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The exponentiated Half-Logistic-Type II Topp-Leone-G family of distribution with applications

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Abstract

The development and application of a new family of distributions is of tremendous practical importance. In this note, a new distribution called the exponentiated Half-Logistic-Type II Topp-Leone-G (EHL-TIITL-G) family is proposed and studied. We derive the statistical properties of this new family of distributions, including the moments, probability weighted moments, distribution of the order statistics and Rényi entropy. We investigated the various density and hazard rate function shapes of the EHL-TIITL-G model. The parameters of this new model have been estimated using the maximum likelihood technique. We conducted a simulation study to examine the bias and mean square error of the maximum likelihood estimators, and lastly, the potentiality of this new model has been demonstrated through the use of two real data sets.

Subject Classification: *62E99, 60E05.*

Keywords: *Generalized-G distribution, Exponentiated distributions, Half-logistic distribution, Maximum likelihood estimation.*

1. Introduction

The extension of classical distributions to obtain new and flexible families of distributions has attracted many researchers in recent years. Generalized families of distributions play a crucial role in the modeling of monotone and non-monotone hazard rate functions, particularly

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in reliability and survival analysis. These extended and generalized distributions are important in explaining some desirable characteristics of the underlying data generating process or event. The new models assist practitioners in various scientific fields to make informed decisions and inferences about the process or event they will be studying, for example, Dey et. al. [10] developed a two parameter exponentiated Gumbel distribution and applied it to flood data. Al-Dubaicy and Yassin [1] illustrated the usefulness and applicability of the exponentiated Gumbel distribution to censored data. There are a number of newly developed families of distributions in the statistical literature, including the Weibull-G family of distributions by Bourguignon et. al. [6], the odd power generalized Weibull-G family of distributions by Moakofi et. al. [12], the generalized beta-generated distributions by Alexander et. al.[3] and a new power generalized Weibull-G family of distributions by Oluyede et. al. [13]. Sengweni et. al. [16] developed the exponentiated half logistic-odd Lindley-G family of distributions in an effort to improve the performance of the half-logistic distribution in data modeling. The authors employed the transformer (T-X) by Alzaghal et. al. [4] to come up with the exponentiated half logistic-type II Topp-Leone-G family of distributions.

Our main motivation for developing this new family of distributions is the flexibility gained when it comes to modeling lifetime data with monotonic and non-monotonic failure rates. The new model produced various shapes for the density function as well as the hazard rate function. We noted that the new model captures a variety of real data sets and provides consistently better fits than other generated distributions with the same underlying model. It is worth noting that the new family of distributions contains some other new families of distributions.

This article is organized in the following manner. In Section 2, some basic results are presented, the exponentiated half-logistic type II Topp-Leone-G (EHL-TIITL-G) family of distribution and its sub-models, the hazard function, and the quantile function. The moments, moment generating function and probability weighted moments are given in Section 3. Section 4 contains some additional useful results on the distribution of order statistics and Rényi entropy. Some special cases are given in Section 5. In Section 6, results on the estimation of the parameters of the EHL-TIITL-G family of distributions via the method of maximum likelihood are presented. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators in Section 7. Applications are given in Section 8, followed by some concluding remarks in Section 9.

2. The New Model

The model synopsis, statistical properties of the exponentiated Half-Logistic Type II Topp Leone-G (EHL-TIITL-G) family of distributions including special cases, expansion of the density, hazard rate function and quantile function are presented in this section.

Cordeiro et. al. [9] developed the exponentiated half logistic family of distributions with the cumulative distribution function (cdf) given by

$$F(x;\alpha,\lambda,\zeta)=\int_0^{-\log(1-G(x;\zeta))}\frac{2\alpha\lambda e^{-\lambda t}[1-e^{-\lambda t}]^{\alpha-1}}{(1+e^{-\lambda t})^{\alpha+1}}dt$$
$$=\left[\frac{1-\overline{G}^{\lambda}(x;\zeta)}{1+\overline{G}^{\lambda}(x;\zeta)}\right]^{\alpha}, \tag{1}$$

where $G(x;\zeta)$ is the baseline cdf depending on a parameter vector ζ and $\alpha,\lambda>0$ are shape parameters. When $\lambda=1,$ equation (1) reduces to

$$F(x;\alpha,\zeta)=\left[\frac{G(x;\zeta)}{1+\overline{G}(x;\zeta)}\right]^{\alpha}.$$

In this note, we consider the Type II Topp-Leone distribution to obtain the new family of EHL-TIITL-G distributions. The cdf and probability density function (pdf) of the proposed model are respectively given by

$$F(x;\alpha,b,\xi)=\left[\frac{1-(1-G^2(x;\xi))^b}{1+(1-G^2(x;\xi))^b}\right]^{\alpha} \tag{2}$$

and

$$f(x;\alpha,b,\xi)=\frac{4b\alpha g(x;\xi)G(x;\xi)(1-G^2(x;\xi))^{b-1}(1-(1-G^2(x;\xi))^b)^{\alpha-1}}{(1+(1-G^2(x;\xi))^b)^{\alpha+1}}, \tag{3}$$

where $b,\alpha>0$ and ξ is a vector of parameters of the baseline cdf G. The hazard rate function (hrf) is given by

$$h(x;\alpha,b,\xi)=\frac{4b\alpha g(x;\xi)G(x;\xi)(1-G^2(x;\xi))^{b-1}(1-(1-G^2(x;\xi))^b)^{\alpha-1}}{(1+(1-G^2(x;\xi))^b)^{\alpha+1}}$$
$$\times\left\{1-\left[\frac{1-(1-G^2(x;\xi))^b}{1+(1-G^2(x;\xi))^b}\right]^{\alpha}\right\}^{-1}. \tag{4}$$

2.1 *Linear Representation*

In this section, we expressed the pdf of the EHL-TIITL-G family of distributions as an infinite linear combination of exponentiated-G densities. This is an important result since it will assist us in the derivation of other statistical properties of this new model. Note that

$$
\begin{aligned}
f(x;\alpha,b,\xi) &= 4b\alpha g(x;\xi)G(x;\xi)(1-G^2(x;\xi))^{b-1}(1-(1-G^2(x;\xi))^b)^{\alpha-1} \\
&\quad \times (1+(1-G^2(x;\xi))^b)^{-\alpha-1} \\
&= 4b\alpha g(x;\xi)G(x;\xi)\sum_{i,j=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}(-1)^j \\
&\quad \times (1-G^2(x;\xi))^{b(j+i+1)-1} \\
&= 4b\alpha g(x,\xi)G(x,\xi)\sum_{i,j,l=0}^{\infty}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}(-1)^{j+l} \\
&\quad \times \binom{b(j+i+1)-1}{l}G^{2l}(x;\xi) \\
&= 4b\alpha g(x;\xi)\sum_{i,j,l=0}^{\infty}(-1)^{j+l}\binom{-\alpha-1}{i}\binom{\alpha-1}{j} \\
&\quad \times \binom{b(j+i+1)-1}{l}G(x;\xi)^{2l+1} \\
&= \sum_{l=0}^{\infty}a_l^{*}h_{2(l+1)}(x;\xi),
\end{aligned}
\tag{5}
$$

where

$$
a_l^{*} = 2\sum_{i,j=0}^{\infty}\frac{(-1)^j b\alpha}{l+1}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}\binom{b(j+i+1)-1}{l}, \tag{6}
$$

and $h_{2(l+1)}(x;\xi)=2(l+1)g(x;\xi)G^{2l+1}(x;\xi)$ is the Exp-G density with power parameter $2(l+1)$. Thus, the EHL-TIITL-G density can be expressed as an infinite linear combination of exponentiated-G (Exp-G) densities.

