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Calibration-based approach in estimation of population mean under stratified two-phase sampling scheme

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Abstract

The calibration method of estimation utilizes ancillary data to enhance the accuracy of the estimates. In this paper, some new calibration estimators have been introduced to estimate the population mean under stratified two-phase (double) sampling scheme. The chi-square distance measure subject to some of the calibration conditions based on one or two auxiliary variables has been applied to get modified stratum weights. The technique of Taylor linearization has been utilized to obtain the equation for the MSE of the proposed calibration estimators. A set of actual data has been considered to check the execution of the suggested calibration estimators. An empirical analysis based on artificially generated data has also been carried out to illustrate the conceptual outputs.

Subject Classification: *62D05.*

Keywords: *Calibration approach, Stratified double sampling, Population mean, Ancillary information, Simulation analysis.*

1. Introduction

The ancillary data plays a vital role in the sample survey. It is to be mentioned that the calibration approach incorporates the auxiliary information in increasing the accuracy of any estimates. The calibration technique is a weighting process that agrees with the known population values. The weighting process under the calibration approach is a

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technique used for adjusting probability sampling weights to make the estimates more accurate. A chi-square distance measure and calibration constraints are generally utilized to assemble appropriate calibration weights. Deville and Särndal (1992) first envisaged the calibration estimator in survey sampling. Singh et al. (1998) suggested a method for more advanced calibration techniques. Tracy et al. (2003) have suggested two limitations to obtaining calibration weights to estimate the mean of a stratified population employing the auxiliary information. Kim et al. (2007) have discussed calibration approach ratio estimators of the population mean in stratified sampling. Kadilar and Koyuncu (2013) have used different distance measures for approximating the population mean under the calibration approach. Chaudhary et al. (2020) have suggested a family of combined-type estimators for estimating the population mean using the stratified double sampling design in the presence of non-response. There are some other researchers such as Farrell and Singh (2005), Rao et al. (2017), Nidhi et al. (2017), Ozgul (2019), and Chaudhary and Dutta (2021) who have contributed a lot to developing the calibration estimators under various sampling techniques.

In the subsequent sections, we have developed some updated calibration estimators of the population mean considering the data on one/two auxiliary variables under a stratified double sampling design. The new stratum weights have been deduced using a chi-square distance measure subject to some calibration constraints based on ancillary information. The essential features of the suggested calibration estimators have been thoroughly examined. We have also conducted a quantitative study along with the computational evaluation to assess the efficacy of the suggested calibration estimators.

2. Procedures, Notations, and Definitions

Consider a finite population constituting N units and stratified into g strata. The a^{th} stratum consists of N_a units $(a = 1,2,...,g)$ s. t. $\sum_{a=1}^{g} N_a = N$. Let y_{ai} be the observation on the i^{th} units in the a^{th} stratums for the study variable $Y\,(i = 1,2,...,N_a)$. Suppose x_{ai} and z_{ai} are the observations on the i^{th} unit in a^{th} stratum of the auxiliary variable X and Z respectively. Thus, $\overline{Y} = \sum_{a=1}^{g} \vartheta_a \overline{Y}_a$, $\overline{X} = \sum_{a=1}^{g} \vartheta_a \overline{X}_a$ and $\overline{Z} = \sum_{a=1}^{g} \vartheta_a \overline{Z}_a$ are the population mean of the variables Y; X and Z. Here, $\overline{Y}_a = \frac{1}{N_a}\sum_{i=1}^{N_a} y_{ai}$, $\overline{X}_a = \frac{1}{N_a}\sum_{i=1}^{N_a} x_{ai}$, $\overline{Z}_a = \frac{1}{N_a}\sum_{i=1}^{N_a} z_{ai}$ and $\vartheta_a = \frac{N_a}{N}$. Now, considering simple random sample

without replacement (SRSWOR), a random sample of n_a units is selected from the a^{th} stratum such that $\sum_{a=1}^{g} n_a = n$. The normal estimator of the population mean $\overline{Y}$ (without using ancillary information) is given by

$$\overline{y}_{usual} = \sum_{a=1}^{g} \vartheta_a \overline{y_a} \tag{1}$$

Where $\overline{y_a} = \frac{1}{n_a} \sum_{i}^{n_a} y_{ai}$.

The expression for the variance of the estimator $\overline{y}_{usual}$ is stated as

$$V(\overline{y}_{usual}) = \sum_{a=1}^{g} \left(\frac{1}{n_a} - \frac{1}{N_a} \right) \vartheta_a^2 S_{aY}^2 \tag{2}$$

Where $S^2_{aY} = \frac{1}{N_a - 1} \sum_{i=1}^{N_a} (y_{ai} - \overline{Y_a})^2$.

3. Proposed Calibration Estimators

In this section, we have suggested some unique calibration estimators of the population mean $\overline{Y}$ with stratified double sampling employing the information on the auxiliary variable(s). First, we have suggested the calibration estimator applying the information on a single auxiliary variable and then the information on two auxiliary variables has been utilized to construct the calibration estimators.

