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Error function Burr XII distribution : Quantile regression, classical and Bayesian applications

Kingsley Kuwubasamni Ajongba *
Suleman Nasiru †
Abdul-Aziz Adam Kobilla §
Abdul Ghaniyyu Abubakari ‡
Department of Statistics and Actuarial Science
School of Mathematical Sciences
C. K. Tedam University of Technology and Applied Sciences
Box 24, West Africa
Navrongo
Ghana

Abstract

Statistical distributions form the bedrock of many parametric analyses. Thus, identifying an apt model is key in parametric statistical inference. This study formulates the error function Burr XII (EFBXII) model. The EFBXII model is parsimonious compared to other extended forms of the Burr XII model. The EFBXII model density and hazard functions display desirable shapes. The bivariate appendage of the EFBXII model was formulated for studying bivariate data. Maximum likelihood estimators of the EFBXII model parameters were proposed and simulations carried out to assess their behaviours. The simulation results revealed the estimators are well behaved. A quantile regression model is contrived by re-parametrizing the EFBXII model. Utility of the EFBXII model and its corresponding regression are illustrated utilizing data. Empirical findings in both cases reveal that the developed models offer better fit compared to some existing distributions for the given data sets. Bayesian illustrations were carried out and in all cases the estimates are quite close to those from the classical method.

Subject Classification: *(2010) 60E05, 62F10, 62E15.*

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* *E-mail:* **kajongba.stu@cktutas.edu.gh (Corresponding Author)**
† *E-mail:* **sulemanstat@gmail.com;**
snasiru@cktutas.edu.gh
§ *E-mail:* **a-aadam@cktutas.edu.gh**
‡ *E-mail:* **aabubakari@cktutas.edu.gh**

1. Introduction

The extension of existing probability distribution models to improve their performance when it comes to modeling data has gained much attention in literature recently. This is due to continuous search of models that provide reasonable parametric fit with less loss of information. Although many distributions have been extended or modified in literature, the Burr XII distribution [9] continues to receive attention due to its applicability in countless jurisdictions such as hydrology, medicine and quality control among others [4].

Quite a number of add-on of the Burr XII distribution have been initiated recently and includes: Harris extended Burr XII [33], equilibirium renewal Burr XII [5], Gompertz modified Burr XII [2], unit Burr XII [20], Weibull Burr XII [16], modified Burr XII [19], Burr XII-Burr XII [13], extended Weibull Burr XII [31], Gompertz Burr XII [1], exponentiated Burr XII [22], gamma Burr XII [15], Burr XII modified Weibull [23], five parameter Burr XII [3], new Weibull Burr XII [24], modified log Burr XII [6], Burr XII power series [32], Kumaraswamy Burr XII [29] and beta Burr XII [30] among others.

Despite the several add-on or modifications of the Burr XII distribution, none of these annexes can dispense good parametric fit to all forms of data due to the complex nature of the data generating process. Thus, proposing new appendage of the Burr XII model is still not out of place. Hence, we develop the error function Burr XII (EFBXII) distribution using the error function generator (erf-G) developed by Fernándz and De Andrade [12]. The motivations for the new development are:

i. The error function improves the flexibility of existing distribution without adding extra parameter to it. Hence, the EFBXII distribution is parsimonious compared to good deal of appendages of the Burr XII model in literature.
ii. The EFBXII model probability density function (PDF) of the exhibit desirable shapes with different degrees of kurtosis.
iii. The EFBXII model hazard rate function (HRF) displays both monotonic and non-monotonic shapes.
iv. Formulate bivariate add-on of the EFBXII model for modeling relationship between bivariate random variables.

v. Propose EFBXII quantile regression model which is robust for modeling heavy-tailed response variable, skewed response variable or data contaminated with outliers.

vi. Illustrate the empirical applications of the EFBXII model and its associated regression employing classical and Bayesian procedures.

The remnant of the paper are structured in this order: Section 2 gives detail on the formulation of the EFBXII model, Section 3 detailed important properties of the proposed model, Section 4 demonstrated the bivariate appendage of the EFBXII model, Section 5 outlines the procedure used in estimating the model parameters, Section 6 dispense details of the simulation results, Section 7 shows the applicability of the EFBXII model, Section 8 presents the development and applications of the quantile regression model and Section 9 proffers conclusion of the research.

2. Error Function Burr XII Distribution

Given that X is a random variable from the erf-G family of distributions. Then from [12], the cumulative distribution function (CDF) and PDF are

$$F(x;\zeta) = erf\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right], x \in \mathbb{R} \tag{1}$$

and

$$f(x;\zeta) = \frac{2g(x;\zeta)\exp\left[-\left(\frac{G(x;\zeta)}{1-G(x;\zeta)}\right)^2\right]}{\sqrt{\pi}(1-G(x;\zeta))^2}, x \in \mathbb{R}, \tag{2}$$

where $g(x;\zeta)$ is the baseline PDF, $G(x;\zeta)$ is the basline CDF and ζ is a $p \times 1$ vector of parameters.

Suppose that X follows the Burr XII (BXII) distribution with CDF and PDF respectively given by

$$G(x;\alpha,\lambda,\theta) = 1 - \left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta}, x > 0 \tag{3}$$

and

$$g(x;\alpha,\lambda,\theta) = \lambda\theta\alpha^{-\lambda}x^{\lambda-1}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta-1}, x > 0, \tag{4}$$

where $\lambda > 0$ and $\theta > 0$ are the shape parameters and $\alpha > 0$ is the scale parameter. Making use of equations (3) and (1), CDF of the EFBXII model is obtained as

$$F(x;\alpha,\lambda,\theta) = erf\left[\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{\theta} - 1\right], x > 0, \alpha > 0, \lambda > 0, \theta > 0. \tag{5}$$

The PDF of the EFBXII model is achieved by utilizing equations (3), (4) and (2). Hence, the PDF is given by

$$f(x;\alpha,\lambda,\theta) = \frac{2\lambda\theta\alpha^{-\lambda}x^{\lambda-1}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta-1}\exp\left[-\left(\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right)^{2}\right]}{\sqrt{\pi}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-2\theta}}, x > 0. \tag{6}$$

In order to establish properties of the EFBXII model, we require expansion of its PDF. Adopting the linear form of the PDF of erf-G family by [12] and simplifying further using binomial expansion, gives

$$f(x;\alpha,\lambda,\theta) = \sum_{k,m=0}^{\infty}\sum_{n=0}^{\infty}\omega_{kmn}\lambda\theta\alpha^{-\lambda}x^{\lambda-1}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta(n+1)-1}, \tag{7}$$

where $a_{k,m} = \frac{2(-1)^k d_{2k+1,m}}{\sqrt{\pi}k!(2k+1)}(m+2k+1)$, $d_{2k+1,0} = 1$, $d_{2k+1,m} = \frac{1}{m}\sum_{j=1}^{m}[2j(k+1)$ $-m]d_{2k+1,m-j}, m \geq 1$ *and* $\omega_{kmn} = a_{k,m}(-1)^n\binom{m+2k}{n}$.

The HRF of the EFBXII model is given by

$$h(x;\alpha,\lambda,\theta) = \frac{2\lambda\theta\alpha^{-\lambda}x^{\lambda-1}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta-1}\exp\left[-\left(\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right)^{2}\right]}{\sqrt{\pi}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-2\theta}\left\{1-erf\left[\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right]\right\}}, x > 0. \tag{8}$$

Figure 1 display the plots of the PDF of the EFBXII model. The PDF displays interesting shapes shown in Figure 1.