2.2 *Sub-Families*

The following sub-families can be obtained from the EHL-TIITL-G family of distributions.

- If $\alpha=1$, we obtain the type II Topp Leone-G family of distributions with cdf given by

$$F(x;b,\xi)=\frac{1-(1-G^2(x;\xi))^b}{1+(1-G^2(x;\xi))^b}.$$

- If $b=1,$ we obtain the exponentiated half logistic-G family of distributions with cdf given by

$$F(x;\alpha,\xi)=\left[\frac{G^2(x;\xi)}{1+\overline{G^2}(x;\xi)}\right]^{\alpha}.$$

- If $\alpha=b=1,$ we obtain a new family of distributions with cdf given by

$$F(x;\xi)=\frac{G^2(x;\xi)}{1+\overline{G^2}(x;\xi)}.$$

2.3 *Quantile Function*

The quantile function, $Q(u),\ 0\le u\le 1,$ of the EHL-TIITL-G family of distributions is obtained by inverting the cdf as follows:

$$X_q=G^{-1}\left[1-\left(\frac{1-u^{\frac{1}{\alpha}}}{1+u^{\frac{1}{\alpha}}}\right)^{\frac{1}{b}}\right]^{\frac{1}{2}}. \qquad (7)$$

This function is useful in generating some random numbers from the EHL-TIITL-G distribution and will also help in the simulations when assessing the performance of estimators.

3. Moments

In this section, we assume a random variable $Y_{2(l+1)}$ to be an Exp-G distribution with power parameter $2(l+1),$ and make use of the linear representation results to derive the statistical properties of this new model. Thus, the r^{th} ordinary moment of the EHL-TIITL-G family of distributions is given by

$$\mu_r'=E(X^r)=\sum_{l=0}^{\infty}a_l^*E(Y_{2(l+1)}^r), \qquad (8)$$

where a_l^* is as given in equation (6) and $E(Y_{2(l+1)}^r)$ is the r^{th} moment of the Exp-G distribution. The r^{th} incomplete moment of the EHL-TIITL-G family of distributions is given by

$$\delta_r(y) = \int_{-\infty}^{y} x^r f(x;\alpha,b,\xi)dx = \sum_{l=0}^{\infty} a_l^* \int_{-\infty}^{y} x^r h_{2(l+1)}(x;\xi)dx,$$

where $\int_{-\infty}^{y} x^r h_{2(l+1)}(x;\xi)dx$ is the r^{th} incomplete moment of the Exp-G distribution. The conditional moments can also be obtained from the incomplete moments. The q^{th} central moment of the EHL-TIITL-G family of distributions is given as:

$$\mu_q = \sum_{r=0}^{q} \binom{q}{r} (-\mu_1')^{q-r} E(X^r) = \sum_{r=0}^{q}\sum_{l=0}^{\infty} a_l^* \binom{q}{r} (-\mu_1')^{q-r} E(Y_{2(l+1)}^r).$$

The moment generating function (mgf) $M_X(t) = E(e^{tX})$ of the EHL-TIITL-G family of distributions can be derived from equation (5) and is given by

$$M_X(t) = \sum_{l=0}^{\infty} a_l^* M_{2(l+1)}(t),$$

where $M_{2(l+1)}(t)$ is the mgf of $Y_{2(l+1)}$.

The skewness and kurtosis plots of the EHL-TIITL-W are given in Figure 1 and Figure 2. From the plots, it can be seen that the members of the EHL-TIITL-G family can model various data types in terms of skewness and kurtosis.

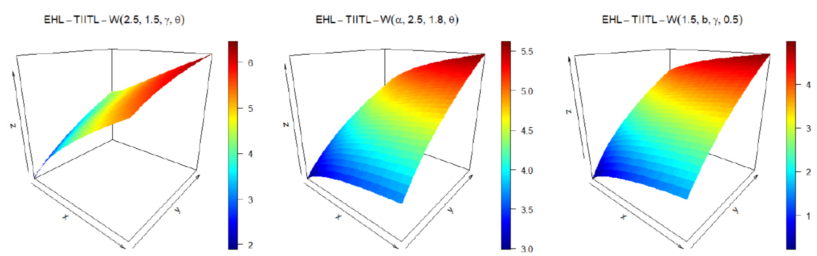

Figure 1

Skewness plots for the EHL-TIITL-W distribution.

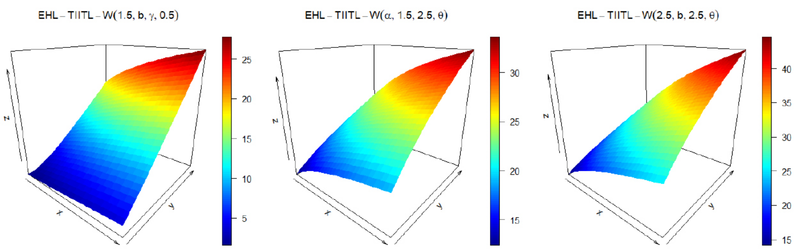

Figure 2

Kurtosis plots for the EHL-TIITL-W distribution.

3.1 *Probability Weighted Moments*

The $(i,j)^{th}$ Probability Weighted Moments (PWM) of a random variable $X \sim EHL-TIITL-G$, say $\wp_{i,j}$ is given by

$$\wp_{i,j} = E(X^i(F(X))^j) = \int_{-\infty}^{\infty} x^i f(x)(F(x))^j \, dx. \tag{9}$$

