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Improved generalized exponential estimators in simple random sampling without replacement using some transformations

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Abstract

This article suggests three generalized exponential estimators using an auxiliary variable for estimating population mean with some transformations. The mean squared error expressions are derived up to the first order approximation for comparing the efficiency of the proposed estimators with the existing estimators in literature. We compared the efficiency of the proposed estimators with the usual mean estimator, [2] ratio estimator, [1] ratio type exponential estimator, [15] and [9] usual regression estimator. It is found that our proposed estimators perform better than these existing estimators in the literature. Findings are empirically verified using three real data sets and a simulation study.

Subject Classification: *62D05.*

Keywords: *Auxiliary variable, Simulation, Mean square error (MSE), Exponential estimator, Regression estimator.*

1. Introduction

In sample surveys, it is widely known that if auxiliary information is used intelligibly then this information provides us with better estimates than those where no auxiliary information is used. [2] was the first who suggested use of auxiliary information in the estimation purpose to increase the precision of estimates. Several authors including [3], [11], [7],

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[10], [14], [13] and [12] proposed improved estimators using auxiliary information.

Let us consider a finite population of size N having units $\theta=\{\theta_1,.\theta_2,...,\theta_N\}$ and a sample of size n is drawn from the population using simple random sampling without replacement (SRSWOR).

Y and X are the study and auxiliary variable respectively.

We use following notations:

$f=\frac{N-n}{Nn}$; ρ is the correlation coefficient between study variable Y and auxiliary variable X.

$\bar{Y}$ and $\bar{X}$ are the population means of the study variable Y and auxiliary variable X respectively.

$\bar{y}$ and $\bar{x}$ are the sample means of the study variable and auxiliary variable respectively.

$$S_y^2=\frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})^2;\ S_x^2=\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})^2$$

$C_y=\frac{S_y}{\bar{Y}}$: Coefficient of variation of study variable Y.

$C_x=\frac{S_x}{\bar{X}}$: Coefficient of variation of auxiliary variable X

Table 1: Showing some of the existing estimators in the literature with their MSE expressions.

Here in this article we have discussed a problem for estimating the population mean using auxiliary information under SRSWOR. And then we have compared the proposed estimators with the existing ones. First of all we discussed properties of our proposed generalized exponential estimators in the section 2.Then we compared the efficiency of the proposed estimators existing estimators given in table 1, using three real data sets in section 3. At last we concluded with the simulation study providing explanation about the results obtained.

2. Proposed Estimators

Motivated by [14], we have proposed the following estimator:

$$t_8=\lambda\,\bar{y}\,a^{\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]}\ \text{for } a>0 \tag{2.1}$$

λ is any constant.

a is a positive real number.

Remark: Estimator t_4 is particular case of the estimator t_8 for a = 2.71 = e.

For comparing the efficiency of our proposed estimator t_8 with the existing estimators we need to calculate the MSE of the estimator t_8 . And for that we need to calculate $E(t_8 - \overline{Y})^2$. We use the following approximations:

$$\overline{y} = \overline{Y}(1+\varepsilon_0)$$
$$\overline{x} = \overline{X}(1+\varepsilon_1)$$

Since population mean is estimated under SRSWOR, So

$$E(\varepsilon_0) = 0$$
$$E(\varepsilon_1) = 0$$
$$E(\varepsilon_0^2) = E\left(\frac{\overline{y}-\overline{Y}}{\overline{Y}}\right)^2 = f\,C_y^2$$
$$E(\varepsilon_1^2) = E\left(\frac{\overline{x}-\overline{X}}{\overline{X}}\right)^2 = f\,C_x^2$$
$$E(\varepsilon_0\varepsilon_1) = E\left[\left(\frac{\overline{x}-\overline{X}}{\overline{X}}\right)\left(\frac{\overline{y}-\overline{Y}}{\overline{Y}}\right)\right] = f\,\rho C_y\, C_x$$

Using above approximations we write estimator t_8 as:

$$\begin{aligned} t_8 &= \lambda\ \overline{Y}(1+\varepsilon_0)\ a^{\left[\frac{\overline{X}-\overline{X}(1+\varepsilon_1)}{\overline{X}+\overline{X}(1+\varepsilon_1)}\right]} \\ &= \lambda\ \overline{Y}(1+\varepsilon_0)\ a^{\left[\frac{-\varepsilon_1}{2}\left(1+\frac{\varepsilon_1}{2}\right)^{-1}\right]} \end{aligned} \tag{2.2}$$

Expanding $\left(1+\frac{\varepsilon_1}{2}\right)^{-1}$ and ignoring $\varepsilon's$ higher order terms as they become negligibly small, we get:

$$t_8 = \lambda\overline{Y}\left(1+\varepsilon_0 - \frac{\varepsilon_1}{2}\log_e a + \frac{\varepsilon_1^2}{4}\log_e a - \frac{\varepsilon_0\varepsilon_1}{2}\log_e a + \frac{\varepsilon_1^2}{8}(\log_e a)^2\right) \tag{2.3}$$

Subtracting $\overline{Y}$ from both the sides of the equation (2.3) and squaring both the sides, we get:

$$(t_8 - \overline{Y})^2 = \left[\lambda\overline{Y}\left(1+\varepsilon_0 - \frac{\varepsilon_1}{2}\log_e a + \frac{\varepsilon_1^2}{4}\log_e a - \frac{\varepsilon_0\varepsilon_1}{2}\log_e a + \frac{\varepsilon_1^2}{8}(\log_e a)^2\right) - \overline{Y}\right]^2 \tag{2.4}$$

