{(4)}$	$2.0181^{(6)}$	$2.0117^{(3)}$	$2.0116^{(2)}$
		η	$0.5073^{(2)}$	$0.5320^{(6)}$	$0.5129^{(3)}$	$0.5071^{(1)}$	$0.5236^{(5)}$	$0.5152^{(4)}$
	AVAB	φ	$0.0155^{(5)}$	$0.0093^{(1)}$	$0.0125^{(4)}$	$0.0181^{(6)}$	$0.0117^{(3)}$	$0.0116^{(2)}$
		η	$0.0073^{(2)}$	$0.0320^{(6)}$	$0.0129^{(3)}$	$0.0071^{(1)}$	$0.0236^{(5)}$	$0.0152^{(4)}$
	MSE	φ	$0.0339^{(2)}$	$0.0323^{(1)}$	$0.0357^{(3)}$	$0.0392^{(6)}$	$0.0376^{(5)}$	$0.0358^{(4)}$
		η	$0.0135^{(1)}$	$0.0160^{(6)}$	$0.0140^{(2)}$	$0.0145^{(4)}$	$0.0159^{(5)}$	$0.0142^{(3)}$
	Σranks		$\mathbf{17^{(1)}}$	$\mathbf{21^{(4)}}$	$\mathbf{19^{(2.5)}}$	$\mathbf{24^{(5)}}$	$\mathbf{26^{(6)}}$	$\mathbf{19^{(2.5)}}$
300	AVE	φ	$2.0031^{(5)}$	$1.9922^{(1)}$	$2.0022^{(4)}$	$2.0041^{(6)}$	$2.0021^{(2.5)}$	$2.0021^{(2.5)}$
		η	$0.4996^{(2)}$	$0.5074^{(6)}$	$0.5015^{(3)}$	$0.4995^{(1)}$	$0.5049^{(5)}$	$0.5016^{(4)}$
	AVAB	φ	$0.0031^{(4)}$	$0.0078^{(6)}$	$0.0022^{(3)}$	$0.0041^{(5)}$	$0.0021^{(1.5)}$	$0.0021^{(1.5)}$
		η	$0.0004^{(1)}$	$0.0074^{(6)}$	$0.0015^{(3)}$	$0.0005^{(2)}$	$0.0049^{(5)}$	$0.0016^{(4)}$
	MSE	φ	$0.0111^{(2)}$	$0.0110^{(1)}$	$0.0114^{(3)}$	$0.0123^{(6)}$	$0.0121^{(5)}$	$0.0115^{(4)}$
		η	$0.0039^{(1)}$	$0.0041^{(4)}$	$0.0040^{(2.5)}$	$0.0043^{(5)}$	$0.0044^{(6)}$	$0.0040^{(2.5)}$
	Σranks		$\mathbf{15^{(1)}}$	$\mathbf{24^{(4)}}$	$\mathbf{18.5^{(2.5)}}$	$\mathbf{25^{(5.5)}}$	$\mathbf{25^{(5.5)}}$	$\mathbf{18^{(2.5)}}$

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1000	AVE	φ	$1.9992^{(5)}$	$1.9950^{(1)}$	$1.9989^{(4)}$	$1.9993^{(6)}$	$1.9987^{(2)}$	$1.9988^{(3)}$
		η	$0.4981^{(1)}$	$0.5004^{(6)}$	$0.4987^{(3.5)}$	$0.4985^{(2)}$	$0.5001^{(5)}$	$0.4987^{(3.5)}$
	AVAB	φ	$0.0008^{(3)}$	$0.0005^{(1)}$	$0.0011^{(5)}$	$0.0007^{(2)}$	$0.0013^{(6)}$	$0.0012^{(5)}$
		η	$0.0019^{(6)}$	$0.00004^{(1)}$	$0.0013^{(3.5)}$	$0.0015^{(5)}$	$0.0001^{(2)}$	$0.0013^{(3.5)}$
	MSE	φ	$0.0031^{(1.5)}$	$0.0031^{(1.5)}$	$0.0033^{(3.5)}$	$0.0035^{(5.5)}$	$0.0035^{(5.5)}$	$0.0033^{(3.5)}$
		η	$0.0012^{(2.5)}$	$0.0012^{(2.5)}$	$0.0012^{(2.5)}$	$0.0013^{(5.5)}$	$0.0013^{(5.5)}$	$0.0012^{(2.5)}$
	Σranks		**$19^{(2)}$**	**$13^{(1)}$**	**$22^{(4)}$**	**$26^{(5.5)}$**	**$26^{(5.5)}$**	**$21^{(3)}$**

Table 2

Simulation results of the six estimation methods for $\varphi = 2, \eta = 1.5$.