Note that $f(x)(F(x))^j$ can be written as follows:

$$\begin{aligned}
f(x)(F(x))^j &= 4b\alpha g(x;\xi)G(x;\xi)(1-G^2(x;\xi))^{b-1}(1-(1-G^2(x;\xi))^b)^{\alpha(j+1)-1} \\
&\quad \times (1+(1-G^2(x;\xi))^b)^{-\alpha(j+1)-1} \\
&= 4b\alpha g(x;\xi)G(x;\xi)\sum_{i,q=0}^{\infty}(-1)^q \binom{-\alpha(j+1)-1}{i}\binom{\alpha(j+1)-1}{q} \\
&\quad \times (1-G^2(x;\xi))^{b(i+q+1)-1} \\
&= 4b\alpha g(x;\xi)\sum_{i,q,l=0}^{\infty}(-1)^{q+l}\binom{-\alpha(j+1)-1}{i}\binom{\alpha(j+1)-1}{q} \\
&\quad \times \binom{b(i+q+1)-1}{l}(G(x;\xi))^{2l+1} \\
&= \sum_{l=0}^{\infty}\varphi^{*}_{2(l+1)}h_{2(l+1)}(x;\xi),
\end{aligned}$$

where $h_{2(l+1)}(x;\xi) = 2(l+1)g(x;\xi)G^{2l+1}(x;\xi)$ and

$$\varphi^{*}_{2(l+1)} = 2b\alpha\sum_{i,q=0}^{\infty}\frac{(-1)^{q+l}}{l+1}\binom{-\alpha(j+1)-1}{i}\binom{\alpha(j+1)-1}{q}\binom{b(i+q+1)-1}{l}.$$

Consequently, the PWM of X can be written as:

$$\rho_{j,i} = \sum_{l=0}^{\infty}\varphi^{*}_{2(l+1)}\int_{-\infty}^{\infty}x^j h_{2(l+1)}(x;\xi)dx = \sum_{l=0}^{\infty}\varphi^{*}_{2(l+1)}E(Y^{j}_{2(l+1)}). \tag{10}$$

4. Order Statistics and Rényi Entropy

In this section, expressions for the distribution of order statistics and the Rényi entropy are presented.

4.1 *Order Statistics*

Suppose $X_1, X_2, X_3, ..., X_n$ are independent and identically EHL-TIITL-G distributed random variables. Then, the pdf of the i^{th} order statistics $(X_{i:n})$ can be written as:

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (F(x))^{j+i-1}, \tag{11}$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)(F(x))^{j+i-1} &= 4b\alpha g(x;\xi)G(x;\xi)(1-G^2(x;\xi))^{b-1}(1-(1-G^2(x;\xi))^b)^{\alpha(j+i-1)} \\ &\times (1+(1-G^2(x;\xi))^b)^{-\alpha(j+i-1)} \\ &= 4b\alpha g(x;\xi)G(x;\xi) \sum_{q,p=0}^{\infty} (-1)^q \binom{\alpha(j+i-1)}{q} \binom{-\alpha(j+i-1)}{p} \\ &\times (1-G^2(x;\xi))^{b(q+p+1)-1} \\ &= 4b\alpha g(x;\xi) \sum_{q,p,l=0}^{\infty} (-1)^{q+l} \binom{\alpha(j+i-1)}{q} \binom{-\alpha(j+i-1)}{p} \\ &\times \binom{b(q+p+1)-1}{l} G(x;\xi)^{2l+1}. \end{aligned} \tag{12}$$

Substituting equation (12) into equation (11), we obtain

$$f_{i:n}(x) = \sum_{l=0}^{\infty} \omega^*_{2(l+1)} h_{2(l+1)}(x;\xi),$$

where $h_{2(l+1)}(x;\xi) = 2(l+1)g(x;\xi)G^{2l+1}(x;\xi)$ is the Exp-G density with power parameter $2(l+1)$ and

$$\begin{aligned} \omega^*_{2(l+1)} &= 2b\alpha \sum_{q,p=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{q+l+j}}{B(i,n-i+1)(l+1)} \binom{n-i}{j} \binom{\alpha(j+i-1)}{q} \\ &\times \binom{-\alpha(j+i-1)}{p} \binom{b(q+p+1)-1}{l}. \end{aligned}$$

4.2 *Rényi Entropy*

Rényi entropy (Rényi [14]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(\nu)=\frac{1}{1-\nu}\log\left(\int_0^\infty [f(x)]^\nu dx\right), \nu \neq 1, \nu > 0. \tag{13}$$

From the EHL-TIITL-G pdf, $f^\nu(x)$ can be expressed as:

$$\begin{aligned}
f^\nu(x) &= (4b\alpha)^\nu g^\nu(x,\xi)G^\nu(x,\xi)(1-G^2(x,\xi))^{\nu(b-1)}(1-(1-G^2(x,\xi))^b)^{\nu(\alpha-1)} \\
&\times (1+(1-G^2(x,\xi))^b)^{-\nu(\alpha+1)} \\
&= (4b\alpha)^\nu g^\nu(x,\xi)G^\nu(x,\xi)\sum_{i,j=0}^{\infty}\binom{-\nu(\alpha+1)}{i}\binom{\nu(\alpha-1)}{j}(-1)^j \\
&\times (1-G^2(x,\xi))^{b(j+i+\nu)-\nu} \\
&= (4b\alpha)^\nu g^\nu(x,\xi)\sum_{i,j=0}^{\infty}(-1)^{j+l}\binom{-\nu(\alpha+1)}{i}\binom{\nu(\alpha-1)}{j} \\
&\times\binom{b(j+i+\nu)-\nu}{l}G^{2l+\nu}(x;\xi).
\end{aligned}$$

Thus, the Rényi entropy of the EHL-TIITL-G family of distributions in terms of the Rényi entropy of the Exp-G distribution as follows:

$$I_R(\nu)=\frac{1}{1-\nu}\log\left[\sum_{l=0}^{\infty}c_l^* e^{(1-\nu)I_{REG}}\right],$$

where

$$c_l^* = (4b\alpha)^\nu \sum_{i,j=0}^{\infty}(-1)^{j+l}\binom{-\nu(\alpha+1)}{i}\binom{\nu(\alpha-1)}{j}\binom{b(j+i+\nu)-\nu}{l}\times\left(\frac{2l+\nu}{\nu}+1\right)$$

and $I_{REG} = \int_0^\infty [(\frac{2l+\nu}{\nu}+1)g(x;\xi)G(x;\xi)^{\frac{2l+\nu}{\nu}}]^\nu dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{2l+\nu}{\nu}$.

5. Some Special Cases

In this section, we present three special cases of the EHL-TIITL-G family of distributions. We considered cases where the underlying distribution is Burr XII, Weibull and Uniform distributions.