Taking expectation on both the sides of the equation (2.4), we get the MSE expression of the estimator t_8 as:

$$MSE(t_8) = E(t_8 - \bar{Y})^2$$
$$= E\left[\lambda \bar{Y}\left(1+\varepsilon_0 - \frac{\varepsilon_1}{2}\log_e a + \frac{\varepsilon_1^2}{4}\log_e a - \frac{\varepsilon_0\varepsilon_1}{2}\log_e a + \frac{\varepsilon_1^2}{8}(\log_e a)^2\right) - \bar{Y}\right]^2 \quad (2.5)$$

Simplifying equation (2.5) we get the MSE expression for the estimator t_8 as follows:

$$MSE(t_8) = \bar{Y}^2(1+\lambda^2 A'' - 2\lambda B'') \quad (2.6)$$

Where

$$A'' = 1 + f\left(C_y^2 + \frac{C_x^2}{2}(\log a)^2 - 2\rho C_y C_x(\log a) + \frac{C_x^2}{2}(\log a)\right)$$

$$B'' = 1 + f\left(\frac{C_x^2}{4}(\log a) - \frac{\rho C_y C_x}{2}(\log a) + \frac{C_x^2}{8}(\log a)^2\right)$$

Differentiating equation (2.6) with respect to λ and equating it to zero, we get the optimum value of λ as:

$$\lambda = \frac{B''}{A''}$$

Substituting the optimum value of λ in equation (2.6), we get the minimum MSE expression for the estimator t_8 as:

$$Min.MSE(t_8) = \bar{Y}^2\left(1 - \frac{B''^2}{A''}\right) \quad (2.7)$$

Motivated by [4], we have proposed another estimator t_9 as:

$$t_9 = [\lambda_1 \bar{y} + \lambda_2(\bar{X} - \bar{x})]\ a^{\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]} \text{ for } a > 0 \quad (2.8)$$

Where a is any real positive number.

λ_1 and λ_2 are constants and $\lambda_1 + \lambda_2 \neq 1$.

For calculating MSE of the estimator t_9, we write estimator t_9 in terms of ε_0 and ε_1 as follows:

$$t_9 = [\lambda_1 \bar{Y}(1+\varepsilon_0) + \lambda_2(\bar{X} - \bar{X}(1+\varepsilon_1))]\ a^{\left[\frac{\bar{X}-\bar{X}(1+\varepsilon_1)}{\bar{X}+\bar{X}(1+\varepsilon_1)}\right]} \quad (2.9)$$

Expanding the right hand side of the equation (2.9) and ignoring the higher order terms of ε_0 and ε_1, we get

$$t_9 = [\lambda_1 \bar{Y}(1+\varepsilon_0) - \lambda_2 \bar{X}\varepsilon_1]\left(1 - \tfrac{\varepsilon_1}{2}\log a + \tfrac{\varepsilon_1^2}{4}\log a + \tfrac{\varepsilon_1^2}{8}(\log a)^2\right) \quad (2.10)$$

Simplifying equation (2.10), we obtain:

$$t_9 = \lambda_1 \bar{Y}\left(1 + \varepsilon_0 - \tfrac{\varepsilon_1}{2}\log a + \tfrac{\varepsilon_1^2}{4}\log a - \tfrac{\varepsilon_0\varepsilon_1}{2}\log a + \tfrac{\varepsilon_1^2}{8}(\log a)^2\right)$$
$$-\lambda_2 \bar{X}\left(\varepsilon_1 - \tfrac{\varepsilon_1^2}{2}\log a\right) \quad (2.11)$$

Subtracting $\bar{Y}$ from both the sides of equation (2.11), squaring it and taking expectation on both the sides we get the MSE of the estimator t_9 as:

$$MSE(t_9) = E(t_9 - \bar{Y})^2$$
$$= E\left\{\lambda_1 \bar{Y}\left(1 + \varepsilon_0 - \tfrac{\varepsilon_1}{2}\log a + \tfrac{\varepsilon_1^2}{4}\log a - \tfrac{\varepsilon_0\varepsilon_1}{2}\log a + \tfrac{\varepsilon_1^2}{8}(\log a)^2\right)\right.$$
$$\left.-\lambda_2 \bar{X}\left(\varepsilon_1 - \tfrac{\varepsilon_1^2}{2}\log a\right) - \bar{Y}\right\}^2 \quad (2.12)$$

Further MSE expression for the estimator t_9 t_9 is written as:

$$MSE(t_9) = \bar{Y}^2 + \lambda_1^2 A_3 + \lambda_2^2 B_3 + 2\lambda_1\lambda_2 C_3 - 2\lambda_1 D_3 - 2\lambda_2 E_3 \quad (2.13)$$

Where,

$$A_3 = \bar{Y}^2\left(1 + f\left\{C_y^2 + \tfrac{C_x^2}{2}(\log a)^2 - 2\rho C_y C_x \log a + \tfrac{C_x^2}{2}\log a\right\}\right)$$
$$B_3 = \bar{X}^2 f C_x^2$$
$$C_3 = \bar{X}\bar{Y} f (C_x^2 \log a - \rho C_y C_x)$$
$$D_3 = \bar{Y}^2\left(1 - f\left\{\tfrac{1}{2}\rho C_y C_x \log a + \tfrac{1}{4}C_x^2 \log a + \tfrac{1}{8}C_x^2(\log a)^2\right\}\right)$$
$$E_3 = \bar{X}\bar{Y} f \tfrac{C_x^2}{2}\log a$$

