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The Type I Half Logistic-Burr XII lifetime distribution : Properties and application to survival data

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Abstract

Real life data cannot fit to the commonly known statistical model structures and therefore, different new models are needed in the modeling of this type of data. In this paper, we propose a new lifetime distribution called the Type I Half Logistic-Burr XII that can model different data from engineering and biomedicine science with its flexible structure. For this new distribution, we obtain the statistical properties including the hazard function, survival function, moment generating function, variance, quantile function, skewness and kurtosis. Moreover, the unknown parameters of the proposed type I half logistic-Burr XII distribution are obtained by using the maximum likelihood method. Application with two survival time data is presented to illustrate the effectiveness of the new distributions and

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it is shown to be better than the Burr XII, Topp-Leone Burr XII and Extended Weibull Log-Logistic distribution.

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1. Introduction

Classical probability distributions cannot be sometimes enough for modelling the real life data. Therefore, new probability distributions which can apply in the real life data because of their flexibility are derived from existing distributions. In this paper, we propose a new distribution obtained from the type 1 half logistic and burr XII distributions.

The type I half logistic distribution is a kind of continuous distribution. It define the cumulative distribution function (cdf) of the type I half-logistic distributions by

$$F(x;\lambda,\xi)=\frac{1-[1-G(x;\xi)]^{\lambda}}{1+[1-G(x;\xi)]^{\lambda}} \tag{1}$$

where G(x; ξ) is the baseline cdf depending on a parameter vector ξ. λ>0 is an additional shape parameter [1]. Olapade (2014) obtained a generalized form of half logistic distribution through a transformation of an exponential random variable and named it type I generalized half logistic distribution [2]. Kumar, Jain and Gupta (2015), type I generalized half-logistics distribution uses to derive some new explicit expressions and repetition relationships for marginal and common moment generating functions of parent record values [3]. Cordeiro, Alizadeh and Marinho (2016) studied the type I half-logistic family. Ogunde, Oseghale and Audu (2017) developed a three parameter distribution known as Type 1 half logistic Gompertz distribution [4]. El-Sherpieny and Elsehetry (2019) studied a new family of distributions named the Kumaraswamy type I half logistic [5]. Shrahili, Elbatal and Muhammad (2019) investigated type I half-logistic Burr X distribution [6]. The probability density function (pdf) related to the type i half logistic distribution in which various studies are carried out,

$$F(x;\lambda,\xi)=\frac{2\lambda g(x;\xi)[1-[1-G(x;\xi)]^{\lambda-1}}{\{1+[1-G(x;\xi)]^{\lambda}\}^{2}}. \tag{1}$$

where g(x; ξ) is the baseline probability density function (pdf). The hazard and survival function of The Type I Half Logistic Distribution are as in equations 3 and 4, respectively

$$S(x) = 1 - F(x) = 1 - \frac{1-[1-G(x;\xi)]^{\lambda}}{1+[1-G(x;\xi)]^{\lambda}}, \tag{3}$$

$$h(x) = \frac{f(x)}{S(x)} = \frac{\lambda g(x;\xi)}{[1-G(x;\xi)]\{1+[1-G(x;\xi)]^{\lambda}\}}. \tag{4}$$

The Burr type XII distribution was first introduced in the literature by Burr (1942) and has gained special attention in the last two decades or so, due to its broad applications in different fields including the area of reliability, failure time modeling and acceptance sampling plan and so on [7]. Widely used to model household income, crop prices, insurance risk, travel time, flood levels and failure data. It is a flexible distribution family that can express a wide range of distribution shapes. The Burr distribution includes, overlaps, or as a limiting case, has many commonly used distributions such as gamma, lognormal, loglogistic, bell-shaped and J-shaped beta distributions (but not U-shaped) [8]. Some studies in recent years need to be mentioned; Paranaíba, Cordeiro and Pascoa (2013) made studies on the theory and application of the The Kumaraswamy Burr XII distribution [9]. Al-Saiari, Baharith and Mousa (2014) studied the Marshall-Olkin extended-Burr type XII distribution [10]. Da Silva, Gomes-Silva, Ramos and Cordeiro (2015) worked applications on the lifetime data with The exponentiated Burr XII Poisson distribution [11]. Rahneva, Terzieva and Golev (2019) worked on the subject of Zubair-G-distribution with baseline Ghosh–Bourguignon's extended-Burr XII [12]. Burr XII distribution has cumulative distribution function (cdf) and probability density function (pdf) given by

$$F(x) = 1-(1+x^{\alpha})^{-\beta},\ x>0,\ \alpha,\beta>0 \tag{5}$$

$$f(x) = \alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-(\beta+1)}, x>0, \alpha,\beta>0 \tag{6}$$

where α and β are shape and scale parameters. The hazard and survival function of the Burr XII distribution are as in equations 7 and 8, respectively,

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-(\beta+1)}}{(1+x^{\alpha})^{-\beta}}, \tag{7}$$

$$S(x) = 1-F(x) = (1+x^{\alpha})^{-\beta} \tag{8}$$

In this paper, we propose Type I Half Logistic-Burr XII and explore some of the properties of this new distribution. We examine the different properties of the Type I Half Logistic-Burr XII distribution such as survival function, hazard function, moment generating function, variance, quantile function, skewness-kurtosis, ordered statistics and maximum likelihood estimator [13]. The model fit of the Type I Half Logistic-Burr XII distribution on two real life data set is researched and as a result, it is found that the type I half logistic-Burr XII distribution provide a better fit than the Burr XII, Topp-Leone Burr XII and extended weibull log-logistic distiributions.