n	**Measures**	**Pa.**	**MLE**	**MPSE**	**ANDE**	**CVME**	**LSE**	**WLSE**
50	AVE	φ	$2.0374^{(5)}$	$1.9963^{(1)}$	$2.0306^{(4)}$	$2.0426^{(6)}$	$2.0285^{(2)}$	$2.0287^{(3)}$
		η	$1.5185^{(2)}$	$1.6692^{(6)}$	$1.5539^{(3)}$	$1.5108^{(1)}$	$1.6117^{(5)}$	$1.5736^{(4)}$
	AVAB	φ	$0.0374^{(5)}$	$0.0037^{(1)}$	$0.0306^{(4)}$	$0.0426^{(6)}$	$0.0285^{(2)}$	$0.0287^{(3)}$
		η	$0.0185^{(2)}$	$0.1692^{(6)}$	$0.0539^{(3)}$	$0.0108^{(1)}$	$0.1117^{(5)}$	$0.0736^{(4)}$
	MSE	φ	$0.0750^{(2)}$	$0.0676^{(1)}$	$0.0763^{(3)}$	$0.0892^{(6)}$	$0.0815^{(5)}$	$0.0775^{(4)}$
		η	$0.2386^{(1)}$	$0.3217^{(6)}$	$0.2528^{(3)}$	$0.2441^{(2)}$	$0.2880^{(5)}$	$0.2648^{(4)}$
	Σranks		**$17^{(1)}$**	**$21^{(3)}$**	**$20^{(2)}$**	**$22^{(4.5)}$**	**$24^{(6)}$**	**$22^{(4.5)}$**
100	AVE	φ	$2.0154^{(5)}$	$1.9907^{(1)}$	$2.0125^{(4)}$	$2.0181^{(6)}$	$2.0117^{(3)}$	$2.0116^{(2)}$
		η	$1.5219^{(2)}$	$1.5965^{(6)}$	$1.5388^{(3)}$	$1.5212^{(1)}$	$1.5709^{(5)}$	$1.5456^{(4)}$
	AVAB	φ	$0.0154^{(5)}$	$0.0093^{(1)}$	$0.0125^{(4)}$	$0.0181^{(6)}$	$0.0117^{(3)}$	$0.0116^{(2)}$
		η	$0.0219^{(2)}$	$0.0965^{(6)}$	$0.0388^{(3)}$	$0.0212^{(1)}$	$0.0709^{(5)}$	$0.0456^{(4)}$
	MSE	φ	$0.0339^{(2)}$	$0.0323^{(1)}$	$0.0357^{(3)}$	$0.0392^{(6)}$	$0.0376^{(5)}$	$0.0358^{(4)}$
		η	$0.1218^{(1)}$	$0.1440^{(6)}$	$0.1262^{(2)}$	$0.1306^{(4)}$	$0.1431^{(5)}$	$0.1282^{(3)}$
	Σranks		**$17^{(1)}$**	**$21^{(4)}$**	**$19^{(2.5)}$**	**$24^{(5)}$**	**$26^{(6)}$**	**$19^{(2.5)}$**
300	AVE	φ	$2.0031^{(5)}$	$1.9922^{(1)}$	$2.0022^{(4)}$	$2.0041^{(6)}$	$2.0021^{(2.5)}$	$2.0021^{(2.5)}$
		η	$1.4988^{(2)}$	$1.5219^{(6)}$	$1.5044^{(3)}$	$1.4986^{(1)}$	$1.5147^{(5)}$	$1.5049^{(4)}$
	AVAB	φ	$0.0031^{(4)}$	$0.0077^{(6)}$	$0.0022^{(3)}$	$0.0041^{(5)}$	$0.0021^{(1.5)}$	$0.0021^{(1.5)}$
		η	$0.0012^{(1)}$	$0.0219^{(6)}$	$0.0044^{(3)}$	$0.0014^{(2)}$	$0.0147^{(5)}$	$0.0049^{(4)}$
	MSE	φ	$0.0111^{(2)}$	$0.0110^{(1)}$	$0.0114^{(3)}$	$0.0123^{(6)}$	$0.0121^{(5)}$	$0.0115^{(4)}$
		η	$0.0355^{(1)}$	$0.0371^{(4)}$	$0.0362^{(3)}$	$0.0390^{(5)}$	$0.0400^{(6)}$	$0.0361^{(2)}$
	Σranks		**$15^{(1)}$**	**$24^{(4)}$**	**$19^{(3)}$**	**$25^{(5.5)}$**	**$25^{(5.5)}$**	**$18^{(2)}$**