Differentiating equation (2.13) with respect to λ_1 and λ_2, equating it to zero we get the optimum values of λ_1 and λ_2 as:

$$\lambda_1 = \frac{C_3E_3 - B_3D_3}{C_3{}^2 - A_3B_3} = \lambda_1^*(Say) \quad (2.14)$$

$$\lambda_2 = \frac{C_3D_3 - A_3E_3}{C_3{}^2 - A_3B_3} = \lambda_2^*(Say) \quad (2.15)$$

Substituting the values of equation (2.14) and (2.15) in equation (2.13), we obtain the minimum MSE expression for the estimator t_9 in terms of a as:

$$Min.MSE(t_9) = \overline{Y}^2 + \lambda_1^{*2} A_3 + \lambda_2^{*2} B_3 + 2\lambda_1^* \lambda_2^* C_3 - 2\lambda_1^* D_3 - 2\lambda_2^* E_3 \tag{2.16}$$

For $\lambda_1 + \lambda_2 = 1$, estimator t_9 becomes:

$$t_{10} = [\lambda^* \overline{y} + (1-\lambda^*)(\overline{X} - \overline{x})]\, a^{\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right]} \text{ for } a > 0 \tag{2.17}$$

And we write $\lambda_1 = \lambda^*$ and $\lambda_2 = (1-\lambda^*)$ in equation (2.13).

For calculating MSE of the estimator t_{10} we proceed in the same way as above and we get the MSE expression for the estimator t_{10} as:

$$MSE(t_{10}) = \overline{Y}^2 + \lambda^{*2} A^* - 2\lambda^* B^* + C^* \tag{2.18}$$

Where

$$A^* = \overline{Y}^2 \left[1 + f\left(C_y^2 + \frac{C_x^2}{2}(\log a)^2 - 2\rho C_y C_x \log a + \frac{C_x^2}{2} \log a \right) \right] + \overline{X}^2 f C_x^2 + 2\overline{X}\overline{Y} f (\rho C_y C_x - C_x^2 \log a)$$

$$B^* = \overline{Y}^2 \left[1 + f\left(\frac{C_x^2}{4} \log a - \frac{\rho C_y C_x}{2} \log a + \frac{C_x^2}{8} (\log a)^2 \right) \right] + \overline{X}^2 f C_x^2 - \overline{X}\overline{Y} \frac{e_1^2}{2} \log a + \overline{X}\overline{Y} f (\rho C_y C_x - C_x^2 \log a)$$

$$C^* = f C_x^2 \overline{X} (\overline{X} - \overline{Y} \log a)$$

Differentiating equation (2.18) with respect to λ^* and equating it to zero we get the optimum value as:

$$\lambda^* = \frac{\overline{Y}^2 \left[1 + f\left(C_y^2 + \frac{C_x^2}{2}(\log a)^2 - 2\rho C_y C_x \log a + \frac{C_x^2}{2} \log a \right) \right] + \overline{X}^2 f C_x^2 + 2\overline{XY} f \left(\rho C_y C_x - C_x^2 \log a \right)}{\overline{Y}^2 \left[1 + f\left(\frac{C_x^2}{4} \log a - \frac{\rho C_y C_x}{2} \log a + \frac{C_x^2}{8} (\log a)^2 \right) \right] \overline{X}^2 f C_x^2 - \overline{XY} f \frac{C_x^2}{2} \log a + \overline{XY} f \left(\rho C_y C_x - C_x^2 \log a \right)} = \lambda^{*'} (say) \tag{2.19}$$

Substituting the value of equation (2.19) in equation (2.18), we get minimum MSE expression of estimator t_{10} in terms of constant a:

$$Min.MSE(t_{10}) = \overline{Y}^2 + \lambda^{*'2} A^* - 2\lambda^{*'} B^* + C^* \tag{2.20}$$

3. Empirical Study

For the empirical study, we have used three data sets:

Table 2: Data sets for the empirical study.

Using the three data sets given in table 2 we have compared the efficiency of the proposed estimators with the existing estimators in the literature.

Percent relative efficiency (PRE) of the estimators is calculated by using the following formula.

$$PRE(t_i) = \frac{MSE(t_0)}{MSE(t_i)} \times 100 \tag{3.1}$$

Where $i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10

Table 3: Table representing the MSE and the PRE of the existing estimators and the proposed estimators.

We have calculated the PRE and the MSE of the proposed estimators t_8, t_9 and t_{10} for each and every value of a (as a is a positive real number) for the efficiency comparison. But in the table 3, we have only shown the PRE and the MSE for a = e = 2.716 (that is same as the existing exponential estimators) and for the value of a where PRE is maximum. We have not shown the PRE and the MSE for each and every value of a as it will unnecessarily make the table lengthier.