5.1 *Exponentiated Half-Logistic Type II Topp-Leone-Weibull (EHL-TIITL-W) Distribution*

If we take the Weibull distribution with cdf and pdf given by $G(x;\gamma,\theta)=1-\exp(-\theta x^\gamma)$ and $g(x;\gamma,\theta)=\gamma\theta x^{\gamma-1}\exp(-\theta x^\gamma)$, respectively,

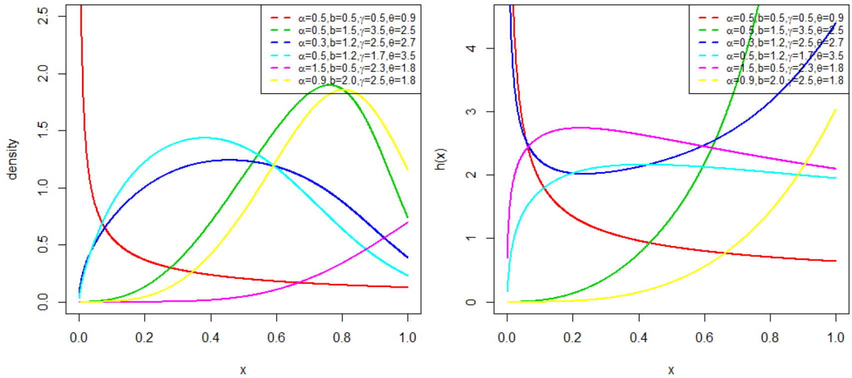

Figure 3

Plots of the pdf and hrf of the EHL-TIITL-W distribution.

for $\theta,\gamma>0$ and $x>0$, then, the EHL-TIITL-W distribution has cdf and pdf given by

$$F(x;\alpha,b,\gamma,\theta)=\left[\frac{1-(1-(1-e^{-\theta x^{\gamma}})^2)^b}{1+(1-(1-e^{-\theta x^{\gamma}})^2)^b}\right]^{\alpha}$$

and

$$f(x;\alpha,b,\gamma,\theta)=\frac{4b\alpha\gamma\theta x^{\gamma-1}e^{-\theta x^{\gamma}}(1-e^{-\theta x^{\gamma}})(1-(1-e^{-\theta x^{\gamma}})^2)^{b-1}}{(1+(1-(1-e^{-\theta x^{\gamma}})^2)^b)^{\alpha+1}}$$
$$\times(1-(1-(1-e^{-\theta x^{\gamma}})^2)^b)^{\alpha-1},$$

for $\alpha,b,\gamma,\theta>0$.

The versatility and flexibility of the EHL-TIITL-W distribution can be seen in Figure 3. From the plots of the pdf, the EHL-TIITL-W distribution can handle a variety of data sets including skewed to the left and skewed to the right. In terms of the hrf, it can be seen that the shapes include bathtub, upside-down bathtub increasing and decreasing shapes.

5.2 *Exponentiated Half-Logistic Type II Topp-Leone-Burr XII (EHL-TIITL-BXII) Distribution*

Taking the Burr XII distribution with cdf $G(x;\theta,\gamma)=1-(1+x^{\theta})^{-\gamma}$ and pdf $g(x;\theta,\gamma)=\theta\gamma x^{\theta-1}(1+x^{\theta})^{-\gamma-1}$ for $\theta,\gamma>0$ and $x>0$ as a baseline distribution, we obtain the EHL-TIITL-BXII distribution with cdf and pdf given by

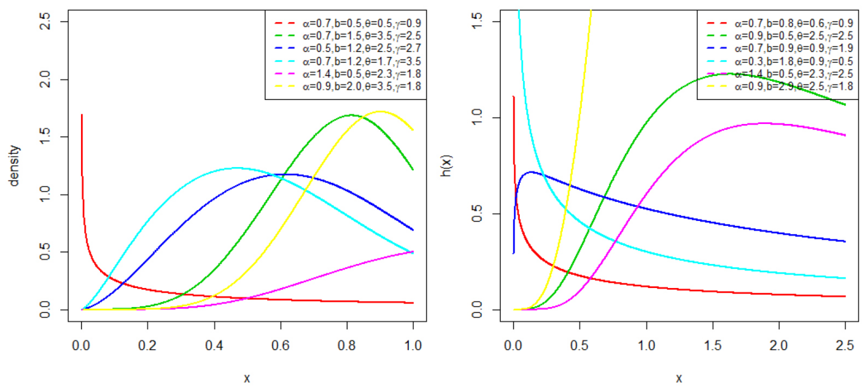

Figure 4

Plots of the pdf and hrf of the EHL-TIITL-BXII distribution..

$$F(x;\alpha,b,\theta,\gamma)=\left[\frac{1-(1-(1-(1+x^{\theta})^{-\gamma})^{2})^{b}}{1+(1-(1-(1+x^{\theta})^{-\gamma})^{2})^{b}}\right]^{\alpha}$$

and

$$f(x;\alpha,b,\theta,\gamma)=\frac{4b\alpha\theta\gamma x^{\theta-1}(1+x^{\theta})^{-\gamma-1}(1-(1+x^{\theta})^{-\gamma})(1-(1-(1+x^{\theta})^{-\gamma})^{2})^{b-1}}{(1+(1-(1-(1+x^{\theta})^{-\gamma})^{2})^{b})^{\alpha+1}}$$
$$\times(1-(1-(1-(1+x^{\theta})^{-\gamma})^{2})^{b})^{\alpha-1},$$

for $\alpha,b,\theta,\gamma>0.$ Note that, if we set $\gamma=1$ we obtain the exponentiated Half Logistic-Type II Topp Leone-Log-Logistic (EHL-TIITL-LLoG) distribution. Also, when $\theta=1,$ we obtain the exponentiated Half Logistic-Type II Topp Leone-Lomax (EHL-TIITL-L) distribution.

Presented in Figure 4 are the plots of the pdf and the hrf of the EHL-TIITL-BXII distribution. It can be seen that the pdf can take various shapes including almost symmetric, the reverse-J, left and right-skewed. Graphs of the hazard rate function shows decreasing, increasing, bathtub and upside-down bathtub shapes.

5.3 *Exponentiated Half-Logistic Type II-Topp-Leone-Uniform (EHL-TIITL-U) Distribution*

Suppose we take the uniform distribution with cdf and pdf given by $G(x;\gamma)=\frac{x}{\gamma}$ and $g(x;\gamma)=\frac{1}{\gamma}$ respectively, for $0<x<\gamma,$ we obtain the EHL-TIITL-U distribution with cdf and pdf given by

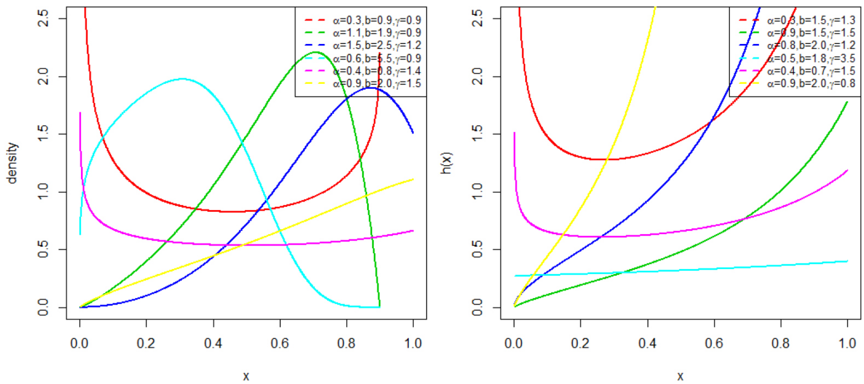

Figure 5

Plots of the pdf and hrf of the EHL-TIITL-U distribution.

$$F(x;\alpha,b,\gamma)=\left[\frac{1-\left(1-\left(\frac{x}{\gamma}\right)^2\right)^b}{1+\left(1-\left(\frac{x}{\gamma}\right)^2\right)^b}\right]^{\alpha}$$

and

$$f(x;\alpha,b,\gamma)=\frac{4b\alpha\dfrac{x}{\gamma^2}\left(1-\left(\frac{x}{\gamma}\right)^2\right)^{b-1}\left(1-\left(1-\left(\frac{x}{\gamma}\right)^2\right)^b\right)^{\alpha-1}}{\left(1+\left(1-\left(\frac{x}{\gamma}\right)^2\right)^b\right)^{\alpha+1}},$$

for $\alpha,b,\gamma>0.$

Figure 5 shows the plots of the pdf and the hrf of the EHL-TIITL-U distribution for different parameter values. The pdf can take various shapes including almost symmetric, the reverse-J, left and right-skewed, U-shaped. Graphs of the hazard rate function exhibits increasing and bathtub shapes.