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1000	AVE	φ	1.9992$^{(5)}$	1.9950$^{(1)}$	1.9989$^{(4)}$	1.9993$^{(6)}$	1.9987$^{(2)}$	1.9988$^{(3)}$
		η	1.4943$^{(1)}$	1.5007$^{(6)}$	1.4962$^{(4)}$	1.4954$^{(2)}$	1.5002$^{(5)}$	1.4961$^{(3)}$
	AVAB	φ	0.0008$^{(3)}$	0.0005$^{(1)}$	0.0011$^{(4)}$	0.0007$^{(2)}$	0.0013$^{(6)}$	0.0012$^{(5)}$
		η	0.0057$^{(6)}$	0.00007$^{(1)}$	0.0038$^{(3)}$	0.0046$^{(5)}$	0.0002$^{(2)}$	0.0039$^{(4)}$
	MSE	φ	0.0031$^{(1.5)}$	0.0031$^{(1.5)}$	0.0033$^{(3.5)}$	0.0035$^{(5.5)}$	0.0035$^{(5.5)}$	0.0033$^{(3.5)}$
		η	0.0106$^{(1)}$	0.0107$^{(3)}$	0.0107$^{(3)}$	0.0117$^{(5.5)}$	0.0117$^{(5.5)}$	0.0107$^{(3)}$
	Σranks		**17.5$^{(2)}$**	**13.5$^{(1)}$**	**21.5$^{(3.5)}$**	**26$^{(5.5)}$**	**26$^{(5.5)}$**	**21.5$^{(3.5)}$**

Table 3

Simulation results of the six estimation methods for $\varphi = 2, \eta = 2$.

n	**Measures**	**Pa.**	**MLE**	**MPSE**	**ANDE**	**CVME**	**LSE**	**WLSE**
50	AVE	φ	2.0374$^{(5)}$	1.9963$^{(1)}$	2.0306$^{(4)}$	2.0426$^{(6)}$	2.0285$^{(2)}$	2.0287$^{(3)}$
		η	2.0247$^{(2)}$	2.2254$^{(6)}$	2.0719$^{(3)}$	2.0144$^{(1)}$	2.1490$^{(5)}$	2.0982$^{(4)}$
	AVAB	φ	0.0374$^{(5)}$	0.0037$^{(1)}$	0.0306$^{(4)}$	0.0426$^{(6)}$	0.0285$^{(2)}$	0.0287$^{(3)}$
		η	0.0247$^{(2)}$	0.2254$^{(6)}$	0.0719$^{(3)}$	0.0144$^{(1)}$	0.1490$^{(5)}$	0.0982$^{(4)}$
	MSE	φ	0.0750$^{(2)}$	0.0676$^{(1)}$	0.0763$^{(3)}$	0.0892$^{(6)}$	0.0815$^{(5)}$	0.0775$^{(4)}$
		η	0.4241$^{(1)}$	0.5713$^{(6)}$	0.4494$^{(3)}$	0.4339$^{(2)}$	0.5120$^{(5)}$	0.4707$^{(4)}$
	Σranks		**17$^{(1)}$**	**21$^{(3)}$**	**20$^{(2)}$**	**22$^{(4.5)}$**	**24$^{(6)}$**	**22$^{(4.5)}$**
100	AVE	φ	2.0154$^{(5)}$	1.9907$^{(1)}$	2.0125$^{(4)}$	2.0181$^{(6)}$	2.0117$^{(3)}$	2.0116$^{(2)}$
		η	2.0292$^{(2)}$	2.1285$^{(6)}$	2.0517$^{(3)}$	2.0282$^{(1)}$	2.0945$^{(5)}$	2.0608$^{(4)}$
	AVAB	φ	0.0154$^{(5)}$	0.0093$^{(1)}$	0.0125$^{(4)}$	0.0181$^{(6)}$	0.0117$^{(3)}$	0.0116$^{(2)}$
		η	0.0292$^{(2)}$	0.1285$^{(6)}$	0.0517$^{(3)}$	0.0282$^{(1)}$	0.0945$^{(5)}$	0.0608$^{(4)}$
	MSE	φ	0.0339$^{(2)}$	0.0323$^{(1)}$	0.0357$^{(3)}$	0.0392$^{(6)}$	0.0376$^{(5)}$	0.0358$^{(4)}$
		η	0.2162$^{(1)}$	0.2565$^{(6)}$	0.2243$^{(2)}$	0.2321$^{(4)}$	0.2545$^{(5)}$	0.2280$^{(3)}$
	Σranks		**17$^{(1)}$**	**21$^{(4)}$**	**19$^{(2.5)}$**	**24$^{(5)}$**	**26$^{(6)}$**	**19$^{(2.5)}$**
300	AVE	φ	2.0031$^{(5)}$	1.9922$^{(1)}$	2.0022$^{(4)}$	2.0041$^{(6)}$	2.0021$^{(2.5)}$	2.0021$^{(2.5)}$
		η	1.9985$^{(2)}$	2.0299$^{(6)}$	2.0058$^{(3)}$	1.9981$^{(1)}$	2.0196$^{(5)}$	2.0065$^{(4)}$
	AVAB	φ	0.0031$^{(4)}$	0.0078$^{(6)}$	0.0022$^{(3)}$	0.0041$^{(5)}$	0.0021$^{(1.5)}$	0.0021$^{(1.5)}$
		η	0.0015$^{(1)}$	0.0299$^{(6)}$	0.0058$^{(3)}$	0.0019$^{(2)}$	0.0196$^{(5)}$	0.0065$^{(4)}$
	MSE	φ	0.0111$^{(2)}$	0.0110$^{(1)}$	0.0114$^{(3)}$	0.0123$^{(6)}$	0.0121$^{(5)}$	0.0115$^{(4)}$
		η	0.0631$^{(1)}$	0.0665$^{(4)}$	0.0643$^{(2.5)}$	0.0693$^{(5)}$	0.0711$^{(6)}$	0.0643$^{(2.5)}$
	Σranks		**15$^{(1)}$**	**24$^{(4)}$**	**18.5$^{(2.5)}$**	**25$^{(5.5)}$**	**25$^{(5.5)}$**	**18.5$^{(2.5)}$**