From the above table 3, we observe that

- PRE of the estimator t_7 is same as the estimator t_2 for a = e = 2.716. It means that estimator t_7 behaves as the exponential estimator t_2 for a = e = 2.716. This shows that t_2 is particular case of t_7 for a = e = 2.716.Maximum PRE of the estimator t_7 is attained at a = 71.910 (for population I) which is $a \approx \exp\left(\frac{2\rho C_y}{C_x}\right)$. Same is for population II and III.
- PRE of the proposed estimator t_8 is same as estimator t_4 for a = e = 2.716. This shows that t_4 is particular case of t_8 for a = e = 2.716. But as we increase the value of a from 2.716 PRE of the estimator t_8 also increases up to the value of a = 73 (for Population I) and after this value of a PRE starts decreasing. So, maximum PRE of the estimator t_8 is attained at a = 73 which is $a \approx \exp\left(\frac{2\rho C_y}{C_x}\right)$. Same is for population II and III.
- PRE of estimator t_9 is same as estimator t_5 for a = e = 2.716. It means that estimator t_9 behaves as the exponential estimator t_5 for a = e = 2.716. This shows that t_5 is particular case of t_9 for a = e = 2.716.

- PRE of estimator t_{10} is same as estimator t_6 for a = e = 2.716. It means that estimator t_{10} behaves as the exponential estimator t_6 for a = e = 2.716. This shows that t_{10} is particular case of t_6 for a = e = 2.716.
- As we increase the value of a, PRE of the proposed estimators t_8 and t_{10} increases and becomes maximum at a point (point where maxima is attained for estimator t_7) i.e. $a \approx \exp\left(\frac{2\rho C_y}{C_x}\right)$.
- Estimator t_9 is anomalous in nature, as it behaves in a different way. PRE of the estimator t_9 increases as the value of a moves towards 0 i.e. a = 0.1, 0.2, 0.5…. For estimator t_9, on increasing the value of a from a = e = 2.716 to 3, 4, 5…we observed a decreasing pattern in PRE of the estimator. However decreasing the value of a from a = 2.716 and moving towards zero we observed an increasing pattern in PRE of the estimator t_9.
- As the PRE of our proposed estimators t_8, t_9 and t_{10} is better than the exponential as well as regression estimator. So, it is recommended to use our proposed estimators t_8 and t_{10} for $\left(a \approx \exp\left(\frac{2\rho C_y}{C_x}\right)\right)$ and estimator t_9 (for a < 2.718) over the existing exponential estimators.

4. Simulation Study

For the efficiency comparison of our proposed estimators with the existing estimators considered in this study, we conducted a simulation study. We adopted same procedure as [4]. First of all we have generated bi-variate normal population (X_i, Y_i) of size N=10,000 with means (μ_X, μ_Y) = (100,100) and standard deviation (σ_X, σ_Y) = (200,200). The correlation coefficient between study variable and auxiliary variable is fixed at values (0.9 and 0.7). Then we have drawn samples using SRSWOR of size n=500. The Sampling has been repeated r=1000 times and we get simulated MSE as follows:

$$MSE(t_i) = \frac{1}{r}\sum_{r=1}^{1000} MSE(t_i^r) \tag{4.1}$$

Where $i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10

The performance of the estimators is measured with respect to the usual mean estimator by means of simulated PRE's

$$PRE(t_i) = \frac{MSE(t_0)}{MSE(t_i)} \times 100 \tag{4.2}$$

Where $i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10

Table 4: Table comparing efficiency of different estimators with $\rho = 0.9$ and n= 500 (opt. a = 5.8822)

From the Figure 1, Figure 2 and Figure 4, it is obserbed that PRE of the estimators t_7, t_8 and t_{10} increases upto the point $a \approx \exp\left(2\rho \frac{C_y}{C_x}\right)$ =5.8822 (maximum PRE is attained approximately at this point) and after this point PRE starts decreasing. While from the Figure 3 it is observed that estimator t_9 is anomalous in nature, as it shows different behavior than the other estimators. PRE of the estimator t_9 is higher for $a < 2.718$ than PRE of the estimator t_9 for $a = 2.718$. So, it is advised to prefer estimator t_8 and t_{10} for $a \approx \exp\left(2\rho \frac{C_y}{C_x}\right)$ over t_4 and t_6 respectively. Estimator t_9 is preferable for $a < 2.718$ over estimator t_5.

Table 5: Table comparing efficiency of different estimators with $\rho = 0.7$ and n = 500(opt. a = 3.7820)

From the Figure 5, Figure 6 and Figure 8, it is observed that PRE of the estimators t_7, t_8 and t_{10} increases upto the point $a \approx \exp\left(2\rho \frac{C_y}{C_x}\right)$ = 3.7820 (maximum PRE is attained approximately at this point) and after this point PRE starts decreasing. While from the Figure 7 it is observed that estimator t_9 is anomalous in nature, as it shows different behavior than other estimators. PRE of the estimator t_9 is higher for $a < 2.718$ than PRE of the estimator t_9 for $a = 2.718$. So, it is advised to prefer estimator t_8 and t_{10} for $a \approx \exp\left(2\rho \frac{C_y}{C_x}\right)$ over t_4 and t_6 respectively. Estimator t_9 is preferable for $a < 2.718$ over estimator t_5.

For $\rho = 0.7$, optimum point a changes as we change the value of ρ. As the optimum point a depends on the correlation coefficient so, if the correlation between Y and X is small optimum point a shifts towards e=2.718 and our proposed estimators becomes equally efficient as exponential estimators discussed here in this article.

Results: From the simulation study conducted here, we conclude that our proposed generalized exponential estimators t_8, t_9 and t_{10} perform better than the other exponential estimators considered here. From the above Table 4 and 5 it is observed that as we increase correlation coefficient from 0.7 to 0.9, MSE decreases and PRE of estimators increases. From the Table 4 and 5, it is clearly shown that PRE of estimators t_8 and t_{10} increases up to a certain point and after that it starts decreasing. This point where PRE for the estimators t_8 and t_{10} is maximum is approximately same as the optimum point that is obtained for the estimator t_7 i.e. $a \approx \exp\left(\frac{2\rho c_y}{c_x}\right)$. But estimator t_{10} behaves differently as this estimator performs better than the existing exponential estimators for the values of $a < 2.718$.