2. The type i half logistic - burr xii distribution

Variety approaches based on $G(x;\xi)$ are used to create a new distribution. Likewise for each baseline G, it can be generated the type I half logistic-burr xii distribution by the cdf in equation (1). We will introduce the type I half logistic burr xii distribution which is shortly TIHLBXII, in this section. We define according to the cumulative distribution function given as,

$$G(x)=\frac{1-(1+x^{\alpha})^{-\beta\lambda}}{1+(1+x^{\alpha})^{-\beta\lambda}},\ x>0,\ \alpha,\beta,\lambda>0 \tag{9}$$

where α, β and λ are shape and scale parameters and the probability density function of the TIHLBXII distribution is given as,

$$g(x)=2\alpha\beta\lambda\frac{x^{\alpha-1}(1+x^{\alpha})^{\beta\lambda-1}}{[(1+x^{\alpha})^{\beta\lambda}+1]^{2}},x>0,\ \alpha,\beta,\lambda>0 \tag{10}$$

where α, β and λ are shape and scale parameters.

Since the cdf of the TIHLBXII distribution is in closed form, we can use it to generate simulated data by using the following formula

$$x=\left(1-\left(\frac{1-u}{u+1}\right)^{-\frac{1}{\beta\lambda}}\right)^{1/\alpha}, \tag{11}$$

where, u is a random variable which follows a standard uniform distribution on (0,1) interval.

The values obtained for the probability density function and cumulative function of the TIHLBXII distribution for several parameter values are as shown in Figure 1 and Figure 2.

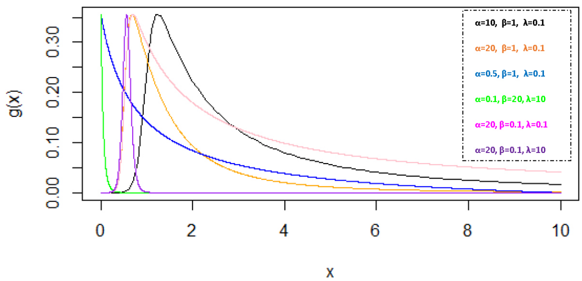

Figure 1

Plots of the TIHLBXII distribution with several parameter values

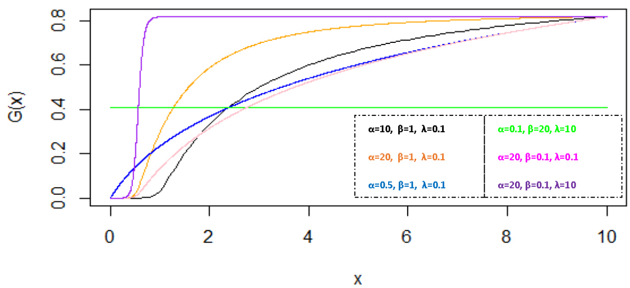

Figure 2

Plots of the cdf for TIHLBXII distribution with several parameter values

2.1 *Survival and Hazard Functions of The TIHLBXII Distribution*

The survival function describes the probability that a variate X takes on a value greater than a number x [14]. The survival function is interrelated to the probability density function. This relationship can be illustrated as follows:

$$S(x) = 1 - G(x). \tag{12}$$

Based on equation 12, the survival function can be rewritten as follows:

$$S(x) = 1 - \frac{1-(1+x^{\alpha})^{-\beta\lambda}}{1+(1+x^{\alpha})^{-\beta\lambda}} \tag{13}$$

Moreover, we present the survival plot of our new distribution in Figure 3 for different parameter values which show flexibility of proposed distribution.