6. Estimation and Inferences

In this section, we used the maximum likelihood estimation technique to obtain the parameters of the EHL-TIITL-G family of distributions. Suppose a random variable $X\sim EHL-TIITL-G(\alpha,b,\xi)$ and $\Delta=(\alpha,b,\xi)^T$

the parameter vector for the model. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n is

$$\begin{aligned}\ell_n(\Delta) &= n\ln(4b) + n\ln\alpha + \sum_{i=1}^{n}\ln(g(x_i;\xi)) + \sum_{i=1}^{n}\ln(G(x_i;\xi)) \\ &\quad + (b-1)\sum_{i=1}^{n}\ln(1-G^2(x_i;\xi)) + (\alpha-1)\sum_{i=1}^{n}\ln(1-(1-G^2(x_i;\xi))^b) \\ &\quad - (\alpha-1)\sum_{i=1}^{n}\ln(1+(1-G^2(x_i;\xi))^b). \end{aligned} \tag{14}$$

By differentiating the log-likelihood function with respect to each component of the parameter vector $\Delta = (\alpha, b, \xi)^T$ we obtain the elements of the score vector $U(\Delta)$ which are given by

$$U_\alpha(\Delta) = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-(1-G^2(x_i;\xi))^b) - \sum_{i=1}^{n}\ln(1+(1-G^2(x_i;\xi))^b), \tag{15}$$

$$\begin{aligned}U_b(\Delta) &= \frac{n}{b} - (\alpha-1)\sum_{i=1}^{n}\frac{(1-G^2(x_i;\xi))^b\ln(1-G^2(x_i;\xi))}{(1-(1-G^2(x_i;\xi))^b)} \\ &\quad + \sum_{i=1}^{n}\ln(1-G^2(x_i;\xi)) - (\alpha+1)\sum_{i=1}^{n}\frac{(1-G^2(x_i;\xi))^b\ln(1-G^2(x_i;\xi))}{(1+(1-G^2(x_i;\xi))^b)} \end{aligned} \tag{16}$$

and

$$\begin{aligned}U_{\xi_k}(\Delta) &= \sum_{i=1}^{n}\frac{\partial g(x_i;\xi)/\partial\xi_k}{g(x_i;\xi)} + (b-1)\sum_{i=1}^{n}\frac{\partial(1-G^2(x_i;\xi))/\partial\xi_k}{(1-G^2(x_i;\xi))} \\ &\quad + \sum_{i=1}^{n}\frac{\partial G(x_i;\xi)/\partial\xi_k}{G(x_i;\xi)} + (\alpha-1)\sum_{i=1}^{n}\frac{\partial(1-(1-G^2(x_i;\xi))^b)/\partial\xi_k}{(1-(1-G^2(x_i;\xi))^b)} \\ &\quad - (\alpha+1)\sum_{i=1}^{n}\frac{\partial(1+(1-G^2(x_i;\xi))^b)/\partial\xi_k}{(1+(1-G^2(x_i;\xi))^b)}. \end{aligned} \tag{17}$$

By solving the non-linear equation $\left(\frac{\partial\ell_n}{\partial\alpha}, \frac{\partial\ell_n}{\partial b}, \frac{\partial\ell_n}{\partial\xi_k}\right)^T = \mathbf{0}$, we obtain the estimates of the parameters $\hat{\Delta}$. Since we can not obtain explicit solutions of the likelihood equations, we make use of some numerical method such as Newton-Raphson procedure to maximize or optimize the likelihood function. For purposes of inferences to construct confidence intervals and confidence regions of the individual parameters, we assumed that under standard regulatory conditions the the MLEs have approximately a multivariate normal distribution $N_{p+2}(\underline{0}, J(\Delta)^{-1})$, where $\underline{0} = (0,0,\underline{0})^T$ is the mean vector and $J(\Delta)^{-1}$ is the observed Fisher information matrix evaluated at Δ.

7. Simulations

A simulation study was conducted and results are presented in this section to examine the performance of the EHL-TIITL-W distribution. We simulated $N = 1000$ samples for different sizes (n=50, 100, 200, 400, 800, 1000), this was done using the R package. Table 1 and Table 2 show the simulation results and it can be seen that the mean estimates of the parameters are approaching the true parameter values as the sample size is increasing. The Average Bias (ABias) and the Root Mean Squared Errors (RMSEs) are also decreasing towards zero as we increase the sample size. This shows that the MLE of the model parameters are strongly consistent and the results from the estimation can be trusted in making inferences about model parameters. The ABias and RMSE for the estimated parameter, say $\hat{\kappa}$, are given by:

$$ABias(\hat{\kappa}) = \frac{\sum_{i=1}^{N}\hat{\kappa}_i}{N} - \kappa, \quad \text{and} \quad RMSE(\hat{\kappa}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\kappa}_i - \kappa)^2}{N}},$$

respectively.

Table 1

Simulation Results

		(0.7, 0.3, 0.3, 0.9)			(0.5, 0.3, 0.3, 0.2)		
parameter	**Sample Size**	**Mean**	**RMSE**	**ABias**	**Mean**	**RMSE**	**ABias**
	50	2.0716	3.8409	1.3716	1.1240	1.1035	0.6240
	100	1.3069	1.0641	0.6069	0.9022	0.7422	0.4022
	200	0.9583	0.4057	0.2583	0.6936	0.3234	0.1936
α	400	0.8467	0.2423	0.1467	0.6046	0.1903	0.1046
	800	0.7768	0.1583	0.0768	0.5735	0.1111	0.0735
	1000	0.7520	0.1050	0.0520	0.5042	0.0741	0.0042
	50	4.0168	6.7624	3.7168	5.8827	10.3558	5.5827
	100	2.1943	4.0595	1.8943	3.3136	6.8296	3.0136
	200	1.0892	1.8621	0.7892	1.6606	3.2189	1.3606
b	400	0.7795	1.2219	0.4795	0.6840	1.0101	0.3840
	800	0.4029	0.3358	0.1029	0.4368	0.3029	0.1368
	1000	0.3204	0.1183	0.0204	0.3366	0.1470	0.0366
	50	0.1927	0.1237	-0.1073	0.2142	0.1309	-0.0858
	100	0.2156	0.1013	-0.0844	0.2395	0.1110	-0.0605
	200	0.2484	0.0692	-0.0516	0.2538	0.0823	-0.0462
γ	400	0.2642	0.0544	-0.0358	0.2721	0.0542	-0.0279

Contd...