5. Conclusion

In this article we have proposed three improved generalized exponential estimators $(t_8, t_9 \text{ and } t_{10})$ for estimating population mean using auxiliary variable under SRSWOR. Our proposed estimators $t_8, t_9 \text{ and } t_{10}$ show improvement over the existing [1], [2], [9] and [4] estimators. We found that our proposed estimators $t_8 \text{ and } t_{10}$ acquire the highest efficiency than exponential estimators for $a \approx \exp\left(\frac{2\rho c_y}{c_x}\right)$. Among all estimators, t_9 acquire the highest efficiency for the values of $a < 2.718$. Results are verified empirically as well as using a simulation study. It is advised to use our proposed estimators $t_8 \text{ and } t_{10}$ for $a \approx \exp\left(\frac{2\rho c_y}{c_x}\right)$ and t_9 for $a < 2.718$ over existing exponential estimators considered in this article.

Appendix

Table 1

Showing some of the existing estimators in the literature with their MSE expressions.

Estimators	MSE expressions
$t_0 = \bar{y}$ Usual mean	$MSE(t_0) = \bar{Y}^2 f C_y^2$
$t_1 = \frac{\bar{y}}{\bar{x}} \bar{X}$ (Ratio Estimator [4])	$MSE(t_1) = \bar{Y}^2 f (C_y^2 + C_x^2 - 2\rho C_y C_x)$
$t_2 = \bar{y} \exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]$ (by [1])	$MSE(t_2) = \bar{Y}^2 f \left(C_y^2 + \frac{1}{4} C_x^2 - \rho C_y C_x\right)$
$t_3 = \bar{y} + \beta (\bar{X} - \bar{x})$ (by [9])	$MSE(t_3) = \bar{Y}^2 f (1-\rho^2) C_y^2$

Contd...

$t_4 = k\,\bar{y}\,\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]$ (by [4])	$MSE(t_4) = \bar{Y}^2(1+k^2A-2kB)$ Where $A = 1+f(C_y^2+C_x^2-2\rho C_y C_x)$ $B = 1-f\left(\frac{1}{2}\rho C_y C_x - \frac{3}{8}C_x^2\right)$ and $k=\frac{B}{A}$
$t_5 = [k_1\bar{y}+k_2(\bar{X}-\bar{x})]\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]$ (by [4])	$MSE(t_5) = \bar{Y}^2 + k_1^2A_1 + k_2^2B_1 + 2k_1k_2C_1 - 2k_1D_1 - 2k_2E_1$ Where $A_1 = \bar{Y}^2(1+f(C_y^2+C_x^2-2\rho C_y C_x))$ $B_1 = \bar{X}^2 f C_x^2$ $C_1 = \bar{X}\bar{Y} f(C_x^2-\rho C_y C_x)$ $D_1 = \bar{Y}^2\left(1-f\left(\frac{\rho C_y C_x}{2}-\frac{3}{8}C_x^2\right)\right)$ $E_1 = \bar{X}\bar{Y}f\frac{C_x^2}{2}$; $k_1 = \frac{C_1E_1-B_1D_1}{C_1^2-A_1B_1}$ and $k_2 = \frac{C_1D_1-A_1E_1}{C_1^2-A_1B_1}$
$t_6 = [k''\bar{y}+(1-k'')(\bar{X}-\bar{x})]$ $\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]$ for $k_1+k_2=1$ (by [4])	$MSE(t_6) = \bar{Y}^2 + k''^2A_2 - 2k''B_2 + C_2$ Where $A_2 = A_1+B_1-2C_1$ $B_2 = B_1-C_1+D_1-E_1$ $C_2 = B_1-2E_1$ and $k''=\frac{B_2}{A_2}$
$t_7 = \bar{y}\,a^{\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]}$ (by [14])	$MSE(t_7) = \bar{Y}^2$ $f(C_y^2+\frac{1}{4}C_x^2(\log a)^2-\rho C_y C_x(\log a))$ Optimum $a \approx \exp\left(\frac{2\rho C_y}{C_x}\right)$

Table 2

Data sets for the empirical study.

	Population I (taken from [6])	**Population II (taken from [5])**	**Population III (taken from [8])**
N	106	104	923
N	20	20	180
$\overline{Y}$	2212.59	625.37	436.4345
$\overline{X}$	27421.70	13.93	11440.5
C_y	5.22	1.866	1.7183
C_x	2.10	1.653	1.8645
ρ	0.86	0.865	0.9543

Table 3

Table representing the MSE and the PRE of the existing estimators and the proposed estimators.