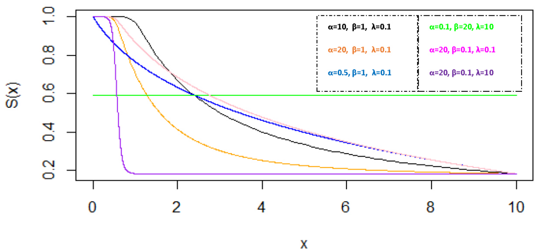

Figure 3

Survival function of the TIHLBXII distribution with several parameter values

The hazard function is defined as the risk rate at time x conditional on survival until time x or later. Other important characteristic of the TIHLBXII distribution is the hazard function which is given by

$$h(x) = \frac{g(x)}{S(x)} = \frac{g(x)}{1-G(x)} \tag{14}$$

The hazard rate function can be shown as follows:

$$h(x) = \frac{2\alpha\beta\lambda\dfrac{x^{\alpha-1}(1+x^{\alpha})^{\beta\lambda-1}}{[(1+x^{\alpha})^{\beta\lambda}+1]^{2}}}{1-\dfrac{1-(1+x^{\alpha})^{-\beta\lambda}}{1+(1+x^{\alpha})^{-\beta\lambda}}}$$

$$h(x) = \frac{\alpha\beta\lambda x^{\alpha-1}(1+x^{\alpha})^{\beta\lambda-1}}{[(1+x^{\alpha})^{\beta\lambda}+1]} \tag{15}$$

In Figure 4, we illustrate the hazard graph of the TIHLBXII distribution for different parameter values. According to the hazard ratio graph, the proposed TIHLBXII distribution has a flexibility structure because parameter values have shown a diverse curve.

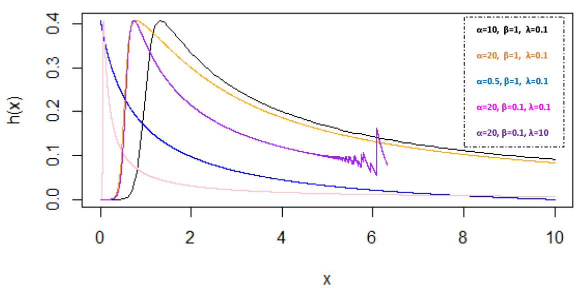

Figure 4

Hazard function of the TIHLBXII distribution with several parameter values

2.2 *Quantile Function of The TIHLBXII Distribution*

The quantile function of a random variable X is the inverse of its distribution function. The quantile function gives information about the description of the statistical properties of the random variable.

$$Q(u) = G^{-1}(x) \tag{16}$$

As mentioned in equation 11 and based on equation 16, the quantile function is as follows:

$$u = \frac{1-(1+x^{\alpha})^{-\beta\lambda}}{1+(1+x^{\alpha})^{-\beta\lambda}}$$

$$Q(u) = \left(1-\left(\frac{1-u}{u+1}\right)^{-\frac{1}{\beta\lambda}}\right)^{1/\alpha}, 0 < u < 1 \tag{17}$$

where u takes the value in the range of (0,1). Quantile functions are widely used in general statistics and often find representations in terms of lookup tables for key percentiles [15]. In equation 17, when u=0.5, the median value of the distribution is obtained and at the same time, it is the second quarter value. Likewise, 0.25, 0.75 and the first and third quarter values are obtained by using the quantile function. The quarter values can be shown as follows:

$$Q(u=0.25) = Q_1 = \left(1-\left(\frac{1-0.25}{0.25+1}\right)^{-\frac{1}{\beta\lambda}}\right)^{1/\alpha}$$

$$Q(u=0.50) = Q_2 = \left(1-\left(\frac{1-0.50}{0.50+1}\right)^{-\frac{1}{\beta\lambda}}\right)^{1/\alpha}$$

$$Q(u=0.75) = Q_3 = \left(1-\left(\frac{1-0.75}{0.75+1}\right)^{-\frac{1}{\beta\lambda}}\right)^{1/\alpha}. \tag{18}$$

2.3 *Skewness and Kurtosis Coefficients of The TIHLBXII Distribution*

One of the two important values that give some information about the distribution is skewness which is a measure of the symmetry in a distribution and the other is kurtosis which is a measure of the combined sizes of the two tails. Bowley's Skewness [16] coefficient obtained from the values of the quartiles is calculated as follows:

$$S = \frac{Q\,{}^{3}\!/_{4} - 2Q\left({}^{1}\!/_{2}\right) + Q\left({}^{1}\!/_{4}\right)}{Q\left({}^{3}\!/_{4}\right) - Q\left({}^{1}\!/_{4}\right)}. \tag{19}$$

Likewise, in order to find the kurtosis value, Moors' kurtosis [17] coefficient is calculated as follows:

$$K = \frac{Q\left({}^{7}\!/_{8}\right) - Q\left({}^{5}\!/_{8}\right) - Q\left({}^{3}\!/_{8}\right) + Q\left({}^{1}\!/_{8}\right)}{Q\left({}^{6}\!/_{8}\right) - Q\left({}^{2}\!/_{8}\right)}. \tag{20}$$

In both equations 19 and 20, Q(.) values represent the quantile function.