	800	0.2800	0.0335	-0.0200	0.2755	0.0391	-0.0245
	1000	0.2892	0.0246	-0.0108	0.3051	0.0382	0.0051
γ	50	1.3875	1.5774	0.4875	0.5084	0.8340	0.3084
	100	1.2561	1.1734	0.3561	0.4502	0.6683	0.2502
	200	1.2186	0.9386	0.3186	0.3248	0.3835	0.1248
	400	1.0177	0.5698	0.1177	0.2719	0.2475	0.0719
	800	1.0332	0.4644	0.1332	0.2473	0.1311	0.0473
	1000	0.9987	0.2908	0.0987	0.1863	0.0662	-0.0137

Table 2

Simulation Results

		(0.7, 0.5, 0.5, 0.2)			(1.0, 1.5, 0.6, 1.8)		
parameter	**Sample Size**	**Mean**	**RMSE**	**ABias**	**Mean**	**RMSE**	**ABias**
α	50	1.5667	2.0830	0.8667	2.0523	4.7999	1.0523
	100	1.1225	0.9514	0.4225	1.3621	0.9659	0.3621
	200	0.8952	0.4028	0.1952	1.1338	0.4464	0.1338
	400	0.8245	0.2416	0.1245	1.0917	0.3004	0.0917
	800	0.7722	0.1677	0.0722	1.0545	0.2114	0.0545
	1000	0.7584	0.1399	0.0584	1.0492	0.1972	0.0492
b	50	6.1019	12.2763	5.6019	8.6312	17.8288	7.1312
	100	4.4174	8.2893	3.9174	7.5343	13.2828	6.0343
	200	2.9775	5.5220	2.4775	4.9808	7.9097	3.4808
	400	2.5918	4.2024	2.0918	4.1267	5.7617	2.6267
	800	1.8521	3.0269	1.3521	3.6179	4.7746	2.1179
	1000	1.4822	2.2386	0.9822	3.3955	4.1966	1.8955
γ	50	0.4019	0.2112	-0.0981	0.6053	0.2945	0.0053
	100	0.4314	0.1901	-0.0686	0.6083	0.2406	0.0083
	200	0.4452	0.1455	-0.0548	0.6266	0.1940	0.0266
	400	0.4400	0.1148	-0.0600	0.6095	0.1524	0.0095
	800	0.4581	0.0956	-0.0419	0.6024	0.1253	0.0024
	1000	0.4618	0.0851	-0.0382	0.6012	0.1236	0.0012
θ	50	0.6323	1.0010	0.4323	4.5573	5.1293	2.7573
	100	0.3934	0.6376	0.1934	3.6014	4.0646	1.8014
	200	0.2745	0.3283	0.0745	3.3119	3.4814	1.5119
	400	0.2150	0.1768	0.0150	3.0342	3.0279	1.2342
	800	0.1897	0.0988	-0.0103	2.6131	2.3250	0.8131
	1000	0.1916	0.0899	-0.0084	2.5160	2.2084	0.7160

8. Applicability of the EHL-TIITL-G Distribution

In this section, we illustrate the applicability of the EHL-TIITL-G family of distributions by fitting the EHL-TIITL-W distribution to two real data sets and these fits are compared to the fits of the exponentiated Half Logistic (EHL) distribution, the Half Logistic distribution (Johnson et. al. [11]), the Type II Topp-Leone Power Lomax distribution by Al-Marzouki et. al [5], the Type II Topp-Leone Dagum (TIITLD) distribution (Sakthivel and Dhivakar [15]), the Odd exponentiated Half-Logistic Burr XII (OEHLBXII) by Aldahlam et. al [2]. To assess the performance of these models, the goodness-of-fit statistics -2loglikelihood $(-2\log L)$, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-von Mises (W^*), Andersen-Darling (A^*), the Kolmogorov-Smirnov (K-S) statistic and its p-value, as described by Chen and Balakrishnan [8] and the sum of squared errors (SS) are presented in Table 3 and Table 4. These goodness-of-fit statistics are used to verify which distribution fits the data better. The general rule is the smaller the values of $(-2logL)$, AIC, AICC, BIC, W^*, A^* and SS the better the fit. The pdfs of these distributions are:

$$f_{EHL}(x)=\frac{2a\lambda e^{-\lambda x}}{1-e^{-2\lambda x}}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^{a-1},$$

for $a,\lambda,x>0$,

$$f_{HL}(x)=\frac{2\lambda e^{-\lambda x}}{(1+e^{-2\lambda x})^2},$$

for $\lambda,x>0$,

$$f_W(x)=\frac{a}{\lambda}\left(\frac{x}{\lambda}\right)^{a-1}e^{\left(\frac{x}{\lambda}\right)^a},$$

for $a,\lambda,x>0$,

$$f_{TIITLPL}(x)=\frac{2\theta\alpha\beta}{\lambda}x^{\beta-1}\left(1+\frac{x^\beta}{\lambda}\right)^{-\alpha-1}\left[1-\left(1+\frac{x^\beta}{\lambda}\right)^{-\alpha}\right]$$
$$\times\left\{1-\left[1-\left(1+\frac{x^\beta}{\lambda}\right)^{-\alpha}\right]^2\right\}^{\theta-1},$$

for $\theta, \alpha, \beta, \lambda, x > 0$,

$$f_{TIITLD}(x) = 2a\tau\delta\theta\beta x^{-\theta-1}(1+\delta x^{-\theta})^{-2\beta-1}[1-(1+\delta x^{-\theta})^{-2\beta}]^{\tau-1},$$

for $\tau, \delta, \theta, \beta, x > 0$, and

$$f_{OEHLBXII}(x;\alpha,\lambda,a,b) = \frac{2\alpha\lambda abx^{a-1}\exp\{\lambda[1-(1+x^a)^b]\}}{(1+x^a)^{-b+1}(1+\exp\{\lambda[1-(1+x^a)^b]\})^{\alpha+1}} \times (1-\exp\{\lambda[1-(1+x^a)^b]\})^{\alpha-1},$$

for $\alpha, \lambda, a, b, x > 0$.