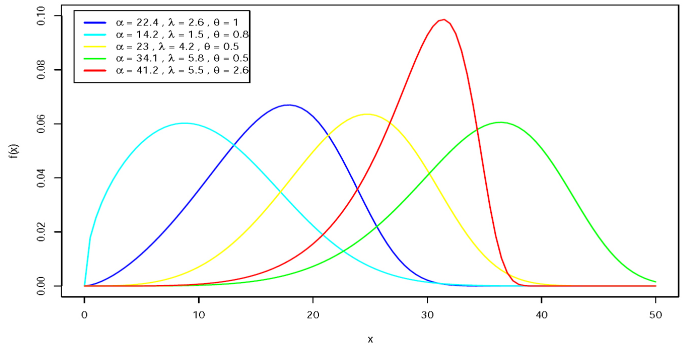

Figure 1

Plots of the PDF of the EFBXII model

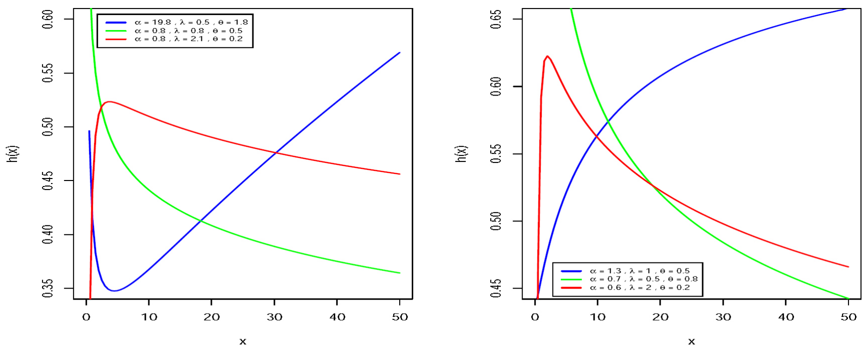

Figure 2

Plots of the HRF of the EFBXII model

Figure 2 displays the HRF plots of the EFBXII model showing desirable shapes.

3. Statistical Properties

Some statistical properties are established in this section.

3.1 *Quantile Function*

The EFBXII model quantile function $Q_X(u)$ is acquired by equating its CDF in equation (5) to $u \in [0,1]$ and solving for x. Thus

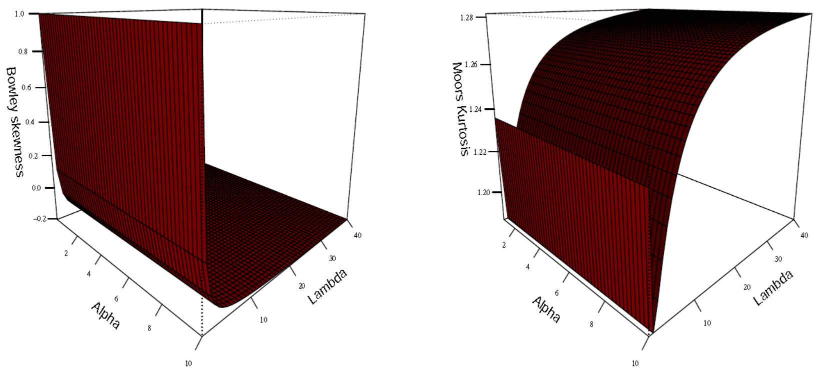

Figure 3

Bowley's skewness(left) and Moor's kurtosis(right) plots

$$erf\left[\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right]=u.$$

After some algebraic manipulations, the quantile of the EFBXII model is

$$Q_X(u)=\alpha((1+erf^{-1}(u))^{\frac{1}{\theta}}-1)^{\frac{1}{\lambda}}, u\in[0,1]. \tag{9}$$

Bowley's coefficient of skewness (BS) and Moor's coefficient of kurtosis (MK) are calculated respectively as follows;

$$BS=\frac{Q_X(0.75)+Q_X(0.25)-2Q_X(0.5)}{Q_X(0.75)-Q_X(0.25)},$$

and

$$MK=\frac{Q_X(0.375)-Q_X(0.125)+Q_X(0.875)-Q_X(0.625)}{Q_X(0.75)-Q_X(0.25)}.$$

Figure 3 displays the plots of BS and MK. From Figure 3, the EFBXII model can be negatively or positively skewed and changes in parameter figures affect both the skewness and kurtosis.

3.3 *Moments*

Moments are very crucial in statistical data analysis since they enable the researcher to obtain the mean, standard deviation, variance, kurtosis and skewness of statistical models. The r^{th} non-central moment of the EFBXII model is defined as

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x)$$
$$= \sum_{k,m=0}^{\infty} \sum_{n=0}^{\infty} \omega_{kmn} \lambda\theta\alpha^{-\lambda} \int_0^\infty x^r x^{\lambda-1} \left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta(n+1)-1} dx.$$

Simplifying further, yields

$$\mu'_r = \sum_{k,m=0}^{n} \sum_{n=0}^{n} \omega_{kmn} \theta\alpha^r B\left(\theta(n+1) - \frac{r}{\lambda}, \frac{r}{\lambda}+1\right), r < \lambda\theta(n+1), r = 1,2,\ldots. \quad (10)$$

Table 1 gives the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the EFBXII random variable for the following sets of parameter values: I : $\alpha = 40$, $\lambda = 3$ and $\theta = 0.5$, II: $\alpha = 25$, $\lambda = 0.2$ and $\theta = 3$, III: $\alpha = 31$, $\lambda = 11$ and $\theta = 5$, IV: $\alpha = 5$, $\lambda = 2$ and $\theta = 11$. Table 1 intimate that the EFBXII model can be negatively or positively skewed. Again the CK values also show that the EFBXII model can display different peaks.

Table 1

Moments, SD, CV, CS and CK

μ'_r	I	II	III	IV
μ'_1	34.1311	0.0210	24.2052	0.9212
μ'_2	1.4589×10^3	0.0045	590.1656	0.9642
μ'_3	6.6389×10^4	0.0023	1.4480×10^4	1.0903
μ'_4	3.1592×10^6	0.0021	3.5725×10^5	1.3012
μ'_5	1.5543×10^8	0.0028	8.8571×10^6	1.6178
μ'_6	7.8466×10^9	0.0051	2.2055×10^8	2.0779
SD	17.1442	0.0635	2.0678	0.3400
CV	0.5023	3.0221	0.0854	0.0854
CS	-0.6881	7.9619	-1.3433	-0.2768
CK	2.5600	115.5372	5.4823	2.4435

3.3 *Moment Generating Function*

The moment generating function (MGF) of the EFBXII model is

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$
$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_{r'}.$$

Hence, the MGF of the EFBXII model is

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k,m=0}^{\infty}\frac{t^r}{r!}\omega_{kmn}\theta\alpha^r B\left(\theta(n+1)-\frac{r}{\lambda},\frac{r}{\lambda}+1\right), r<\lambda\theta(n+1). \tag{11}$$

3.4 *Incomplete Moments*

Incomplete moments also known as conditional moment are crucial for describing the shapes of many distributions. They are essential in estimating the Lorenz (LC) and Bonferroni (BC) curves. The Incomplete moment $m_r(y)$ of the EFBXII model is

$$m_r(y)=\int_0^y x^r f(x)dx,$$

$$=\sum_{k,m=0}^{\infty}\sum_{n=0}^{\infty}\omega_{kmn}\lambda\theta\alpha^{-\lambda}\int_0^y x^r x^{\lambda-1}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta(n+1)-1}dx.$$