Estimators	**Population I**			**Population II**			**Population III**		
	a	**MSE**	**PRE**	***a***	**MSE**	**PRE**	***a***	**MSE**	**PRE**
t_0		5411348	100		54993.7	100		2515.07	100
t_1		2542740	212.82		13869.9	396.49		267.64	939.71
t_2		3758095	143.99		23642.9	232.60		651.04	386.31
t_3		1409115	384.02		13846.0	397.18		224.62	1119.68
t_4		2423766	223.26		22774.2	241.47		649.9361	386.97
t_5		1043336.3	518.66		12931.8	425.25		222.76	1129.00
t_6		1130449	478.69		22731.2	241.93		245.50	1024.439
t_7	**2.7182**	**3758095**	**143.99**	**2.718**	**23642.9**	**232.60**	**2.718**	**651.04**	**386.31**
	71.910	1409115	384.02	7.049	13846.0	397.18	5.806	224.62	1119.68
t_8	**2.7182**	**2423767**	**223.26**	**2.718**	**22774.0**	**241.47**	**2.718**	**649.8909**	**386.999**
	71.910	1380751	391.91	7.049	13338.9	412.27	4.5	272.305	1112.16
	73	1380289	**392.44**	7.5	13329.3	**412.57**	5.806	223.829	**1123.65**
	75	1379638	392.22	8	13395.0	410.55	6.5	232.82	1080.25

Contd...

t_9	**2.7182**	**1043338**	**518.68**	**2.718**	**12931.4**	**425.27**	**2.718**	**222.73**	**1129.17**
	3.5	1036951	521.85	3	12931.1	425.28	3	222.75	1129.06
	4	1035387	522.64	4	12948.7	424.70	4	223.05	1127.53
	4.5	1034732	**522.97**	0.1	7335.49	**749.69**	0.1	157.54	**1586.37**
	5	1034722	**522.97**	0.2	12872.2	427.22	0.2	204.48	1229.93
	5.5	1035178	522.74	0.3	13706.1	401.23	0.3	216.83	1159.882
	6	1035979	522.34	2	12988.8	423.39	0.5	223.44	1125.61
t_{10}	**2.7182**	**1130449**	**478.69**	**2.718**	**22731.0**	**241.93**	**2.718**	**245.473**	**1024.58**
	1.5	1121767	482.39	7	13342.0	412.18	4.5	224.21	1121.73
	2	1122630	482.024	**7.5**	**13328.8**	**412.59**	**5.5**	**223.90**	**1123.25**
	3	1132607	477.77	8	13396.6	410.50	6.5	226.6765	1109.54

Table 4

Table comparing efficiency of different estimators with $\rho = 0.9$ and n= 500

(opt. $a \approx \exp\left(\frac{2\rho C_y}{C_x}\right)$ =5.8822)

Estimator	MSE	PRE
t_0	0.3707834	100
t_1	0.0765659	484.2666
t_2	0.131256	284.8606
t_3	0.07219518	513.5847
t_4	0.1301620	284.8630
t_5	0.0721938	513.5947
t_6	0.1301568	284.8744

a	Estimator t_7		Estimator t_8		Estimator t_9		Estimator t_{10}	
	MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
0.1	1.6350188	22.6776	1.6345965	22.6835	**0.0721798**	**513.6938**	1.6342980	22.6876
0.3	0.9082604	40.8235	0.9081467	40.8286	0.0721940	513.5931	0.9080165	40.8344
0.5	0.6471404	57.2957	0.6470883	57.3003	0.0721951	513.5849	0.6470066	57.3075
0.7	0.5017780	73.8939	0.5017494	73.8981	0.0721951	513.5855	0.5016922	73.9065
0.9	0.4070051	91.1004	0.4069878	91.1043	0.0721948	513.5874	0.4069460	91.1137
1.1	0.3398035	109.1170	0.3397924	109.1206	0.0721945	513.5893	0.3397614	109.1305
1.3	0.2895983	128.0337	0.2895908	128.0370	0.0721943	513.5909	0.2895680	128.0471

Contd...

1.5	0.2507381	147.8767	0.2507329	147.8798	0.0721941	513.5921	0.2507167	147.8894
1.7	0.2198857	168.6255	0.2198819	168.6284	0.0721940	513.5931	0.2198712	168.6366
1.9	0.1949251	190.2184	0.1949223	190.2211	0.0721939	513.5938	0.1949163	190.2270
2.1	0.1744406	212.5557	0.1744385	212.5582	0.0721938	513.5942	0.1744367	212.5605
2.3	0.1574447	235.5007	0.1574431	235.5032	0.0721938	513.5945	0.1574450	235.5003
2.5	0.1432250	258.8818	0.1432236	258.8842	0.0721938	513.5947	0.1432290	258.8745
2.7	**0.1312530**	**282.4952**	**0.1312519**	**282.4976**	**0.0721938**	**513.5947**	**0.1312604**	**282.4793**
2.9	0.1211277	306.1095	0.1211268	306.1119	0.0721938	513.5947	0.1211382	306.0829
3.1	0.1125386	329.4721	0.1125378	329.4745	0.0721938	513.5946	0.1125520	329.4328
3.3	0.1052414	352.3170	0.1052407	352.3194	0.0721938	513.5945	0.1052576	352.2630
3.5	0.0990408	374.3744	0.0990402	374.3768	0.0721938	513.5943	0.0990595	374.3036
3.7	0.0937788	395.3809	0.0937782	395.3834	0.0721939	513.5940	0.0937999	395.2917
3.9	0.0893260	415.0899	0.0893255	415.0926	0.0721940	513.5937	0.0893495	415.9807
4.1	0.0855756	433.2816	0.0855750	433.2844	0.0721940	513.5935	0.0856014	433.1512
4.3	0.0824383	449.7709	0.0824377	449.7740	0.0721940	513.5932	0.0824662	449.6186
4.5	0.0798389	464.4141	0.0798384	464.4174	0.0721940	513.5928	0.0798690	464.2396
4.7	0.0777140	477.1127	0.0777134	477.1162	0.0721941	513.5925	0.0777461	477.9159
4.9	0.0760092	487.8141	0.0760086	487.8179	0.0721941	513.5922	0.0760432	487.5957
5.1	0.0746778	496.5111	0.0746771	496.5152	0.0721942	513.5919	0.0747137	496.2720
5.3	0.0736796	503.2376	0.0736789	503.2421	0.0721942	513.5915	0.0737174	503.9792
5.5	0.0729797	508.0640	0.0729790	508.0689	0.0721943	513.5912	0.0730194	508.7878
5.7	0.0725475	511.0904	0.0725468	511.0956	0.0721943	513.5909	0.0725890	511.7982
5.9	**0.0723564**	**512.4400**	**0.0723557**	**512.4456**	0.0721944	513.5906	0.0723997	512.1338
6.1	0.0723830	512.2522	0.0723821	512.2583	0.0721944	513.5902	**0.0724280**	**512.9341**
6.3	0.0726063	510.6764	0.0726054	510.6828	0.0721944	513.5899	0.0726530	510.3483
6.5	0.0730081	507.8658	0.0730072	507.8726	0.0721945	513.5896	0.0730565	507.5297