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1000	AVE	φ	$1.9992^{(5)}$	$1.9951^{(1)}$	$1.9989^{(4)}$	$1.9993^{(6)}$	$1.9987^{(2)}$	$1.9988^{(3)}$
		η	$1.9924^{(1)}$	$2.0017^{(6)}$	$1.9949^{(4)}$	$1.9938^{(2)}$	$2.0002^{(5)}$	$1.9948^{(3)}$
	AVAB	φ	$0.0008^{(2)}$	$0.0049^{(6)}$	$0.0011^{(3)}$	$0.0007^{(1)}$	$0.0013^{(5)}$	$0.0012^{(4)}$
		η	$0.0076^{(6)}$	$0.0017^{(2)}$	$0.0051^{(3)}$	$0.0062^{(5)}$	$0.0002^{(1)}$	$0.0052^{(4)}$
	MSE	φ	$0.0031^{(1.5)}$	$0.0031^{(1.5)}$	$0.0033^{(3.5)}$	$0.0035^{(5.5)}$	$0.0035^{(5.5)}$	$0.0033^{(3.5)}$
		η	$0.0189^{(1)}$	$0.0191^{(3)}$	$0.0191^{(3)}$	$0.0207^{(5)}$	$0.0208^{(6)}$	$0.0191^{(3)}$
	Σranks		$\mathbf{16.5^{(1)}}$	$\mathbf{19.5^{(2)}}$	$\mathbf{20.5^{(3.5)}}$	$\mathbf{24.5^{(6)}}$	$\mathbf{22.5^{(5)}}$	$\mathbf{20.5^{(3.5)}}$

Table 4

Simulation results of the six estimation methods for $\varphi = 3, \eta = 0.5.$

n	Measures	Pa.	MLEs	MPSEs	ANDE	CVME	LSE	WLSE
50	AVE	φ	$3.0815^{(5)}$	$2.9776^{(1)}$	$3.0586^{(4)}$	$3.1025^{(6)}$	$3.0428^{(2)}$	$3.0500^{(3)}$
		η	$0.5028^{(1)}$	$0.5629^{(6)}$	$0.5190^{(3)}$	$0.5044^{(2)}$	$0.5379^{(5)}$	$0.5246^{(4)}$
	AVAB	φ	$0.0815^{(5)}$	$0.0224^{(1)}$	$0.0586^{(4)}$	$0.1025^{(6)}$	$0.0428^{(2)}$	$0.0500^{(3)}$
		η	$0.0028^{(1)}$	$0.0629^{(6)}$	$0.0190^{(3)}$	$0.0044^{(2)}$	$0.0379^{(5)}$	$0.0246^{(4)}$
	MSE	φ	$0.1673^{(3)}$	$0.1371^{(1)}$	$0.1629^{(2)}$	$0.2318^{(6)}$	$0.1944^{(5)}$	$0.1694^{(4)}$
		η	$0.0237^{(1)}$	$0.0338^{(6)}$	$0.0265^{(2)}$	$0.0273^{(3)}$	$0.0324^{(5)}$	$0.0280^{(4)}$
	Σranks		$\mathbf{16^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{18^{(2)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{22^{(4)}}$
100	AVE	φ	$3.0318^{(5)}$	$2.9741^{(1)}$	$3.0218^{(2.5)}$	$3.0408^{(6)}$	$3.0128^{(2.5)}$	$3.0186^{(4)}$
		η	$0.5055^{(1)}$	$0.5372^{(6)}$	$0.5137^{(3)}$	$0.5079^{(2)}$	$0.5245^{(5)}$	$0.5153^{(4)}$
	AVAB	φ	$0.0318^{(5)}$	$0.0259^{(1)}$	$0.0218^{(1.5)}$	$0.0408^{(6)}$	$0.0128^{(1.5)}$	$0.0186^{(4)}$
		η	$0.0055^{(1)}$	$0.0372^{(6)}$	$0.0137^{(3)}$	$0.0079^{(2)}$	$0.0245^{(5)}$	$0.0153^{(4)}$
	MSE	φ	$0.0699^{(2)}$	$0.0637^{(1)}$	$0.0738^{(3)}$	$0.0909^{(6)}$	$0.0844^{(5)}$	$0.0756^{(4)}$
		η	$0.0118^{(1)}$	$0.0147^{(5)}$	$0.0130^{(2)}$	$0.0144^{(4)}$	$0.0158^{(6)}$	$0.0133^{(3)}$
	Σranks		$\mathbf{15^{(1.5)}}$	$\mathbf{20^{(3)}}$	$\mathbf{15^{(1.5)}}$	$\mathbf{26^{(6)}}$	$\mathbf{25^{(5)}}$	$\mathbf{23^{(4)}}$
300	AVE	φ	$3.0094^{(5)}$	$2.9866^{(1)}$	$3.0064^{(4)}$	$3.0129^{(6)}$	$3.0039^{(2)}$	$3.0061^{(3)}$
		η	$0.4994^{(1)}$	$0.5108^{(6)}$	$0.5019^{(4)}$	$0.4999^{(2)}$	$0.5053^{(5)}$	$0.5017^{(3)}$
	AVAB	φ	$0.0094^{(4)}$	$0.0134^{(6)}$	$0.0064^{(3)}$	$0.0129^{(5)}$	$0.0039^{(1)}$	$0.0061^{(2)}$
		η	$0.0006^{(2)}$	$0.0108^{(6)}$	$0.0019^{(4)}$	$0.0001^{(1)}$	$0.0053^{(5)}$	$0.0017^{(3)}$
	MSE	φ	$0.0223^{(2)}$	$0.0218^{(1)}$	$0.0233^{(3)}$	$0.0266^{(6)}$	$0.0260^{(5)}$	$0.0235^{(4)}$
		η	$0.0034^{(1)}$	$0.0037^{(4)}$	$0.0037^{(2.5)}$	$0.0043^{(5)}$	$0.0044^{(6)}$	$0.0037^{(2.5)}$