There are two types of Skewness. If S is greater than 0, it is Positive other Negative Skewness. Negative skewness refers to a longer or fatter tail on the left side of the distribution, while positive skewness refers to a longer or fatter tail on the right. Also, If S is equal to 0 means that it is a symmetrical distribution that has not a skewness. There are two types of Kurtosis if K greater than 3, it is High other Low Kurtosis. High kurtosis is another name "Leptokurtic", means in a data set is an indicator that data has heavy tails or outliers, while low kurtosis is another name "Platykurtic", means in a data set is an indicator that data has light tails or lack of outliers. Also, if K is equal to 3 means that it is a symmetrical distribution that has not a Kurtosis.

Table 1

Skewness, kurtosis, median and quartiles of the TIHLBXII distribution for different parameter values

α	β	λ	Q_1	Median	Q_3	Skewness	Kurtosis
0.1	1	0.1	1.440573e+22	5.152902e+47	3.234476e+84	9.993936e-01	2.628235e+02
		2	4.353585e-06	4.419898e-02	1.457613e+02	9.993936e-01	2.628235e+02
		5	2.073787e-10	8.027835e-07	5.942913e-04	9.972990e-01	6.189593e+01
		10	1.563591e-13	4.458502e-10	2.092313e-07	9.957397e-01	4.043457e+01
	2	0.1	5.506518e+10	6.888960e+23	1.797395e+42	9.957397e-01	4.043457e+01
		2	2.199827e-09	9.951485e-06	9.326989e-03	9.993936e-01	2.628235e+02
		5	1.563591e-13	4.458502e-10	2.092313e-07	9.993936e-01	2.628235e+02
		10	1.342787e-16	3.295876e-13	1.241410e-10	9.993936e-01	2.628235e+02

Contd...

0.5	1	2	0.08467777	0.53589838	2.70849738	0.99939360	262.82346639
		5	0.01157052	0.06038369	0.22636010	0.99939360	262.82346639
		10	2.746785e-03	1.348459e-02	4.614507e-02	9.993936e-01	2.628235e+02
	5	1	0.01157052	0.06038369	0.22636010	0.99939360	262.82346639
		2	2.746785e-03	1.348459e-02	4.614507e-02	9.993936e-01	2.628235e+02
		10	1.054499e-04	4.935246e-04	1.574934e-03	9.993936e-01	2.628235e+02
1	0.01	2	-1.237193e+11	-7.178980e+23	-1.79846e+42	9.957397e-01	4.043457e+01
		5	-2.735011e+04	-3.486784e+09	-7.97922e+16	9.957397e-01	4.043457e+01
		10	-1.643817e+02	-5.904800e+04	-2.82475e+08	9.957397e-01	4.043457e+01
	2	0.1	-1.186008e+01	-2.420000e+02	-1.68060e+04	9.957397e-01	4.043457e+01
		1	-0.2909944	-0.7320508	-1.6457513	0.9957397	40.4345674
		5	-0.05240978	-0.11612317	-0.21481404	0.99573970	40.43456739

According to Table 1, investigating the results of different parameter values, we can say that as the λ parameter value increases, the skewness coefficient decreases and likewise the kurtosis coefficient decreases. In addition, we can deduce that the values of Q_1, Q_2 and Q_3 decrease as the β parameter value increases. When the parameter α takes a value between 0 and 1, while Q_1, Q_2 and Q_3 values increase; as the parameter α has a greater than 1, quantile values seem negatively decreasing.

2.5 *Ordered Statistics*

Ordered statistics are often used in the fields of reliability and survival tests. On the other hand, it has a crucial importance in statistical estimation [18]. If a random variable X has density and distribution functions as g(x) and G(x), respectively, then the r-th order statistic of X is defined as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} g(x_i)[G(x_i)]^{i-1}[1-G(x_i)]^{n-1}, \tag{21}$$

where g(x) and G(x) are given in equation 10 and 9, respectively. Therefore, the pdf for the ith order statistics becomes

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} 2\alpha\beta\lambda \frac{x_i^{\alpha-1}(1+x_i^{\alpha})^{\beta\lambda-1}}{[(1+x_i^{\alpha})^{\beta\lambda}+1]^2} \left[\frac{1-(1+x_i^{\alpha})^{-\beta\lambda}}{1+(1+x_i^{\alpha})^{-\beta\lambda}}\right]^{i-1} \left[1-\frac{1-(1+x_i^{\alpha})^{-\beta\lambda}}{1+(1+x_i^{\alpha})^{-\beta\lambda}}\right]^{n-1} \tag{22}$$

The minimum order statistic value of pdf is as in equation 30,

$$f_{1:n}(x) = n\, 2\alpha\beta\lambda \frac{x_i^{\alpha-1}(1+x_1^{\alpha})^{\beta\lambda-1}}{[(1+x_1^{\alpha})^{\beta\lambda}+1]^2} \left[\frac{1-(1+x_1^{\alpha})^{-\beta\lambda}}{1+(1+x_1^{\alpha})^{-\beta\lambda}}\right]^{i-1} \left[1-\frac{1-(1+x_1^{\alpha})^{-\beta\lambda}}{1+(1+x_1^{\alpha})^{-\beta\lambda}}\right]^{n-1} \tag{23}$$