Simplifying further, yields

$$m_r(y)=\sum_{k,m=0}^{n}\sum_{n=0}^{n}\omega_{kmn}\theta\alpha^r B^*\left(\left(1+\left(\frac{y}{\alpha}\right)^{\lambda}\right)^{-1};\theta(n+1)-\frac{r}{\lambda},\frac{r}{\lambda}+1\right), \tag{12}$$

where $r<\lambda\theta(n+1)$ and $B^*(y;u,v)=\int_y^1 t^{u-1}(1-t)^{v-1}dt$. The LC and BC curves are acquired utilizing

$$L_F(y)=\frac{1}{\mu}\int_0^y x\, f_Y(x;\alpha,\lambda,\theta)dx$$

and

$$B_F(y)=\frac{1}{\mu F_Y(y;\alpha,\lambda,\theta)}\int_0^y x f_Y(x;\alpha,\lambda,\theta)dx.$$

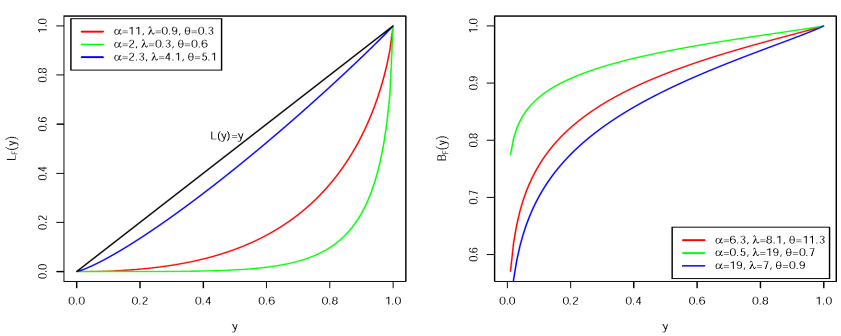

Figure 4

Lorenz curve (left) and Bonferroni curve (right)

Figure 4 shows the inequality curves for the EFBXII model. When $L_F(y)=y,$ the minimal point of inequality is attained for the Lorenz curve and when $B_F(y)=y,$ the equi-distributional line for the Bonferroni curve is attained.

3.5 *Order Statistics*

Let $x_1,x_2,x_3,...,x_n$ be random observations from EFBXII model and $x_{1:n},x_{2:n},...,x_{n:n}$ be the associated order statistic. The PDF of the i^{th} order statistic is given as

$$f_{i:p}(x)=\frac{1}{B(i,p-i+1)}f(x)[F(x)]^{i-1}[1-F(x)]^{p-i},$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the EFBXII model respectively and $B(.,.)$ is the beta function.

Using binomial expansion

$$f_{i:p}(x)=\frac{1}{B(i,p-i+1)}\sum_{q=0}^{p-i}(-1)^q\binom{p-i}{q}[F(x)]^{i+q-1}f(x).$$

Utilizing equations (5) and (7), we obtain the PDF of the i^{th} order statistic as

$$f_{i:p}(x)=\frac{\lambda\theta\alpha^{-\lambda}x^{\lambda-1}\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\theta(n+1)-1}}{B(i,p-i+1)}\sum_{k,m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{p-i}(-1)^q\omega_{kmn}\binom{p-i}{q}$$
$$\times\left[erf\left[\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right]\right]^{i+q-1}. \tag{13}$$

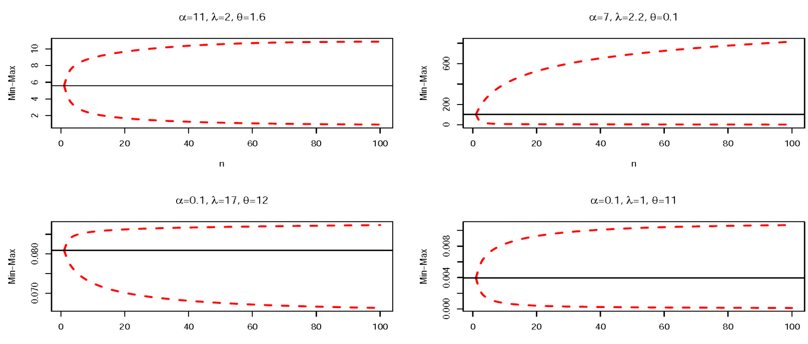

Figure 5

Min-max plots of the EFBXII model

The Minimum and Maximum (min-max) plot is useful in determining whether the distribution is skewed or symmetric. Figure 5 presents the min-max plots of the EFBXII model. It is seen from Figure 5 that the EFBXII model can be negatively-skewed, positively-skewed or symmetric.

4. Bivariate Extension

Development of bivariate model has been of great importance in determining the joint relationship between two random variables. Different methods of developing bivariate models have been proposed in literature. In this study, we use the approach followed by Elhassanein [10], Ganji et al. [14] and Nasiru et al. [26]. The CDF of the bivariate EFBXII (BEFXII) model for $\alpha > 0,\ \lambda > 0,\ \theta > 0,\ -1 < \psi_1 + \psi_3 < 1$ and $-1 < \psi_2 + \psi_3 < 1$ is

$$F_{X,Y}(x, y; \vartheta) = \frac{erf(A)erf(B)}{\{1+(\psi_1+\psi_3)(1-erf(A))+(\psi_2+\psi_3)(1-erf(B))\}^{-1}}, \tag{14}$$

where $x > 0,\ y > 0,\ A = \left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{\theta} - 1,\ B = \left(1+\left(\frac{y}{\alpha}\right)^{\lambda}\right)^{\theta} - 1$ and $\vartheta = (\alpha, \lambda, \psi_1, \psi_2, \psi_3, \theta)^T$.

The related PDF of the BEFBXII model is

$$f_{X,Y}(x, y; \vartheta) = \frac{4\pi^{-1}(\lambda\theta)^2\alpha^{-2}(xy)^{\lambda-1}\left(1+\left(\frac{y}{\alpha}\right)^{\lambda}+\left(\frac{x}{\alpha}\right)^{\lambda}+\left(\frac{xy}{\alpha^2}\right)^{\lambda}\right)^{\theta-1} e^{-A^2-B^2}}{\{1+(\psi_1+\psi_3)(2(1-erf[A])-1)+(\psi_2+\psi_3)(2(1-erf[B])-1)\}^{-1}}. \tag{15}$$

The PDF plots of the BEFBXII model for are given in Figure 6.

(i) $\alpha = 1.1, \lambda = 2.5, \theta = 1.5, \psi_1 = 0.3, \psi_2 = 0.2, \psi_3 = 0.6,$

(ii) $\alpha = 1.2, \lambda = 1.1, \theta = 2.1, \psi_1 = 0.4, \psi_2 = 0.2, \psi_3 = 0.5$ and

(iii) $\alpha = 0.6, \lambda = 4.2, \theta = 0.2, \psi_1 = 0.6, \psi_2 = -0.1, \psi_3 = 0.3.$

These plots display nearly symmetric, positively-skewed and negatively-skewed shapes as seen in Figure 6.