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	Σranks		$\mathbf{15^{(1)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{20.5^{(3)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{17.5^{(2)}}$
1000	AVE	φ	$3.0018^{(5)}$	$2.9940^{(1)}$	$3.0010^{(3.5)}$	$3.0026^{(6)}$	$2.9999^{(2)}$	$3.0010^{(3.5)}$
		η	$0.4981^{(1)}$	$0.5018^{(6)}$	$0.4989^{(4)}$	$0.4985^{(2)}$	$0.5002^{(5)}$	$0.4987^{(3)}$
	AVAB	φ	$0.0018^{(4)}$	$0.0060^{(6)}$	$0.0010^{(2.5)}$	$0.0026^{(5)}$	$0.00001^{(1)}$	$0.0010^{(2.5)}$
		η	$0.0019^{(6)}$	$0.0018^{(5)}$	$0.0011^{(2)}$	$0.0015^{(4)}$	$0.00002^{(1)}$	$0.0013^{(3)}$
	MSE	φ	$0.0063^{(2)}$	$0.0062^{(1)}$	$0.0067^{(3.5)}$	$0.0076^{(5)}$	$0.0075^{(6)}$	$0.0067^{(3.5)}$
		η	$0.0010^{(1.5)}$	$0.0010^{(1.5)}$	$0.0011^{(3.5)}$	$0.0013^{(5.5)}$	$0.0013^{(5.5)}$	$0.0011^{(3.5)}$
	Σranks		$\mathbf{19.5^{(3)}}$	$\mathbf{20.5^{(4.5)}}$	$\mathbf{19^{(1.5)}}$	$\mathbf{27.5^{(6)}}$	$\mathbf{20.5^{(4.5)}}$	$\mathbf{19^{(1.5)}}$

Table 5

Simulation results of the six estimation methods for $\varphi = 3, \eta = 1.5$.