The maximum order statistic value of pdf is as in equation 31,

$$f_{n:n}(x) = n2\alpha\beta\lambda \frac{x_n{}^{\alpha-1}(1+x_n{}^{\alpha})^{\beta\lambda-1}}{[(1+x_n{}^{\alpha})^{\beta\lambda}+1]^2} \left[\frac{1-(1+x_n{}^{\alpha})^{-\beta\lambda}}{1+(1+x_n{}^{\alpha})^{-\beta\lambda}}\right]^{i-1} \left[1-\frac{1-(1+x_n{}^{\alpha})^{-\beta\lambda}}{1+(1+x_n{}^{\alpha})^{-\beta\lambda}}\right]^{n-1} \tag{24}$$

3. Statistical Properties

In this section, we will present the different structural properties of the new TIHLBXII distribution.

3.1 *Moments*

Let X is a random variable following the TIHLBXII distribution with parameters α, β and λ. The r$^{\text{th}}$ order moments $E(X^r)$ for the any probability distribution is obtained as follows:

$$E(X^r) = \mu_r' = \int_0^\infty x^r g(x;\alpha,\beta,\lambda)dx \tag{25}$$

Substituting from (10) into (21),

$$\mu^r = \int_0^\infty x^r 2\alpha\beta\lambda \frac{x^{\alpha-1}(1+x^{\alpha})^{\beta\lambda-1}}{[(1+x^{\alpha})^{\beta\lambda}+1]^2} dx$$

$$= 2\alpha\beta\lambda \int_0^\infty \frac{x^{r+\alpha-1}(1+x^{\alpha})^{\beta\lambda-1}}{[(1+x^{\alpha})^{\beta\lambda}+1]^2} dx$$

First, let is make the following derivative process,

$$\frac{\partial}{\partial x}((1+x^{\alpha})^{\beta\lambda}+1) = \alpha x^{\alpha-1}(1+x^{\alpha})^{\beta\lambda-1} dx$$

$$= \frac{1}{\beta\lambda\alpha}\int_0^\infty x^r \frac{\partial}{\partial x}\left[\frac{1}{[(1+x^{\alpha})^{\beta\lambda}+1]}\right]$$

Here, if we take $u = x^r$ and $dv = [(1+x^{\alpha})^{\beta\lambda}+1]$, we get the integral as below:

$$= -\frac{x^r}{(1+x^{\alpha})^{\beta\lambda}+1}\Bigg|_0^\infty + \frac{r}{\beta\lambda\alpha}\int_0^\infty \frac{x^{r-1}}{[(1+x^{\alpha})^{\beta\lambda}+1]}$$

If integral $\alpha,\beta,\lambda > r$, the first term will be zero. We can define a binomial expression as $\frac{1}{1+u} = \sum_{k=0}^{\infty}(-1)^k u^k$ where $u = (1+x^{\alpha})^{\beta\lambda}$, and we can rewrite the following integral according to this binomial expression.

$$\frac{r}{\beta\alpha\lambda}\sum_{k=0}^{\infty}(-1)^k \int_{k=0}^{\infty} x^r (1+x^{\alpha})^{\beta\lambda k} dx$$

If integral is expressed as gamma function here,

$$\frac{\Gamma\left(\frac{r}{\alpha}+\frac{1}{\alpha}\right)\Gamma\left(-\beta\alpha\lambda-\frac{r}{\alpha}-\frac{1}{\alpha}\right)}{\alpha\Gamma(-\beta\alpha k)}$$

we obtain the r. moment for the TIHLBXII distribution as follows:

$$E(X^r) = 2r\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma\left(\frac{r}{\alpha}+\frac{1}{\alpha}\right)\Gamma\left(-\beta\alpha\lambda-\frac{r}{\alpha}-\frac{1}{\alpha}\right)}{\alpha\Gamma(-\beta\alpha k)} \tag{26}$$

Therefore,

$$E(X) = 2\sum_{0}^{\infty}(-1)^k \frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(-\beta\alpha\lambda-\frac{2}{\alpha}\right)}{\alpha\Gamma(-\beta\alpha k)} \tag{27}$$

and

$$E(X^2) = 4\sum_{0}^{\infty}(-1)^k \frac{\Gamma\left(\frac{3}{\alpha}\right)\Gamma\left(-\beta\alpha\lambda-\frac{3}{\alpha}\right)}{\alpha\Gamma(-\beta\alpha k)} \tag{28}$$