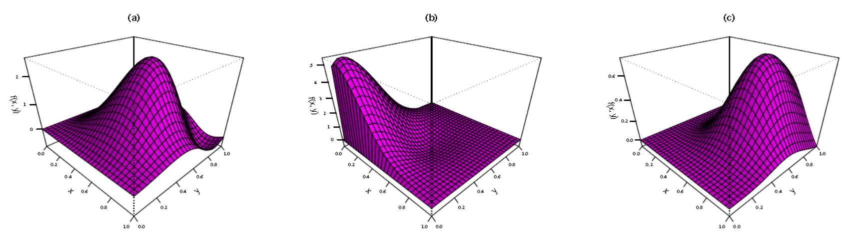

Figure 6

PDF plots of the BEFBXII model

5. Parameter Estimation

The maximum likelihood (ML) procedure is employed to find the estimates of the EFBXII model parameters. Supposed that $x_1, x_2, \ldots, x_n$ are observations from the EFBXII model with vector of parameter $\zeta = (\lambda \ \alpha \ \theta)'$, the log-likelihood function is given by

$$\ell(x;\alpha,\lambda\theta) = n\log(2\lambda\theta) - n\lambda\log(\alpha) + (\lambda-1)\sum_{i=0}^{n}\log(x_i)$$
$$+(\theta-1)\sum_{i=0}^{n}\log\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right) - \sum_{i=0}^{n}\left(\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right)^{2} - \frac{n}{2}\log(\pi). \qquad (16)$$

The score functions are given as

$$\frac{\partial\ell}{\partial\alpha} = -\frac{n\lambda}{\alpha} - \lambda\alpha^{-1}(\theta-1)\sum_{i=0}^{n}\frac{x_i^{\lambda}}{x_i^{\lambda}+\alpha^{\lambda}} + 2\lambda\theta\alpha^{-\lambda-1}$$
$$\times\sum_{i=0}^{n}x_i^{\lambda}\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right)^{\theta-1}\left(\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right), \qquad (17)$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - n\log\alpha + \sum_{i=0}^{n}\log x_i + (\theta-1)\sum_{i=0}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{\lambda}\log\left(\frac{x_i}{\alpha}\right)}{1+\left(\frac{x_i}{\alpha}\right)^{\lambda}}$$
$$-2\theta\sum_{i=0}^{n}\left(\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right)\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right)^{\theta-1}\left(\frac{x_i}{\alpha}\right)\log\left(\frac{x_i}{\alpha}\right) \qquad (18)$$

and

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=0}^{n}\log\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right) - 2\sum_{i=0}^{n}\log\left(\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right)^{\theta}-1\right)\times\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right)^{\theta}\log\left(1+\left(\frac{x_i}{\alpha}\right)^{\lambda}\right). \qquad (19)$$

The ML estimates are secured by equating equations (17), (18) and (19) to zero and evaluating the resulting system algebraically.

6. Simulation Study

Characteristics of the ML estimators of the parameters of the EFBXII model are scrutinized via simulations. The simulation exercise are reproduced 5000 times for different combinations parameter values and sample sizes n = 30, 60, 100, 250, 500 and 600. The sets of parameter values are: I: $(\alpha, \lambda, \theta)$ = (3.5,1.3,0.2) and II: $(\alpha, \lambda, \theta)$ = (1.6,3.1,0.9). The

Table 2

EFBXII model simulation results

Parameters	n	I				II			
		AE	AB	RMSE	CP	AE	AB	RMSE	CP
	30	3.6183	1.2339	1.3627	0.8680	2.1351	0.8932	1.1672	0.8848
	60	3.6275	1.0409	1.1779	0.8974	2.0434	0.7414	1.0202	0.8808
α	100	3.6476	0.9040	1.0473	0.9222	1.9541	0.6093	0.8855	0.8940
	250	3.6205	0.6577	0.8012	0.9414	1.7733	0.3588	0.5649	0.9218
	500	3.5918	0.4898	0.6185	0.9486	1.6769	0.2254	0.3306	0.9302
	600	3.5672	0.4444	0.5648	0.9496	1.6572	0.1942	0.2849	0.9338
	30	1.9321	0.7989	1.3486	0.9810	3.4755	0.7846	0.9988	0.9646
	60	1.5700	0.4269	0.7498	0.9816	3.2966	0.5857	0.7668	0.9606
λ	100	1.4238	0.2639	0.4112	0.9806	3.2113	0.4726	0.6185	0.9496
	250	1.3407	0.1519	0.2019	0.9666	3.1314	0.2957	0.3776	0.9460
	500	1.3170	0.1087	0.1382	0.9570	3.1192	0.2150	0.2705	0.9480
	600	1.3144	0.0989	0.1255	0.9512	3.1187	0.1908	0.2428	0.9516
	30	0.1823	0.0750	0.0897	0.9999	2.2255	1.7179	2.4221	0.6478
	60	0.1929	0.0571	0.0689	0.9999	1.9357	1.3666	2.0871	0.7008
θ	100	0.1978	0.0454	0.0552	0.9999	1.6803	1.0654	1.7561	0.7428
	250	0.2008	0.0314	0.0385	0.9999	1.2467	0.5575	1.0433	0.8168
	500	0.2017	0.0237	0.0294	0.9999	1.0373	0.3085	0.5240	0.8432
	600	0.2013	0.0216	0.0271	0.9999	1.0008	0.2594	0.4424	0.8706

deportment of the ML estimators are explored by looking the average estimate (AE), absolute bias (AB), root mean square error (RMSE) and 95% coverage probability (CP). Table 2 intimate that the AEs converges to the true values as $n \to \infty$, the ABs and RMSEs decreases as $n \to \infty$. This is a manifestation that we have consistent estimators. The high CP values affirms the good traits of the estimators.

7. Applications

The classical and Bayesian utility of the EFBXII model are given here. The data used to demonstrate the applications of the EFBXII model is the lean body mass of Australian athletes data obtained from Esmaeili et al. [11].

7.1 *Classical Applications*

The univariate illustration of the utility of the EFBXII model are given in this subsection. Its superiority is juxtapose with Burr XII (BXII) distribution [28], Burr XII-Burr XII (BXII-BXII) distribution [13], Kumaraswamy Burr XII (KwBXII) distribution [29], Mordified Burr XII-Power (MBXII-Power) distribution [7], Lindley distribution [21], new Weibull-Pareto (NWP) distribution [27], Weibull-Dagum (WD) distribution [34], and Weibull exponential (WEx) distribution [8] based on their log-likelihood (ℓ), Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICc) and Bayesian Information Criterion (BIC) and as well on Kolmogorov-Smirnov (K-S), Anderson-Darling (AD) and Cramér-von Mises (CVM) goodness-of-fit test. The descriptive statistics of lean body mass in kilograms of Australian Sports Athletes is presented in Table 3, It is seen in Table 3 that the data is platykurtic since the coefficient of kurtosis is less than 3. The data is right-skewed due to the positive coefficient of skewness.

Table 4 gives the ML estimates with their respective standard errors in parentheses, information criteria and other measures of fit for the competing models. The EFBXII model offers the best fit for dataset I as is favored by the information criteria and other measures of fit.

Table 3

Descriptive statistics of Australian athletes data

Min.	Max.	Median	Mean	SD	Skewness	Kurtosis
34.36	106	63.03	64.87	13.0702	0.3585	2.7599

Table 4

Parameter estimates and model selection criteria for the Australian athletes data

Model	Parameter	ℓ	AIC	AICc	BIC	K-S	AD	CVM
EFBXII	$\alpha = 49.2527$ (1.4470)	-801.1800	1608.3700	1608.4900	1618.2900	0.0566	0.6107	0.1029
	$\theta = 0.1252$ (0.0282)							
	$\lambda = 11.9638$ (2.1833)							
BXII	$\alpha = 70.1366$ (6.7056)	-805.9800	1617.9700	1618.0900	1627.8900	0.0775	1.2322	0.2101
	$k = 1.6457$ (0.7595)							
	$c = 7.3335$ (0.9336)							
BXII-BXII	$c = 8584.5000$ (1.4200×10^{-5})	-812.8000	1635.6000	1635.9000	1652.1400	0.0726	2.0188	0.2963
	$b = 5519.6000$ (7.7000×10^{-6})							
	$a = 2.8389$(0.0734)							
	$\alpha = 1.7927$ (0.1294)							
	$\beta = 8203.6000$ (2.2000×10^{-6})							
KwBXII	$s = 7608.4183$ (1.9700×10^{-5})	-811.2100	1632.4100	1632.7200	1648.9500	0.0793	1.9853	0.3262
	$k = 9251.2860$ (6×10^{-6})							
	c =2.7079 (0.0587)							
	$a = 1.9843$ (0.1548)							
	$b = 1189.6898$ (2.8100×10^{-5})							

Contd...