n	**Measures**	**Pa.**	**MLEs**	**MPSEs**	**ANDE**	**CVME**	**LSE**	**WLSE**
50	AVE	φ	$3.0815^{(5)}$	$2.9775^{(1)}$	$3.0586^{(4)}$	$3.1025^{(6)}$	$3.0428^{(2)}$	$3.0500^{(3)}$
		η	$1.5083^{(1)}$	$1.6886^{(6)}$	$1.5571^{(3)}$	$1.5133^{(2)}$	$1.6137^{(5)}$	$1.5737^{(4)}$
	AVAB	φ	$0.0815^{(5)}$	$0.0225^{(1)}$	$0.0586^{(4)}$	$0.1025^{(6)}$	$0.0428^{(2)}$	$0.0500^{(3)}$
		η	$0.0083^{(1)}$	$0.1886^{(6)}$	$0.0571^{(3)}$	$0.0133^{(2)}$	$0.1137^{(5)}$	$0.0737^{(4)}$
	MSE	φ	$0.1673^{(3)}$	$0.1371^{(1)}$	$0.1629^{(2)}$	$0.2318^{(6)}$	$0.1944^{(5)}$	$0.1694^{(4)}$
		η	$0.2130^{(1)}$	$0.3034^{(6)}$	$0.2384^{(2)}$	$0.2459^{(3)}$	$0.2920^{(5)}$	$0.2517^{(4)}$
	Σranks		$\mathbf{16^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{18^{(2)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{22^{(4)}}$
100	AVE	φ	$3.0319^{(5)}$	$2.9739^{(1)}$	$3.0218^{(2.5)}$	$3.0408^{(6)}$	$3.0128^{(2.5)}$	$3.0186^{(4)}$
		η	$1.5166^{(1)}$	$1.6120^{(6)}$	$1.5410^{(3)}$	$1.5236^{(2)}$	$1.5736^{(5)}$	$1.5458^{(4)}$
	AVAB	φ	$0.0319^{(5)}$	$0.0261^{(1)}$	$0.0218^{(2.5)}$	$0.0408^{(6)}$	$0.0128^{(2.5)}$	$0.0186^{(4)}$
		η	$0.0166^{(1)}$	$0.1120^{(6)}$	$0.0410^{(3)}$	$0.0236^{(2)}$	$0.0736^{(5)}$	$0.0458^{(4)}$
	MSE	φ	$0.0701^{(2)}$	$0.0638^{(1)}$	$0.0738^{(3)}$	$0.0909^{(6)}$	$0.0844^{(5)}$	$0.0756^{(4)}$
		η	$0.1062^{(1)}$	$0.1322^{(5)}$	$0.1172^{(2)}$	$0.1295^{(4)}$	$0.1423^{(6)}$	$0.1201^{(3)}$
	Σranks		$\mathbf{15^{(1)}}$	$\mathbf{20^{(3)}}$	$\mathbf{16^{(2)}}$	$\mathbf{26^{(5.5)}}$	$\mathbf{26^{(5.5)}}$	$\mathbf{23^{(4)}}$
300	AVE	φ	$3.0093^{(5)}$	$2.9869^{(1)}$	$3.0064^{(4)}$	$3.0129^{(6)}$	$3.0039^{(2)}$	$3.0061^{(3)}$
		η	$1.4983^{(1)}$	$1.5329^{(6)}$	$1.5057^{(4)}$	$1.4996^{(2)}$	$1.5158^{(5)}$	$1.5051^{(3)}$
	AVAB	φ	$0.0093^{(4)}$	$0.0131^{(6)}$	$0.0064^{(3)}$	$0.0129^{(5)}$	$0.0039^{(1)}$	$0.0061^{(2)}$
		η	$0.0017^{(2)}$	$0.0329^{(6)}$	$0.0057^{(4)}$	$0.0004^{(1)}$	$0.0158^{(5)}$	$0.0051^{(3)}$

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	MSE	φ	$0.0223^{(2)}$	$0.0217^{(1)}$	$0.0233^{(3)}$	$0.0266^{(6)}$	$0.0260^{(5)}$	$0.0235^{(4)}$
		η	$0.0307^{(1)}$	$0.0330^{(4)}$	$0.0332^{(2.5)}$	$0.0385^{(5)}$	$0.0395^{(6)}$	$0.0335^{(2.5)}$
	Σranks		$\mathbf{15^{(1)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{20.5^{(3)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{17.5^{(2)}}$
1000	AVE	φ	$3.0018^{(5)}$	$2.9930^{(1)}$	$3.0010^{(3.5)}$	$3.0026^{(6)}$	$2.9999^{(2)}$	$3.0010^{(3.5)}$
		η	$1.4943^{(1)}$	$1.5049^{(6)}$	$1.4966^{(4)}$	$1.4956^{(2)}$	$1.5005^{(5)}$	$1.4962^{(3)}$
	AVAB	φ	$0.0018^{(4)}$	$0.0070^{(6)}$	$0.0010^{(2.5)}$	$0.0026^{(5)}$	$0.00001^{(1)}$	$0.0010^{(2.5)}$
		η	$0.0057^{(6)}$	$0.0049^{(5)}$	$0.0034^{(2)}$	$0.0044^{(4)}$	$0.00005^{(1)}$	$0.0038^{(3)}$
	MSE	φ	$0.0063^{(2)}$	$0.0062^{(1)}$	$0.0067^{(3.5)}$	$0.0076^{(6)}$	$0.0075^{(5)}$	$0.0067^{(3.5)}$
		η	$0.0091^{(1.5)}$	$0.0091^{(1.5)}$	$0.0098^{(3.5)}$	$0.0114^{(5.5)}$	$0.0114^{(5.5)}$	$0.0098^{(3.5)}$
	Σranks		$\mathbf{19.5^{(3.5)}}$	$\mathbf{20.5^{(5)}}$	$\mathbf{19^{(1.5)}}$	$\mathbf{28.5^{(6)}}$	$\mathbf{19.5^{(3.5)}}$	$\mathbf{19^{(1.5)}}$