Therefore, the variance of TIHLBXII can be obtained as

$$V(X) = E(X^2) - [E(X)]^2$$

So we can simply write the variance for the TIHLBXII distribution as follows:

$$= 4\left[\sum_{0}^{\infty}(-1)^k \frac{\Gamma\left(\frac{3}{\alpha}\right)\Gamma\left(-\beta\alpha\lambda-\frac{3}{\alpha}\right)}{\alpha\Gamma(-\beta\alpha k)} - \left(\sum_{0}^{\infty}(-1)^k \frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(-\beta\alpha\lambda-\frac{2}{\alpha}\right)}{\alpha\Gamma(-\beta\alpha k)}\right)^2\right] \tag{26}$$

3.2 *Moment generating function and Characteristic Function*

Suppose X have TIHLBXII distribution and then, the moment generating function of X can be written as follow:

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} g(x;\alpha,\beta,\lambda)dx \tag{30}$$

Using Taylor's series,

$$M_x(t) = \int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2}+\dots\right)g(x;\alpha,\beta,\lambda)dx$$

$$= \int_0^{\infty}\sum_{n=0}^{\infty}\frac{t^n}{n!}x^n g(x;\alpha,\beta,\lambda)dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mu'_n$$

$$= 2n \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{t^n}{n!} \frac{\tau\left(\frac{n}{\alpha}+\frac{1}{\alpha}\right)\tau\left(-\beta\lambda k-\frac{n}{\alpha}-\frac{1}{\alpha}\right)}{\alpha\,\tau(-\beta\lambda k)} \tag{31}$$

Similarly the characteristic function of TIHLBXII distribution is obtained as,

$$\varphi_x(t) = E(e^{rtx}) = \int_0^{\infty} e^{rtx} g(x;\alpha,\beta,\lambda)dx$$

$$= 2n \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{(rt)^n}{n!} \frac{\tau\left(\frac{n}{\alpha}+\frac{1}{\alpha}\right)\tau\left(-\beta\lambda k-\frac{n}{\alpha}-\frac{1}{\alpha}\right)}{\alpha\,\tau(-\beta\lambda k)}$$

4. Estimation

Let $X_1, X_2, \ldots X_n$ be the random sample from the TIHLBXII distribution with the sample size of *n*. Then, the likelihood function of the TIHLBXII distribution is given by

$$L(x,\alpha,\beta,\lambda) = \prod_{i=1}^{n} 2\alpha\beta\lambda \frac{x^{\alpha-1}(1+x^{\alpha})^{\beta\lambda-1}}{[(1+x^{\alpha})^{\beta\lambda}+1]^2}$$

$$= (2\alpha\beta\lambda)^n \frac{\Pi_{i=1}^{n} x_i^{\alpha-1} \Pi_{i=1}^{n}(1+x_i^{\alpha})^{\beta\lambda-1}}{\Pi_{i=1}^{n}[(1+x^{\alpha})^{\beta\lambda}+1]^2} \tag{32}$$

and from Eq. (32) log-likelihood function is defined as follows:

$$\log(x,\alpha,\beta,\lambda) = n \ln(2\alpha\beta\lambda) + (\alpha-1)\ln\sum_{i=1}^{n} x_i + (\beta\lambda-1)\ln\sum_{i=1}^{n}(1+x_i^{\alpha})$$
$$-2\ln\sum_{i=1}^{n}[(1+x_i^{\alpha})^{\beta\lambda}+1]. \tag{33}$$

By taking the derivative of the log-likelihood function with respect to the parameter values, we obtain the score functions for each parameter value, as given below [13].

$$\frac{\partial \log L}{\partial \alpha} = \log \Sigma_{i=1}^{n} x_i - 2\beta\lambda \log \Sigma_{i=1}^{n} x_i + \frac{n}{\alpha} + (\beta\lambda-1)\log \beta\lambda - 1\Sigma_{i=1}^{n} x_i - 2 \tag{34}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \lambda(\log n + \alpha \log \Sigma_{i=1}^{n} x_i) \tag{35}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \beta(\log n + \alpha \log \Sigma_{i=1}^{n} x_i) \tag{36}$$

5. Application

We make two applications to research the flexibility and consistency of the TIHLBXII distribution. For modeling the TIHLBXII distribution, we firstly use survival time in days of 72 guinea pigs infected with virulent tubercle bacilli [19]. Moreover, we use survival times of 33 patients suffering from acute Myelogeneous Leukaemia as a second data set. For this data, we compare the TIHLBXII distribution with Burr XII (BXII), Topp-Leone Burr XII (TLBXII) and Weibul Generalized Log Logistic (WGLL) distributions. The two data sets and probability density functions of the distributions are given in Appendix A (Table 5 and Table 6) and Appendix B, respectively

We obtain parameter estimations by optim packages in R and to compare these distributions, we use the following goodness-of-fit statistics: the Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC):

$$\begin{aligned} AIC &= -2L(\hat{\theta}) + 2k \\ BIC &= -2L(\hat{\theta}) + klog(n) \\ HQIC &= -2L(\hat{\theta}) + 2klog[\log(n)] \end{aligned} \tag{37}$$

where k and n are the number of parameters and the sample size, respectively and $L(\hat{\theta})$ is the maximized log-likelihood. AIC, BIC, HQIC values of distributions for data I and data II gives in Table 2-3. The descriptive statistics of the data I and data II is given in Table 2 [20].