MBXII-power	$\alpha = 6949.6409$ (0.0009)	-811.2100	1632.4100	1632.7200	1648.9500	0.0793	1.9853	0.3262
	$k = 0.3303$ (0.0082)							
	$\lambda = 1440.4200$ (0.0042)							
	$\gamma = 5468.0361$ (0.0010)							
	$\beta = 15.2478$ (1.0247)							
Lindley	$\theta = 0.0304$ (0.0015)	-973.7700	1949.5440	1949.5640	1952.8520	0.3741	42.6660	8.4496
NWP	$\beta = 5.2611$ (0.2735)	-811.6100	1629.2140	1629.3350	1639.1380	0.0785	2.0196	0.3254
	$\theta = 93.2431$ (0.0918)							
	$\delta = 4.4338$ (0.3670)							
WD	b = 1.1926 (0.0621)	-801.9900	1611.9760	1612.1790	1625.2090	0.0634	0.7242	0.1273
	$\lambda = 616.5191$ (1.2000×10^{-5})							
	$\delta = 3.0369$ (0.0101)							
	$\beta = 431.2228$ (1.7000×10^{-5})							
WEx	$\alpha = 12.9718$ (5.9800×10^{-6})	-811.6100	1631.2140	1631.4170	1644.4470	0.0785	2.0196	0.3254
	$\lambda = 0.0740$ (0.0010)							
	$\gamma = 67.4519$ (1.2000×10^{-6})							
	$c = 5.2611$ (0.2735)							

7.2 *Bayesian Applications*

Bayesian approach of computing the parameter estimates of the EFBXII model is demonstrated in this subsection. In Bayesian estimation, prior distributions for the parameters of a given distribution are very essential. The study uses the gamma distribution as the prior distribution of the parameters of the EFBXII model and are presented as follows;

$$\pi(\alpha) \sim Gamma(a_1, \beta_1) = \frac{\beta_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1 - 1} e^{-\beta_1 \alpha}, a_1 > 0, \beta_1 > 0, \alpha > 0,$$

$$\pi(\lambda) \sim Gamma(a_2, \beta_2) = \frac{\beta_2^{a_2}}{\Gamma(a_2)} \lambda^{a_2 - 1} e^{-\beta_2 \lambda}, a_2 > 0, \beta_2 > 0, \lambda > 0$$

and

$$\pi(\theta) \sim Gamma(a_3, \beta_3) = \frac{\beta_3^{a_3}}{\Gamma(a_3)} \theta^{a_3 - 1} e^{-\beta_3 \theta}, a_3 > 0, \beta_3 > 0, \theta > 0.$$

The joint prior distribution (joint PDF) is

$$\pi(\alpha, \lambda, \theta) = \pi(\alpha)\pi(\lambda)\pi(\theta).$$

Hence, the joint posterior distribution is

$$f(\alpha, \lambda, \theta \mid x) \propto \prod_{i=1}^{n} f_X(x_i; \alpha, \lambda, \theta) \times \pi(\alpha, \lambda, \theta),$$

where $\prod_{i=1}^{n} f_X(x_i; \alpha, \lambda, \theta)$ is the likelihood function of the EFBXII model. Markov Chain Monte Carlo (MCMC) is utilized to generate observation for inference if the joint posterior distribution does not have simple form [26]. Thus, the study uses the MCMC approach with hyper-parameter values $a_1 = a_2 = a_3 = 2$ and $\beta_1 = \beta_2 = \beta_3 = 5$ being considered for the analysis. The Bayesian application is demonstrated for the Australian athletes data. Table 5 presents the posterior estimates. These include: the average estimate, standard error (SE), standard deviation (SD), and other posterior distribution estimates. The estimated deviance information criterion (DIC) of 1703.9860 is almost equal to the AIC in the classical approach. Again it was observed that the classical and Bayesian estimates are close.

Table 5

Posterior summaries for the Australian athletes data

Parameter	Estimates	SE	SD	2.50%	50%	97.50%	$\hat{R}$	Neff
α	38.0229	0.0181	2.2145	33.5409	38.1031	42.1955	1.001	15000
λ	6.2109	0.0061	0.7432	4.8629	6.1753	7.7565	1.001	11000
θ	0.1554	0.0001	0.0181	0.1227	0.1544	0.1932	1.001	15000

Convenience diagnostic test proposed by Heidelberger and Welch [17, 18] is carried out to ascertain whether or not the Markov chain is from a stationary distribution. It is observed from Table 6 that the process is from a stationary distribution for all the parameters.

Table 6

Stationarity and halfwidth test for the Australian athletes data

	Parameter	Stationarity Test	p-value	Halfwidth Test	Halfwidth
Chain 1	α	passed	0.9070	passed	0.0557
	λ	passed	0.4450	passed	0.0210
	θ	passed	0.4100	passed	0.0005
Chain 2	α	passed	0.8610	passed	0.0642
	λ	passed	0.4380	passed	0.0207
	θ	passed	0.4850	passed	0.0005
Chain 3	α	passed	0.3845	passed	0.0589
	λ	passed	0.0608	passed	0.0238
	θ	passed	0.1251	passed	0.0005

Convergence of the chains of the posterior parameters of the EFBXII model for the Australian athletes data are investigated further. Figure 7 presents the time series (trace) plots which affirm a stationary pattern of the chains.

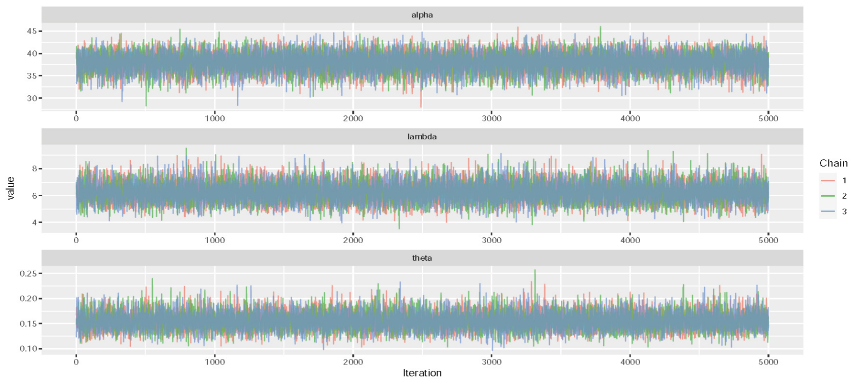

Figure 7

Trace plots for the Australian athletes data

Figure 8 presents the plots of the ergodic mean of the parameters which shows convergence after 1000 iterations.