Table 6

Simulation results of the six estimation methods for $\varphi = 3, \eta = 2.$

n	**Measures**	**Pa.**	**MLEs**	**MPSEs**	**ANDE**	**CVME**	**LSE**	**WLSE**
50	AVE	φ	$3.0815^{(5)}$	$2.9773^{(1)}$	$3.0586^{(4)}$	$3.1025^{(6)}$	$3.0428^{(2)}$	$3.0500^{(3)}$
		η	$2.0110^{(1)}$	$2.2514^{(6)}$	$2.0761^{(3)}$	$2.0178^{(2)}$	$2.1516^{(5)}$	$2.0983^{(4)}$
	AVAB	φ	$0.0814^{(5)}$	$0.0227^{(1)}$	$0.0586^{(4)}$	$0.1025^{(6)}$	$0.0428^{(2)}$	$0.0500^{(3)}$
		η	$0.0110^{(1)}$	$0.2514^{(6)}$	$0.0761^{(3)}$	$0.0178^{(2)}$	$0.1516^{(5)}$	$0.0983^{(4)}$
	MSE	φ	$0.1675^{(3)}$	$0.1373^{(1)}$	$0.1629^{(2)}$	$0.2318^{(6)}$	$0.1944^{(5)}$	$0.1694^{(4)}$
		η	$0.3785^{(1)}$	$0.5400^{(6)}$	$0.4238^{(2)}$	$0.4372^{(3)}$	$0.5191^{(5)}$	$0.4474^{(4)}$
	Σranks		$\mathbf{16^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{18^{(2)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{22^{(4)}}$
100	AVE	φ	$3.0319^{(5)}$	$2.9746^{(1)}$	$3.0218^{(4)}$	$3.0408^{(6)}$	$3.0128^{(2)}$	$3.0186^{(3)}$
		η	$2.0221^{(1)}$	$2.1494^{(6)}$	$2.0547^{(3)}$	$2.0314^{(2)}$	$2.0981^{(5)}$	$2.0611^{(4)}$
	AVAB	φ	$0.0319^{(5)}$	$0.0254^{(4)}$	$0.0218^{(2.5)}$	$0.0408^{(6)}$	$0.0128^{(2.5)}$	$0.0186^{(1)}$
		η	$0.0221^{(1)}$	$0.1494^{(6)}$	$0.0547^{(3)}$	$0.0314^{(2)}$	$0.0981^{(5)}$	$0.0611^{(4)}$
	MSE	φ	$0.0701^{(2)}$	$0.0637^{(1)}$	$0.0738^{(3)}$	$0.0909^{(6)}$	$0.0844^{(5)}$	$0.0756^{(4)}$
		η	$0.1887^{(1)}$	$0.2349^{(5)}$	$0.2084^{(2)}$	$0.2303^{(4)}$	$0.2530^{(6)}$	$0.2135^{(3)}$
	Σranks		$\mathbf{15^{(1)}}$	$\mathbf{23^{(4)}}$	$\mathbf{17.5^{(2)}}$	$\mathbf{26^{(6)}}$	$\mathbf{25.5^{(5)}}$	$\mathbf{19^{(3)}}$

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300	AVE	φ	$3.0093^{(5)}$	$2.9858^{(1)}$	$3.0064^{(4)}$	$3.0129^{(6)}$	$3.0039^{(2)}$	$3.0061^{(3)}$
		η	$1.9977^{(1)}$	$2.0435^{(6)}$	$2.0075^{(4)}$	$1.9994^{(2)}$	$2.0210^{(5)}$	$2.0069^{(3)}$
	AVAB	φ	$0.0093^{(4)}$	$0.0142^{(6)}$	$0.0064^{(3)}$	$0.0129^{(5)}$	$0.0039^{(1)}$	$0.0061^{(2)}$
		η	$0.0023^{(2)}$	$0.0435^{(6)}$	$0.0075^{(4)}$	$0.0006^{(1)}$	$0.0210^{(5)}$	$0.0069^{(3)}$
	MSE	φ	$0.0224^{(2)}$	$0.0216^{(1)}$	$0.0233^{(3)}$	$0.0266^{(6)}$	$0.0260^{(5)}$	$0.0235^{(4)}$
		η	$0.0545^{(1)}$	$0.0591^{(2.5)}$	$0.0591^{(2.5)}$	$0.0684^{(5)}$	$0.0702^{(6)}$	$0.0595^{(4)}$
	Σranks		$\mathbf{15^{(1)}}$	$\mathbf{22.5^{(4)}}$	$\mathbf{20.5^{(3)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{19^{(2)}}$
1000	AVE	φ	$3.0018^{(5)}$	$2.9931^{(1)}$	$3.0010^{(3.5)}$	$3.0026^{(6)}$	$2.9999^{(2)}$	$3.0010^{(3.5)}$
		η	$1.9923^{(1)}$	$2.0071^{(6)}$	$1.9955^{(4)}$	$1.9942^{(2)}$	$2.0006^{(5)}$	$1.9949^{(3)}$
	AVAB	φ	$0.0018^{(4)}$	$0.0069^{(6)}$	$0.0010^{(2.5)}$	$0.0026^{(5)}$	$0.00001^{(1)}$	$0.0010^{(2.5)}$
		η	$0.0077^{(6)}$	$0.0071^{(5)}$	$0.0045^{(2)}$	$0.0058^{(4)}$	$0.00006^{(1)}$	$0.0051^{(3)}$
	MSE	φ	$0.0063^{(2)}$	$0.0061^{(1)}$	$0.0067^{(3.5)}$	$0.0076^{(6)}$	$0.0075^{(5)}$	$0.0067^{(3.5)}$
		η	$0.0161^{(1)}$	$0.0164^{(2)}$	$0.0174^{(3.5)}$	$0.0203^{(5.5)}$	$0.0203^{(5.5)}$	$0.0174^{(3.5)}$
	Σ *ranks*		$\mathbf{20^{(4)}}$	$\mathbf{21^{(5)}}$	$\mathbf{19^{(1.5)}}$	$\mathbf{28.5^{(6)}}$	$\mathbf{19.5^{(3)}}$	$\mathbf{19^{(1.5)}}$