Table 2

Descriptive statistics of data I and data II

	Min	Q_1	Median	Mean	Q_3	Max.
Data I	0.1	1.08	1.56	1.85	2.30	7
Data II	1	4	22	40.88	65	156

Table 3

Model fit statistics the maximum likelihood estimations for the survival time in days of 72 guinea pigs infected with virulent tubercle bacilli

MODEL	Parameter Estimates	-2LL	AIC	BIC	HQIC
TIHLBXII	$\alpha = 2.2009$ $\beta = 0.8059$ $\lambda = 1.0051$	189.464	195.46	198.02	198.18
BXII	$\alpha = 3.102$ $\beta = 0.465$	205.6	209.60	214.15	211.40
TLBXII	$\alpha = 2.393$ $\beta = 0.458$ $\lambda = 1.796$	205.8	211.80	218.63	214.52
WGLL	$\alpha = 0.921$ $\beta = 0.31$ $\lambda = 3.3$	197.86	203.86	210.69	206.58

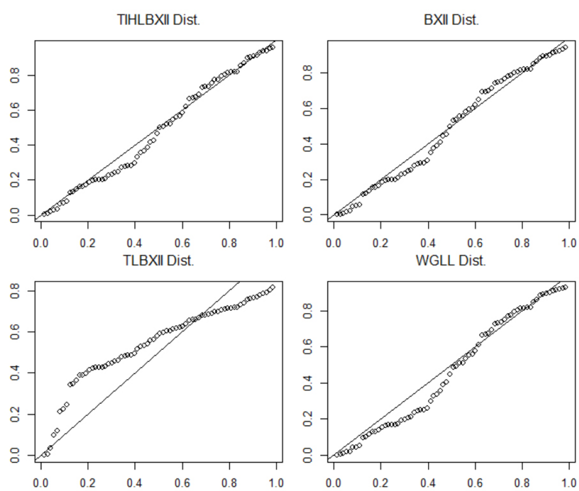

Figure 5

P-P Plots of the distributions for data II

According to the results in Table 3 and Figure 5, we can decide that the proposed distribution shows a better fit over all other distributions mentioned above because AIC, BIC and HQIC values of proposed

TIHLBXII distribution smaller than other given distribution and P-P plot of TIHLBXII distribution has a better fit than other distributions.

Table 4

Model fit statistics the maximum likelihood estimations for the survival times of 33 patients suffering from acute Myelogeneous Leukaemia

MODEL	Parameter Estimates	-2LL	AIC	BIC	HQIC
TIHLBXII	$\alpha = 20.99$ $\beta = 0.79$ $\lambda = 0.01$	225.487	231.49	235.98	232.99
BXII	$\alpha = 58.711$ $\beta = 0.006$	324.2	328.20	331.19	329.19
TLBXII	$\alpha = 0.281$ $\beta = 1.882$ $\lambda = 50.215$	310.26	316.26	320.73	317.76

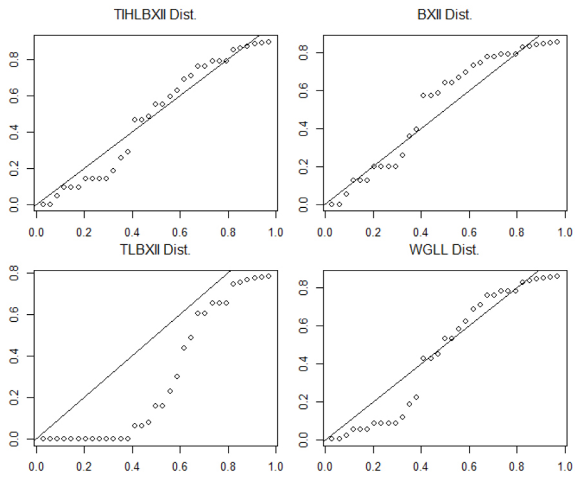

Figure 6

P-P Plots of the distributions for data II

When the results in Table 4 and Figure 6 are examined, it is deducted that the TIHLBXII distribution shows a better fit over all other distributions mentioned above.

6. Conclusions

In this paper, we introduce the Type I Half Logistic-Burr XII distribution with a flexible hazard function and described with different mathematical properties. The survival function, hazard function, moment generating function, variance, quantile function, skewness-kurtosis, ordered statistics and maximum likelihood estimator of the new distribution are presented. An application with two real life data is provided to demonstrate the effectiveness of the new distributions. Consequently, the type I half logistic-Burr XII distribution provides adequate fit values than the Burr XII, Topp-Leone Burr XII and extended weibull log-logistic distributions.