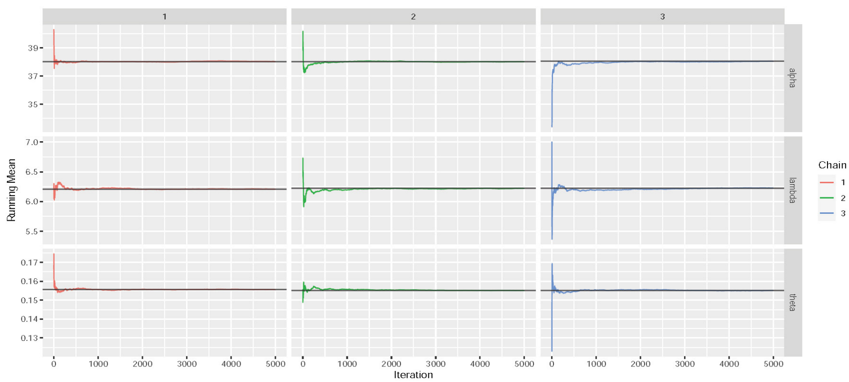

Figure 8

Ergodic mean plots for the Australian athletes data

Figure 9 shows a sharp decay of the autocorrelation. This further suggests convergence of the chains.

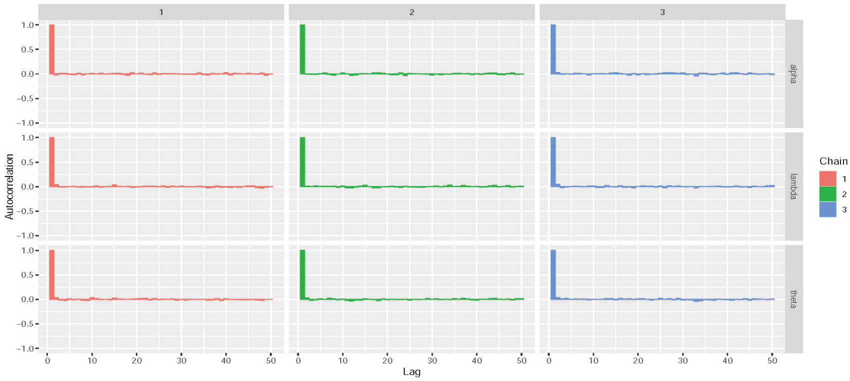

Figure 9

Autocorrelation plots for the Australian athletes data

8. Quantile Regression Model

Unlike the regular linear regression, the quantile regression estimates the conditional quantile of the endogenous variable. When there are outliers or data is skewed, the mean fails to be the best measure of central tendency, hence regression models which relate the exogenous variable to the mean of the endogenous variable fail to capture the variations in the endogenous variable correctly. The EFBXII model is re-parameterized utilizing its quantile function. Suppose $\eta = Q_X\,(u;\ \alpha,\ \lambda,\ \theta),\ \ \eta \in [0,1],$ then $\alpha = \eta((1\ +\ erf^{-1}\ (u))^{\frac{1}{\theta}}\ -1)^{-\frac{1}{\lambda}},$ hence the re-parameterized PDF is given by

$$f(y;\alpha,\lambda,\theta) = \frac{2\lambda\theta D^{-\lambda} y^{\lambda-1}\left(1+\left(\frac{y}{D}\right)^{\lambda}\right)^{-\theta-1} \exp\left[-\left(\left(1+\left(\frac{y}{D}\right)^{\lambda}\right)^{\theta}-1\right)^{2}\right]}{\sqrt{\pi}\left(1+\left(\frac{y}{D}\right)^{\lambda}\right)^{-2\theta}}, x>0, \tag{20}$$

where $D = \eta((1+erf^{-1}(u))^{\frac{1}{\theta}}-1)^{-\frac{1}{\lambda}}$ and η is the quantile parameter.

Let $y_1, y_2, \ldots, y_n$ be random samples from the EFBXII model. Then the quantile regression structure is given as

$$\log(\eta_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}.$$

Hence $\eta_i = \exp(X_i^T \beta), i = 1,2,\ldots,n,$ where $X_i^T = (x_{i1}, x_{i2}, \ldots, x_{ik})$ is the i^{th} vector of covariates and $\beta = (\beta_0, \beta_1, \ldots, \beta_k)^T$ is the vector of regression coefficients. The parameters estimates are secured by maximizing equation (21).

$$\ell(y;\eta,\lambda,\theta) = n\log(2\lambda\theta) - n\lambda\log\eta + n\log D + (\lambda-1)\sum_{i=0}^{n}\log y_i + (\theta-1)$$
$$\times\sum_{i=0}^{n}\log\left(1+\left(\frac{y_i}{\eta D^{-\frac{1}{\lambda}}}\right)^{\lambda}\right) - \sum_{i=0}^{n}\left(\left(1+\left(\frac{y_i}{\eta D^{-\frac{1}{\lambda}}}\right)^{\lambda}\right)^{\theta} - 1\right)^2 - \frac{n}{2}\log\pi, \qquad (21)$$

where $D = (1+erf^{-1}(u))^{\frac{1}{\theta}} - 1$.

8.1 *Residual Analysis*

The randomized quantile residuals (RQR) are employed as a diagnostic tool. The RQR are define as

$$r_i = \Phi^{-1}(F(x_i;\hat{\lambda},\hat{\theta})), i = 1,2,3,...,n,$$

where $\Phi^{-1}(.)$ is the standard normal distribution quantile function [25]. When the model is adequate for the data, the RQR follows standard normal distribution.

8.2 *Quantile Regression simulation*

The section presents simulation results for the ML estimators for the quantile regression model parameters. The exercises are recreated 5000 times using the parameter values:

I: $(\beta_0, \beta_1, \lambda, \theta)$ = (3.2, 2.5, 0.3, 1.8) and II: $(\beta_0, \beta_1, \lambda, \theta)$ = (2.3, 0.3, 0.1, 0.5). The sample sizes n = 30, 60, 100, 250, 500 and 600 are used in generating the samples for the experiments. The simulations are performed using the median regression. The simple regression model used for the simulation is:

$$\eta_i = \exp(\beta_0 + \beta_1 x_{i1}), i = 1,2,...,n,$$

where the exogenous variable x_{i1} is obtained from the standard normal distribution and the values of the endogenous variable are attained utilizing the quantile function of the re-parameterized model. The values of the exogenous variable are held constant during the simulation exercise. Table 7 show that the AEs approaches the actual value as $n \to \infty$. The ABs and RMSEs approaches zero as $n \to \infty$. This affirms the consistency of the ML estimators. The CPs are quite high further affirming that the estimators are well behaved.