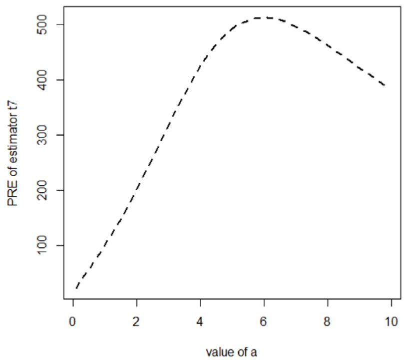

Figure 1

Showing PRE (t_7) for different values of a. (n=500, ρ = 0.9, opt. a =5.8822)

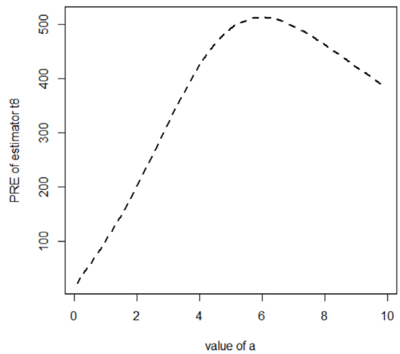

Figure 2

Showing PRE (t_8) for different values of a. (n=500, ρ = 0.9,opt. a =5.8822)

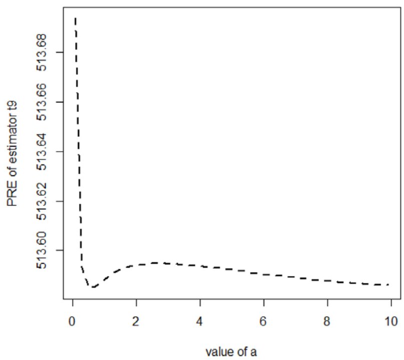

Figure 3

Showing MSE (t_9) for different values of a. (n=500, ρ = 0.9, opt. a = 5.8822)

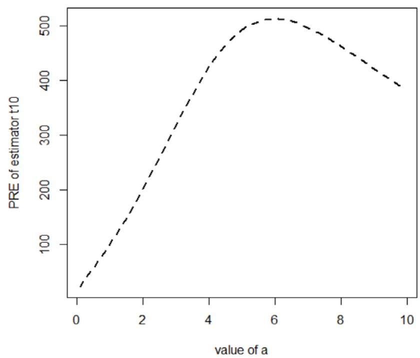

Figure 4

Showing MSE (t_{10}) for different values of a. (n=500, ρ = 0.9, opt. a = 5.8822)

Table 5

Table comparing efficiency of different estimators with $\rho = 0.7$ and n=500(opt. $a \approx \exp\left(\frac{2\rho C_y}{C_x}\right)$ =3.7820)

Estimator	MSE	PRE
t_0	0.3710837	100
t_1	0.2297729	161.5002
t_2	0.2064626	179.7341

Contd...

t_3	0.1927945	192.4763
t_4	0.2064133	179.7771
t_5	0.1927887	192.482
t_6	0.2064097	179.7802