The results in Tables 1–6 demonstrate that MSEs decrease for all parameter combinations as the sample size grows, indicating that all estimators are quite consistent. As a result, the $TIHL_{ST}$ parameter estimates yield reliable MSEs and low AVABs for all parameter combinations when utilizing the six estimation methods. Furthermore, the AVABs approach zero as the sample size increases, demonstrating that the estimators are asymptotically unbiased.

Table 7

Partial and overall ranks

Parameter	n	MLE	MPSE	ANDE	CVME	LSE	WLSE
$\varphi = 2$, $\eta = 0.5$	50	1	3	2	4.5	6	4.5
	100	1	4	2.5	5	6	2.5
	300	1	4	2.5	5.5	5.5	2.5
	1000	2	1	4	5.5	5.5	3
$\varphi = 2$, $\eta = 1.5$	50	1	3	2	4.5	6	4.5
	100	1	4	2.5	5	6	2.5
	300	1	4	3	5.5	5.5	2
	1000	2	1	3.5	5.5	5.5	3.5

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$\varphi = 2$, $\eta = 2$	50	1	3	2	4.5	6	4.5
	100	1	4	2.5	5	6	2.5
	300	1	4	2.5	5.5	5.5	2.5
	1000	1	2	3.5	6	5	3.5
$\varphi = 3$, $\eta = 0.5$	50	1	3	2	6	5	4
	100	1.5	3	1.5	6	5	4
	300	1	4.5	3	6	4.5	2
	1000	3	4.5	1.5	6	4.5	1.5
$\varphi = 3$, $\eta = 1.5$	50	1	3	2	6	5	4
	100	1	3	2	5.5	5.5	4
	300	1	4.5	3	6	4.5	2
	1000	3.5	5	1.5	6	3.5	1.5
$\varphi = 3$, $\eta = 2$	50	1	3	2	6	5	4
	100	1	4	2	6	5	3
	300	1	4	3	6	5	2
	1000	4	5	1.5	6	3	1.5
Σ *ranks*		**34**	**83.5**	**57.5**	**133.5**	**124**	**71.5**
Overall rank		**1**	**4**	**2**	**6**	**5**	**3**

The MLE, ANDE, WLSE, MPSE, LSE, and CVME are listed based on best-to-worst performing estimators for all parameter combinations, according to the overall rankings in Table 7. In a nutshell the simulation results suggest that the six estimators for estimating the $TIHL_{ST}$ parameters perform exceptionally well. We can deduce that MLE outperforms all other estimators, with a total rank of 34. As a result of our findings, we can affirm MLE and ANDE's superiority, with overall $TIHL_{ST}$ distribution rankings of 34 and 57.5, respectively.

4. Conclusion

In this paper, we evaluated the estimation of the Type-I half logistic skew-t ($TIHL_{ST}$) model parameters using six classical estimation procedures. Given that a theoretical comparison of these classical estimating processes is not feasible, an extensive Monte Carlo simulation research is carried out to compare the criteria in terms of AVAB, MSE of each estimate, and AVE. The results demonstrate that MLE, ANDE, WLSE, MPSE, LSE, and CVME rank highest to lowest in terms of performance for all parameter combinations. We may also state that the MLE outperforms all others with an overall rating of 34. As a result, we affirm the dominance of the MLE with an overall rank of 34 and the ANDE with an overall rank of 57.5 in the $\mathrm{TIHL}_{\mathrm{ST}}$ distribution.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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