Table 7

Quantile regression simulation results

Parameters	n	I					II			
		AE	**AB**	**RMSE**	**CP**		**AE**	**AB**	**RMSE**	**CP**
	30	3.0496	0.4619	0.5586	0.9720		2.3258	1.1986	1.4887	0.9990
	60	3.1727	0.2751	0.3480	0.9590		2.3506	1.0637	1.2684	0.9700
β_0	100	3.1674	0.2319	0.2783	0.9990		2.2594	1.0191	1.2415	0.9740
	250	3.1946	0.1494	0.1892	0.9400		2.2313	0.5855	0.7263	0.9690
	500	3.1964	0.1046	0.1302	0.9670		2.2928	0.4518	0.5606	0.9350
	600	3.2007	0.0975	0.1219	0.9460		2.3170	0.3958	0.4982	0.9220
	30	2.5118	0.3269	0.4258	0.9300		0.9119	0.9005	1.3799	0.9540
	60	2.4783	0.2282	0.2898	0.9380		0.6739	0.6372	0.9078	0.9730
β_1	100	2.4888	0.1731	0.2190	0.9380		0.5463	0.5080	0.6917	0.9730
	250	2.5006	0.1087	0.1367	0.9370		0.4394	0.3825	0.4967	0.9620
	500	2.5007	0.0713	0.0899	0.9530		0.3651	0.2908	0.3507	0.9740
	600	2.5009	0.0679	0.0843	0.9460		0.3485	0.2544	0.3126	0.9800
	30	0.3662	0.1090	0.4259	0.9240		0.0929	0.0202	0.0299	0.9260
	60	0.3417	0.0688	0.1020	0.9970		0.1038	0.0182	0.0229	0.9890
λ	100	0.3162	0.0364	0.0464	0.9750		0.1062	0.0204	0.0266	0.9500
	250	0.3107	0.0264	0.0347	0.9710		0.1014	0.0086	0.0111	0.9670
	500	0.3020	0.0206	0.0264	0.9470		0.1004	0.0072	0.0091	0.9360
	600	0.3038	0.0177	0.0219	0.9520		0.1006	0.0060	0.0075	0.9570
	30	4.1742	2.7754	2.9018	0.9230		1.8806	1.4450	2.3170	0.9810
	60	3.0169	2.0476	2.3526	0.7980		0.9827	0.6043	1.2060	0.9390
θ	100	2.7393	1.6502	2.0386	0.8710		0.7562	0.4007	0.8097	0.8890
	250	2.4529	1.3320	1.7558	0.8370		0.5301	0.1269	0.1703	0.9490
	500	2.4182	1.1074	1.5182	0.8870		0.5283	0.0931	0.1459	0.9350
	600	2.2506	0.9471	1.3374	0.8930		0.5186	0.0805	0.1114	0.9570

8.3 *Applications of Regression Model*

The classical and Bayesian applications of the EFBXII quantile regression is exemplified in this subsection. The data represents the survival times (in years) until onset of hypertension from a random sample of 119 patients secured from the Bolgatanga Regional Hospital in the Upper East region of Ghana with gender (Male=1 and Female=0) as a covariate. The endogenous variable, time until the onset of hypertension is modelled with gender as covariate where the endogenous variable follows the EFBXII distribution. The data are found in Zamanah et al. [35]

8.3.1 Frequentist Application

Table 8 presents ML estimates of the parameters of the quantile regression model with their accompanying standard errors in brackets for $u = 0.1, 0.25, 0.5, 0.75, 0.9,$ the AIC, AICc, BIC and ℓ values for the various quantiles. The results in Table 8 indicates that the 10^{th} percentile is the best with minimal information criteria values.

Table 8

Quantile regression model parameter estimates and information criteria

u	Parameter	ℓ	AIC	AICc	BIC
0.1	$\beta_0 = 3.3988$ (0.0731)	-492.83	993.6538	1004.77	994.0047
	$\beta_1 = 0.0190$ (0.0361)				
	$\lambda = 2.5882$ (0.2171)				
	$\theta = 38.2755$ (92.6860)				
0.25	$\beta_0 = 3.7355$ (0.0474)	-492.83	993.654	1004.77	994.0048
	$\beta_1 = 0.0190$ (0.0361)				
	$\lambda = 2.5880$ (0.2170)				
	$\theta = 38.2544$ (92.5808)				
0.5	$\beta_0 = 3.9883$ (0.0310)	-492.83	993.654	1004.77	994.0048
	$\beta_1 = 0.0190$ (0.0361)				
	$\lambda = 2.5880$ (0.2170)				
	$\theta = 38.2393$ (92.5072)				
0.75	$\beta_0 = 4.1528$ (0.0243)	-492.83	993.6541	1004.771	994.005
	$\beta_1 = 0.0190$ (0.0361)				
	$\lambda = 2.5882$ (0.2171)				
	$\theta = 38.2017$ (92.3888)				
0.9	$\beta_0 = 4.2539$ (0.0234)	-492.83	993.6542	1004.771	994.0051
	$\beta_1 = 0.0190$ (0.0361)				
	$\lambda = 2.5882$ (0.2172)				
	$\theta = 38.1845$ (92.3361)				

The RQR plots shown in Figure 10 reveal a good fit of the EFBXII quantile regression model for the u^{th} percentile of survival times until onset of hypertension for u $\in$ (0.10, 0.25, 0.50, 0.75, 0.90).

8.3.2 Bayesian Application

The Bayesian approach of computing the parameters of EFBXII quantile regression is illustrated here. The prior PDFs employed in the study are defined as follows:

$$\pi(\beta_0) \sim N(a_1, b_1) = \frac{1}{b_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\beta_0 - a_1}{b_1}\right)^2}, -\infty \leq a_1 \leq \infty, b_1 > 0, -\infty \leq \beta_0 \leq \infty,$$

$$\pi(\beta_1,) \sim N(a_2, b_2) = \frac{1}{b_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\beta_1 - a_2}{b_2}\right)^2}, -\infty \leq a_2 \leq \infty, b_2 > 0, -\infty \leq \beta_1 \leq \infty,$$

$$\pi(\lambda) \sim Gamma(a_3, b_3) = \frac{b_3^{a_3}}{\Gamma(a_3)} \lambda^{a_3 - 1} e^{-b_3\lambda}, a_3 > 0, b_3 > 0, \lambda > 0$$

and

$$\pi(\theta) \sim Gamma(a_4, b_4) = \frac{b_4^{a_4}}{\Gamma(a_4)} \lambda^{a_4 - 1} e^{-b_4\theta}, a_4 > 0, b_4 > 0, \theta > 0.$$

The joint posterior distribution is given as

$$f(\beta_0, \beta_1, \lambda, \theta \mid x, y) \propto \prod_{i=1}^{n} f_{X,Y}(x_i, y_i; \beta_0, \beta_1, \lambda, \theta) \times \pi(\beta_0, \beta_1, \lambda, \theta),$$

where $\prod_{i=1}^{n} f_{X,Y}(x_i, y_i; \beta_0, \beta_1, \lambda, \theta)$ is the likelihood function of the quantile regression model and $\pi(\beta_0,\ \beta_1,\ \lambda,\ \theta) = \pi(\beta_0)\ \pi(\beta_1)\ \pi(\lambda)\ \pi(\theta)$ is the joint prior distribution. The study used the hyper-parameter values $a_1 = a_2 = b_1 = b_2 = 0.1$, $a_3 = a_4 = 2$, $b_3 = b_4 = 5$. It is seen from Table 9 that $\hat{R}$ is approximately 1 and neff is greater than 400, hence the MCMC algorithm has converged. The DICs for $u = 0.1, 0.25, 0.5, 0.75, 0.9$ are 995.933, 995.951, 995.911, 995.96 and 995.914 respectively which are very close to the AIC values in the classical approach. Using the DIC, the median quantile is the best.