We presented plots of the histograms, probability plots (Chambers et. al. [7]), fitted densities and box plots of the data in Figure 6 and Figure 8. Figure 7 and Figure 9 shows the estimated cumulative density function, Kaplan-Meier survival plots and the estimated hazard rate plot for the EHL-TIITL-W distribution. For the probability plot, we plotted $F(x_{(j)})$ against $\frac{j-0.375}{n+0.25}, j = 1, 2, \cdots, n,$ where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness was obtained and is given by the sum of squares

$$SS = \sum_{j=1}^{n}\left[F(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

8.1 *Strength of Glass Fibre Data Set*

The EHL-TIITL-W was fitted to the strength of glass fibre data set, the data consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. Smith and Naylor [17] also analysed this data set. The data are as follows: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74 ,1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28 ,1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 3

Estimation of the EHL-TIITL-W model for glass fibre data set 4.0cm

	Estimates				Statistics								
Distribution	α	b	γ	θ	-2logL	AIC	CAIC	BIC	W^*	A^*	K–S	P–value	SS
EHL-TIITL-W	0.7920	235.8400	3.0064	0.0173	27.3583	35.3583	36.0480	43.9309	0.1688	0.9393	0.1396	0.1717	0.1633
	(0.2552)	(0.0004)	(0.6392)	(0.0078)									
EHL-TIITL-W $(1, 1, \gamma, \theta)$	-	-	3.5577	0.3808	31.7561	35.7561	35.9561	40.0423	0.2742	1.5030	0.1592	0.0820	0.2392
			(0.3619)	(0.0778)									
EHL-TIITL-W$(\alpha, 1, \gamma, 1)$	1.9855	-	2.2211	-	39.9583	43.9583	44.1583	48.2445	0.4300	2.3622	0.2097	0.0079	0.4962
	(0.2657)		(0.1184)										
EHL	-	-	17.1769	2.6559	61.2453	65.2453	65.4453	69.5316	0.7577	4.1418	0.2237	0.0036	0.7530
			(4.9707)	(0.2297)									
HL	1.0127	-	-	-	155.0062	157.0062	157.0718	159.1493	0.5209	2.8596	0.4137	8.59×10^{-10}	3.6496
	(0.0989)												
W	5.7807	1.6281	-	-	30.4137	34.4137	34.6137	38.7000	0.2372	1.3037	0.1522	0.1078	0.2111
	(0.5761)	(0.0371)											
TIITLPL	2200.3	42.3960	2.9003	25756	30.4442	38.4442	39.1339	47.0168	0.2381	1.3081	0.1524	0.1073	0.2115
	(0.0070)	(7.2957)	(0.2893)	(0.0120)									
TIITLD	16255	10815	4.2938	0.6737	30.4194	38.4194	39.1090	46.9919	0.2374	1.3047	0.1523	0.1076	0.2113
	(6.7509×10^{-06})	(1.3584×10^{-05})	(0.3219)	(0.0185)									
HLBXII-LL	0.3225	0.0030	11.8172	0.8356	50.3244	58.3244	59.0141	66.8969	0.2417	1.3747	0.1423	0.1558	0.3093
	(0.0670)	(0.0036)	(0.0075)	(0.1347)									

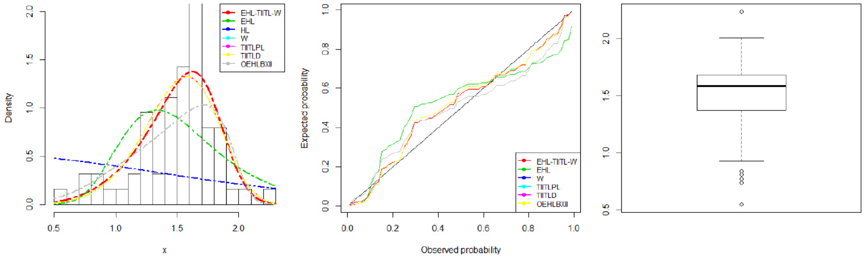

Figure 6

Fitted densities, probability plots and box plot for strength of glass fibre data set.

To assess the efficacy of the EHL-TIITL-W distribution in fitting the glass fibre data set, we used the goodness-of-fit statistics presented in Table 3, plots of the fitted densities as well as the probability plots presented in Figure 6. Based on those statistics and plots, we conclude that the EHL-TIITL-W distribution provides the best fit to the strength of glass fibre data compared to the other models presented.

The estimated variance-covariance matrix for the strength of glass fibre data is given by,

$$\begin{bmatrix} 0.0651 & -8.8430\times10^{-05} & -0.1431 & 0.0018 \\ -8.8430\times10^{-05} & 1.4924\times10^{-07} & 0.0002 & -2.9727\times10^{-06} \\ -0.1431 & 0.0002 & 0.4086 & -0.0049 \\ 0.0018 & -2.9727\times10^{-06} & -0.0049 & 6.0502\times10^{-05} \end{bmatrix}$$

and the 95% asymptotic confidence intervals for the parameters α, b, γ and θ are: 0.7920 ± 0.5002, 235.8400 ± 0.0008, 3.0064 ± 1.2528 and 0.0173 ± 0.0153, respectively.

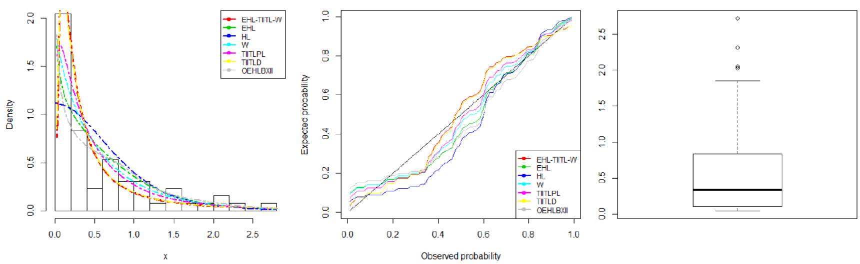

Figure 7

Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate plot of the EHL-TIITL-W distribution for strength of glass fibre data set.