Appendix A

Table 5

The survival time in days of 72 guinea pigs infected with virulent tubercle bacilli

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55

Table 6

The survival times of 33 patients suffering from acute Myelogeneous Leukaemia

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43

Appendix B

Probability density function of the Topp-Leone Burr XII by Reyad and Othman [21] is given follows:

$$f(x) = 2\lambda\alpha\beta x^{\beta-1}(1+x^{\beta})^{-(2\alpha+1)}[1-(1+x^{\beta})^{-2\alpha}]^{\lambda-1} \tag{38}$$

where α, β, $\lambda > 0$ and x>0

The probability density function of the extended Weibull log-logistic proposed Hamed et al. [19] is

$$f(x) = \beta\theta x^{\alpha-1}(1+x^{\alpha})^{\theta-1}[(1+x^{\alpha})^{\theta}-1]^{\beta-1}\exp\{-[(1+x^{\alpha})^{\theta}-1]^{\beta}\}. \tag{39}$$

References

[1] Cordeiro G.M, Alizadeh M. and Marinho P.R.D. The Type I Half – Logistic Family of Distributions, *Journal of Statistical Computation and simulation*, 86 (4), 707-728 (2016).

[2] Olapade A.K., The Type I Generalized Half Logistic Distribution, *Journal of The Iranian Statistical Society*, 13 (1), 69-82 (2014).

[3] Kumar D., Jain N. and Gupta S., The Type I Generalized Half-Logistic Distribution Based on Upper Record Values, Hindawi Publishing Corporation *Journal of Probability and Statistics*, 11 (2015).

[4] Ogunde A. A., Oseghale O. I., Audu A. T., Performance Rating of the Type 1 Half Logistic Gompertz Distribution: An Analytical Approach, *American Journal of Mathematics and Statistics*, 7 (3), 93-98 (2017).

[5] El-Sherpien E., Elsehetry M., Kumaraswamy Type I Half Logistic Family of Distributions with Applications, *Gazi University Journal of Science*, 32 (1), 333-349 (2019).

[6] Shrahili M., Elbatal I. and Muhammad M., The Type I Half-Logistic Burr X Distribution: Theory And Practice, *Journal of Nonlinear Sciences and Applications*, 12, 262-277 (2019).

[7] Kumar D., The Burr Type XII Distribution with Some Statistical Properties, *Journal of Data Science*, 16, 509-534 (2017).

[8] Unknow "https://www.mathworks.com/help/stats/burr-type-xii-distribution.html".

[9] Paranaíba, P. F., Ortega, E. M., Cordeiro, G. M. and Pascoa, M.A.D., The Kumaraswamy Burr XII Distribution: Theory And Practice, *Journal of Statistical Computation and Simulation*, 83 (11), 2117-2143 (2013).

[10] Al-Saiari A.Y., Baharith L.A. and Mousa S.A., Marshall-Olkin Extended Burr Type XII Distribution, *International Journal of Statistics And Probability*, 3 (1), 78 (2014).

[11] Da Silva R.V., Gomes-Silva F., Ramos M.W.A. and Cordeiro G.M., The Exponentiated Burr XII Poisson Distribution with Application To Lifetime Data, *International Journal of Statistics and Probability*, 4 (4), 112 (2015).

[12] Rahneva O., Terzieva T. and Golev A., Investigations On The Zubair– Family with Baseline Ghosh–Bourguignon's Extended Burr

XII Cumulative Sigmoid, Some Applications, Neural, Parallel and Scientific Computations, 27 (1), 11-22 (2019).

[13] Constantinos T. A., Characterizing discrete self-decomposable distributions by incorporating an integral equation for probability generating functions, *Journal of Interdisciplinary Mathematics,* 23 (3), 701-706 (2020).

[14] Evans, M.; Hastings, N.; and Peacock, B. Statistical Distributions, 3rd ed. New York: Wiley, 13 (2000).

[15] Kumar D., The Burr Type XII Distribution With Some Statistical Properties, *Journal of Data Science*, 16, 509-534 (2017).

[16] Kenney, J. F., Mathematics of Statistics Part 1., Chapman & Hall, London (1939).

[17] Moors, J. J. A., A Quantile Alternative For Kurtosis, The Statistician, 37 (1), 25-32 (1988).

[18] Rather, A. A. and Subramanian, C., Exponentiated Ishita Distribution: Properties and Applications, *International Journal of Management and Decision Making IX*, 5, 2473-2484 (2019).

[19] Bjerkedal, T., Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene,* 72, 130-148 (1960).

[20] Hamed, M., Ali, M.M., Yousof and H.M., Korkmaz, M.Ç., Extended Weibull log-logistic distribution. *The Journal of Nonlinear Sciences and Applications* (JNSA), 12 (8), 523-534 (2019).

[21] Reyad, H.M. and Othman, S, A. The Topp-Leone Burr-XII Distribution: Properties and Applications. *British Journal of Mathematics & Computer Science,* 21(5), 1-15 (2017).

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