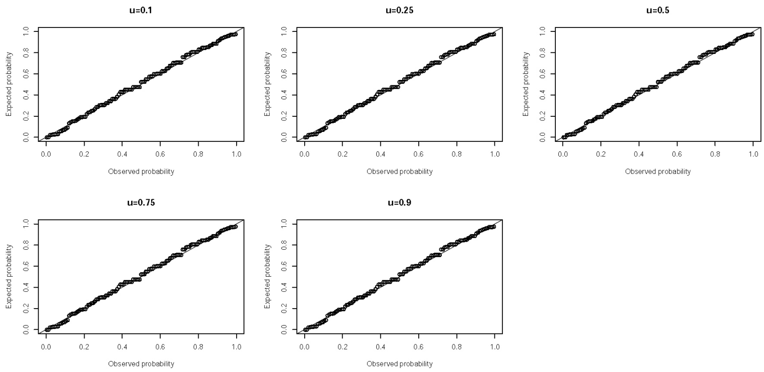

Figure 10

P-P plots for the randomized quantile residuals

Table 9

Posterior summaries of the EFBXII quantile regression model.

u	Parameters	Estimate	SE	SD	2.50%	50%	97.50%	$\hat{R}$	Neff
	β_0	3.3462	0.0006	0.0755	3.1857	3.3501	3.4824	1.002	2800
0.1	β_1	0.0142	0.0004	0.0463	-0.0770	0.0135	0.1055	1.001	1500
	λ	2.7180	0.0020	0.2442	2.2672	2.7074	3.2236	1.002	3200
	θ	1.37478	0.0033	0.4007	0.7547	1.3208	2.3147	1.001	15000
	β_0	3.6870	0.0004	0.0506	3.5810	3.6889	3.7798	1.001	15000
0.25	β_1	0.0144	0.0004	0.0462	-0.0753	0.0142	0.1056	1.001	15000
	λ	2.7230	0.0020	0.2443	2.2716	2.7123	3.2252	1.001	15000
	θ	1.3742	0.0032	0.3969	0.7513	1.3237	2.2826	1.001	15000
	β_0	3.9572	0.0003	0.0355	3.8843	3.9579	4.0242	1.001	8900
0.5	β_1	0.0140	0.0004	0.0459	-0.0770	0.0139	0.1045	1.001	13000
	λ	2.7287	0.0020	0.2434	2.2813	2.7207	3.2380	1.001	4100
	θ	1.3720	0.0032	0.3965	0.7581	1.3132	2.2946	1.001	10000
	β_0	4.1448	0.0002	0.0298	4.0880	4.1442	4.2053	1.001	15000
0.75	β_1	0.0143	0.0004	0.0461	-0.0765	0.0141	0.1054	1.001	15000
	λ	2.7236	0.0020	0.2437	2.2698	2.7168	3.2317	1.001	15000
	θ	1.3673	0.0032	0.3928	0.7550	1.3164	2.2663	1.001	5700
	β_0	4.2690	0.0003	0.0309	4.2146	4.2672	4.3346	1.001	6200
0.9	β_1	0.0139	0.0004	0.0461	-0.0776	0.015	0.1047	1.001	8700
	λ	2.7250	0.0020	0.2431	2.2696	2.7200	3.2173	1.001	8200
	θ	1.3684	0.0032	0.3900	0.7679	1.3157	2.2755	1.001	15000

Stationarity test and other diagnostic plots are further presented for the median quantile since it is the best with minimal DIC value. However upon request from the authors, Stationarity test and plots will be provided for the other quantiles.

Table 10 displays the stationarity and halfwidth test for $u = 0.5$. The results affirms stationarity of the chains.

Table 10

Stationarity and halfwidth test for u = 0.5.

	Parameter	Stationarity Test	p-value	Halfwidth Test	Halfwidth
Chain 1	β_0	passed	0.4980	passed	0.0010
	β_1	passed	0.4410	passed	0.0013
	λ	passed	0.2990	passed	0.0067
	θ	passed	0.4790	passed	0.0107
Chain 2	β_0	passed	0.6700	passed	0.0010
	β_1	passed	0.6630	passed	0.0013
	λ	passed	0.7920	passed	0.0073
	θ	passed	0.4740	passed	0.0110
Chain 3	β_0	passed	0.4800	passed	0.0010
	β_1	passed	0.4240	passed	0.0013
	λ	passed	0.8570	passed	0.0067
	θ	passed	0.5250	passed	0.0111

Figure 11 presents the trace plots of the regression model posterior parameters for $u = 0.5$. The plot shows convergence of the chains.

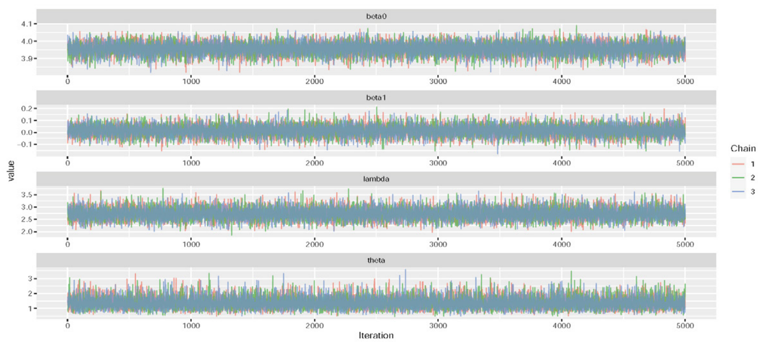

Figure 11

Trace plots for EFBXII quantile regression model

Figure 12 presents the Ergodic mean plots of the parameters for $u = 0.5$. The Figure shows that the chains have converged after 1000 iterations.

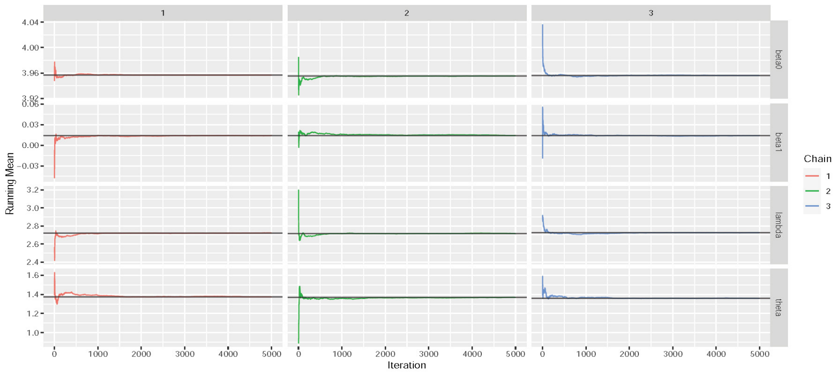

Figure 12

Ergodic mean plots for EFBXII quantile regression model

The autocorrelation plots in Figure 13 shows rapid decay and thus affirms convergence of the chains.

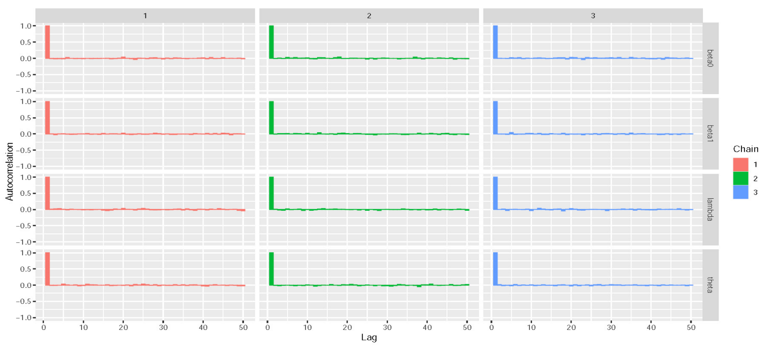

Figure 13

Autocorrelation plots for EFBXII quantile regression model

9. Conclusion

We proposed a new statistical model called the EFBXII model utilizing the erf-G family and the Burr XII model. The density function of the EFBXII model possess interesting shapes like negatively-skewed, positively-skewed or approximately symmetric shapes. The hazard rate function exhibits both monotonic and non-monotonic shapes such as decreasing, increasing, bathtub and upside-down bathtub shapes. Maximum likelihood estimation method is proposed for computing estimates of parameters of the model and simulation studies are executed to scrutinize the traits of the estimators. The simulation results revealed that the maximum likelihood estimation method is consistent. The Classical and Bayesian illustration justifies that the EFBXII model provides competitive fit to given data. The results from the regression analysis showed that the model offers competitive fit for the various quantiles. The residuals analyses for the fitted quantiles justifies the adequacy of the model.

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