Table 4

Estimation of the EHL-TIITL-W model for plasma concentration data set 4.0cm

	Estimates				Statistics								
Distribution	α	b	γ	θ	-2logL	AIC	CAIC	BIC	W^*	A^*	K–S	P–value	SS
EHL-TIITL-W	104.3400	0.4636	0.1703	15.9850	55.7905	63.7905	64.4462	72.5491	0.2237	1.3963	0.1317	0.2024	0.2347
	(7.7125 $\times 10^{-06}$)	(0.0163)	(0.1504)	(0.0004)									
EHL-TIITL-W $(1, 1, \gamma, \theta)$	-	-	0.5971	2.9935	62.2080	66.2080	66.3984	70.5873	0.2504	1.5989	0.140	0.1505	0.2341
			(0.0572)	(0.2480)									
EHL-TIITL-W$(\alpha, 1, \gamma, 1)$	0.2161	-	1.5769	-	68.4827	72.4827	72.6732	76.8620	0.2867	1.8619	0.1426	0.1364	0.4043
	(0.0333)		(0.1879)										
EHL	-	-	0.7653	1.9197	67.2545	71.2545	71.4450	75.6340	0.2927	1.8667	0.1459	0.1203	0.3292
			(0.1156)	(0.2789)									
HL	2.2304	-	-	-	70.5479	72.5479	72.6104	74.7375	0.2942	1.8723	0.2045	0.0080	0.7038
	(0.2377)												
W	0.9546	0.5787	-	-	62.5121	66.5121	66.7026	70.8914	0.2509	1.6080	0.1348	0.1817	0.2263
	(0.0904)	(0.0791)											
TIITLPL	0.5368	1.6471 $\times 10^{05}$	0.7243	4.1860 $\times 10^{04}$	59.1377	67.1376	67.7934	75.8962	0.2279	1.4584	0.1414	0.1430	0.2198
	(0.0664)	(2.5932 $\times 10^{-07}$)	(0.0625)	(1.0203 $\times 10^{-06}$)									
TIITLD	3.4768	0.0054	0.5442	83.4750	56.3737	64.3737	65.0295	73.1324	0.2258	1.4147	0.1345	0.1833	0.2302
	(0.0032)	(0.0006)	(0.0487)	(2.1108 $\times 10^{-06}$)									
HLBXII-LL	0.0777	0.2749	8.0388	0.3081	73.4574	81.4574	82.1132	90.2160	0.3220	2.0260	0.1408	0.1459	0.3733
	(0.0276)	(0.3009)	(2.9359)	(0.1267)									

8.2 *Plasma Concentration*

The EHL-TIITL-W was fitted to the plasma concentration data set which was taken from R base package in the Indometh object. The data are as follows: 1.50, 0.94, 0.78, 0.48, 0.37, 0.19, 0.12, 0.11, 0.08, 0.07, 0.05, 2.03, 1.63, 0.71, 0.70, 0.64, 0.36, 0.32, 0.20, 0.25, 0.12, 0.08, 2.72, 1.49, 1.16, 0.80, 0.80, 0.39, 0.22, 0.12, 0.11, 0.08, 0.08, 1.85, 1.39, 1.02, 0.89, 0.59, 0.40, 0.16 ,0.11, 0.10, 0.07, 0.07, 2.05, 1.04, 0.81, 0.39, 0.30, 0.23, 0.13, 0.11, 0.08, 0.10, 0.06, 2.31, 1.44, 1.03, 0.84, 0.64, 0.42, 0.24, 0.17, 0.13, 0.10, 0.09.

Table 4 and Figure 8 displays the goodness-of-fit statistics, the fitted densities and probability plots, respectively. We note that the EHL-TIITL-W model provides the best fit based on the goodness-of-fit statistics and the plots compared to other competing models presented.

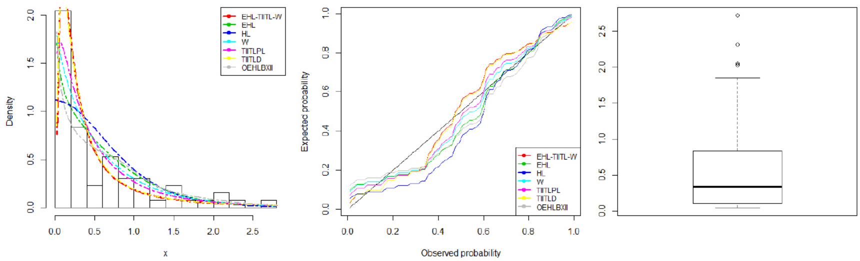

Figure 8

Fitted densities, probability plots and box plot for plasma concentration data set.

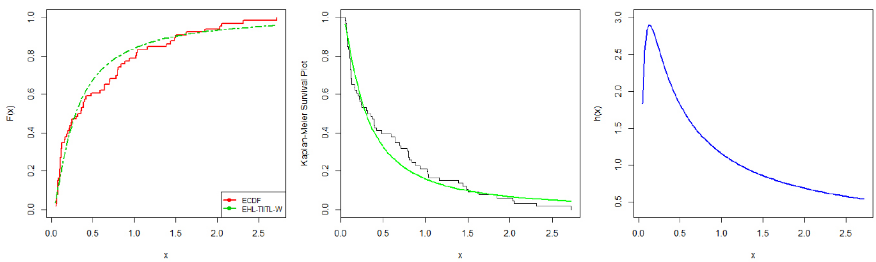

Figure 9

Estimated cdf, Kaplan-Meier survival plots and estimated hazard rate plot of the EHL-TIITL-W distribution for plasma concentration data set.

The estimated variance-covariance matrix for the plasma concentrations data is given by,

$$\begin{bmatrix} 5.9483\times10^{-11} & -1.1668\times10^{-07} & -4.7845\times10^{-08} & -3.5097\times10^{-09} \\ -1.1668\times10^{-07} & 0.0003 & 0.0002 & 7.9935\times10^{-06} \\ -4.7845\times10^{-08} & 0.0002 & 0.0002 & 5.2805\times10^{-06} \\ -3.5097\times10^{-09} & 7.9935\times10^{-06} & 5.2805\times10^{-06} & 2.3927\times10^{-07} \end{bmatrix}$$

and the 95% asymptotic confidence intervals for the parameters α, b, γ and θ are: $104.3400 \pm 1.5117 \times10^{-05}$, 0.4636 ± 0.0319, 0.1703 ± 0.2948 and 15.9850 ± 0.0008, respectively.

Table 5

Likelihood ratio test results

		Strength of glass fibre data set		Plasma concentration data set	
Model	**df**	**Chi-Square**	**p-value**	**Chi-Square**	**p-value**
EHL-TIITL-W $(1,1,\gamma,\theta)$	2	4.3592	0.03681	6.4175	0.0113
EHL-TIITL-W $(\alpha,1,\gamma,1)$	2	12.600	0.000386	12.6922	0.000367
EHL	2	33.887	< 0.00001	11.4640	0.00071
HL	1	127.6479	< 0.00001	14.7574	0.000122

Results from the Likelihood Ratio (LR) test are presented in Table 5. The results show that there is a significant difference between the EHL-TIITL-W distribution and its nested models: (EHL-TIITL-W $(1,1,\gamma,\theta)$, EHL-TIITL-W $(\alpha,1,\gamma,1)$, EHL, HL) for both data sets. Also, based on the lowest values of other goodness-of-fit statistics, we can conclude that the EHL-TIITL-W distribution provides the best fit to the strength of glass fibre data and the plasma concentration data compared to the its nested models.

9. Conclusions and Future Work

We developed a new family of distributions called the exponentiated half-logistic type II Topp-Leone-G (EHL-TIITL-G) distribution. We derived some mathematical and statistical properties of the new model and assessed the performance of the special case of the EHL-TIITL-W distribution by conducting various simulations for different sample sizes. The special case of the EHL-TIITL-W distribution was fitted into two real data sets to demonstrate the potentiality of the proposed family of distributions. Future work looks to use different estimation techniques, for example, Bayesian techniques and the maximum product of spacing technique to estimate the model parameters, application of the developed model to regression analysis, as well as parameter estimation under various censoring schemes.

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