a	Estimator t_7		Estimator t_8		Estimator t_9		Estimator t_{10}	
	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
0.1	1.4646981	25.3352	1.4643749	25.3408	**0.1927869**	**192.4838**	1.4640785	25.3459
0.3	0.8186231	45.3302	0.8185365	45.3350	0.1927944	192.4763	0.8184226	45.3413
0.5	0.5954684	62.3179	0.5954270	62.3223	0.1927934	192.4774	0.5953618	62.3291
0.7	0.4752691	78.0786	0.4752447	78.0827	0.1927921	192.4787	0.4752018	78.0897
0.9	0.3993716	92.9169	0.3993553	92.9207	0.1927911	192.4796	0.3993250	92.9277
1.1	0.3472913	106.8508	0.3472794	106.8545	0.1927904	192.4804	0.3472570	106.8614
1.3	0.3097079	119.8173	0.3096986	119.8209	0.1927899	192.4809	0.3096815	119.8275
1.5	0.2816839	131.7376	0.2816763	131.7412	0.1927895	192.4813	0.2816629	131.7474
1.7	0.2603278	142.5448	0.2603213	142.5483	0.1927892	192.4816	0.2603107	142.5542
1.9	0.2438204	152.1955	0.2438146	152.1991	0.1927890	192.4818	0.2438061	152.2044
2.1	0.2309539	160.6743	0.2309486	160.6781	0.1927889	192.4819	0.2309417	160.6829
2.3	0.2208920	167.9933	0.2208869	167.9971	0.1927888	192.4820	0.2208813	167.0014
2.5	0.2130356	174.1886	0.2130307	174.1926	0.1927887	192.4820	0.2130262	174.1963
2.7	**0.2069437**	**179.3162**	**0.2069389**	**179.3204**	**0.1927887**	**192.4820**	**0.2069353**	**179.3236**
2.9	0.2022839	183.4469	0.2022792	183.4512	0.1927887	192.4820	0.2022763	183.4539
3.1	0.1988008	186.6610	0.1987960	186.6656	0.1927888	192.4820	0.1987938	186.6676
3.3	0.1962946	189.0442	0.1962897	189.0490	0.1927888	192.4820	0.1962881	189.0505
3.5	0.1946069	190.6837	0.1946019	190.6886	0.1927889	192.4819	0.1946008	190.6897
3.7	0.1936105	191.6651	0.1936053	191.6702	0.1927889	192.4818	0.1936047	191.6708
3.9	**0.1932018**	**192.0705**	**0.1931964**	**192.0759**	0.1927890	192.4818	**0.1931963**	**192.0760**
4.1	0.1932960	191.9769	0.1932903	191.9825	0.1927891	192.4817	0.1932906	191.9822
4.3	0.1938226	191.4553	0.1938167	191.4611	0.1927892	192.4816	0.1938174	191.4604
4.5	0.1947230	190.5701	0.1947168	190.5761	0.1927893	192.4815	0.1947179	190.5750
4.7	0.1959476	189.3790	0.1959412	189.3853	0.1927894	192.4814	0.1959427	189.3838
4.9	0.1974547	187.9336	0.1974479	187.9400	0.1927895	192.4813	0.1974499	187.9382

Contd...

5.1	0.1992085	186.2790	0.1992014	186.2857	0.1927896	192.4812	0.1992037	186.2835
5.3	0.2011785	184.4549	0.2011710	184.4618	0.1927897	192.4811	0.2011737	184.4593
5.5	0.2033383	182.4957	0.2033305	182.5027	0.1927898	192.4810	0.2033336	182.5000
5.7	0.2056652	180.4310	0.2056569	180.4382	0.1927899	192.4809	0.2056604	180.4352
5.9	0.2081393	178.2862	0.2081306	178.2836	0.1927900	192.4808	**0.2081345**	**178.2903**
6.1	0.2107433	176.0832	0.2107343	176.0908	0.1927901	192.4807	0.2107385	176.0872
6.3	0.2134623	173.8404	0.2134528	173.8482	0.1927902	192.4805	0.2134574	173.8444
6.5	0.2162827	171.5734	0.2162728	171.5813	0.1927903	192.4804	0.2162778	171.5773

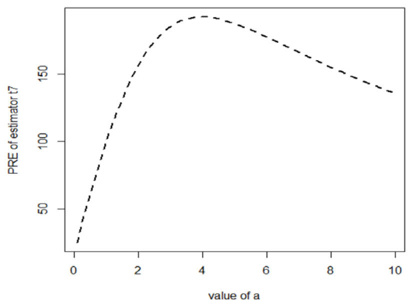

Figure 5

Showing PRE (t_7) for different values of a. (n = 500, ρ = 0.7, opt. a = 3.7820)

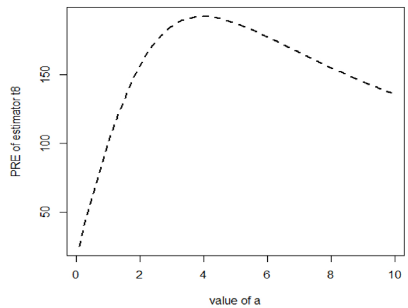

Figure 6

Showing PRE (t_8) for different values of a. (n = 500, ρ = 0.7, opt. a = 3.7820)

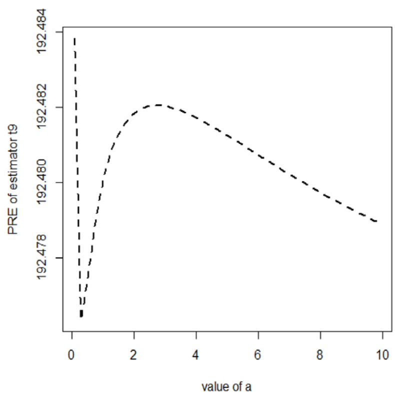

Figure 7

Showing PRE (t_9) for different values of a. (n = 500, ρ = 0.7, opt. a = 3.7820)

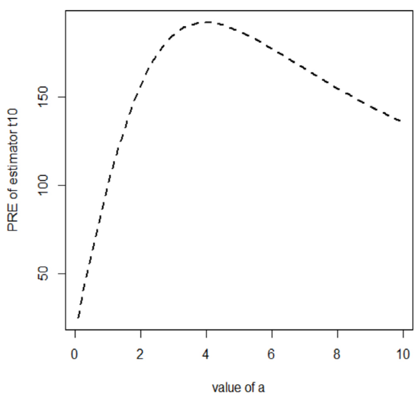

Figure 8

Showing PRE (t_{10}) for different values of a. (n =500, ρ = 0.7, opt. a = 3.7820)

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