Here, a 1st phase sample of m_a units is chosen from the a^{th} stratum of N_a units using SRSWOR such that $\sum_{a=1}^{g} m_a = m$. Now, the information on auxiliary variables X and Z is collected from 1st phase sample. Further, a 2nd phase sample of n_a units are drawn from the 1st phase sample of m_a units using SRSWOR s. t. $\sum_{a=1}^{g} n_a = n$ and the information about variable Y is gathered.

3.1 *Use of Information on Single Auxiliary Variable*

Now, we propose calibration estimator using data on a single auxiliary variable X, which is given as follows

$$\overline{y_{st}} = \sum_{a=1}^{g} w_a \overline{y_a} \tag{3}$$

subject to the calibration constraints

$$\sum_{a=1}^{g} w_a s^2_{ax} = \sum_{a=1}^{g} \vartheta_a s^{*2}_{ax} \tag{4}$$

$$\sum_{a=1}^{g} w_a = \sum_{a=1}^{g} \vartheta_a \tag{5}$$

where $s^{*2}_{ax} = \frac{1}{m_a}\sum_i^{m_a}(x_{ai} - \overline{x}_a^*)^2$, $s^2_{ax} = \frac{1}{n_a}\sum_i^{n_a}(x_{ai} - \overline{x_a})^2$, $\overline{x}_a^* = \frac{1}{m_a}\sum_i^{m_a} x_{ai}$ and $\overline{x_a} = \frac{1}{n_a}\sum_i^{n_a} x_{ai}$. w_a is the calibration weight for the a^{th} stratum.

The calibration weight w_a is so chosen that it minimizes the chi-square type distance $\sum_{a=1}^{g}\frac{(w_a-\vartheta_a)^2}{q_a\vartheta_a}$ subject to the calibration constraints given in equations (4) and (5). Here, q_a is the tuning parameter which decides the form of the estimator.

Let us now define the Lagrange function as

$$\Delta_1 = \sum_{a=1}^{g}\frac{(w_a-\vartheta_a)^2}{q_a\vartheta_a} - 2\lambda_1(\sum_{a=1}^{g} w_a s^2_{ax} - \sum_{a=1}^{g}\vartheta_a s^{*2}_{ax}) - 2\lambda_2(\sum_{a=1}^{g} w_a - \sum_{a=1}^{g}\vartheta_a) \tag{6}$$

where λ_1 and λ_2 are the Lagrange multipliers.

For getting optimum value of w_a, we now differentiate equation (6) concerning w_a and equalize the derivative to zero. Thus, we have

$$w_a = \vartheta_a + q_a\vartheta_a(\lambda_1\sum_{a=1}^{g} s^2_{ax} + \lambda_2) \tag{7}$$

By using equations (4), (5) and (7), we get the values of λ_1 and λ_2 as

$$\lambda_1 = \frac{[\sum_{a=1}^{g}\vartheta_a(s^{*2}_{ax} - s^2_{ax})][\sum_{a=1}^{g} q_a\vartheta_a]}{(\sum_{a=1}^{g} q_a\vartheta_a)[\sum_{a=1}^{g} q_a\vartheta_a(s^2_{ax})^2] - (\sum_{a=1}^{g} q_a\vartheta_a s^2_{ax})^2} \tag{8}$$

$$\lambda_2 = \frac{-[\sum_{a=1}^{g} q_a\vartheta_a s^2_{ax}][\sum_{a=1}^{g}\vartheta_a(s^{*2}_{ax} - s^2_{ax})]}{(\sum_{a=1}^{g} q_a\vartheta_a)[\sum_{a=1}^{g}\vartheta_a(s^2_{ax})^2] - (\sum_{a=1}^{g} q_a\vartheta_a s^2_{ax})^2} \tag{9}$$

Substituting the values of λ_1 and λ_2 from equations (8) and (9) into the equation (7), one can get the optimum value of w_a .as

$$w_a = \vartheta_a + \frac{\left(\sum_{a=1}^{g}\vartheta_a s^{*2}_{ax} - \sum_{a=1}^{g}\vartheta_a s^2_{ax}\right)\sum_{a=1}^{g} q_a^2\vartheta_a^2 s^2_{ax}}{(\sum_{a=1}^{g} q_a\vartheta_a)[\sum_{a=1}^{g} q_a\vartheta_a(s^2_{ax})^2] - (\sum_{a=1}^{g} q_a\vartheta_a s^2_{ax})^2} + \frac{-[\sum_{a=1}^{g} q_a\vartheta_a s^2_{ax})][\sum_{a=1}^{g}\vartheta_a(s^{*2}_{ax} - s^2_{ax})]}{(\sum_{a=1}^{g} q_a\vartheta_a)[\sum_{a=1}^{g}\vartheta_a(s^2_{ax})^2] - (\sum_{a=1}^{g} q_a\vartheta_a s^2_{ax})^2} \tag{10}$$

Thus, the calibration estimator $\overline{y}_{st}$ becomes

$$\overline{y}_{st} = \sum_{a=1}^{g}\vartheta_a\overline{y_a} + \beta\sum_{a=1}^{g}\vartheta_a(s^{*2}_{ax} - s^2_{ax}) \tag{11}$$

where $\beta = \dfrac{\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} \overline{y}_a)(\sum_{a=1}^{g} q_a \vartheta_a) - (\sum_{a=1}^{g} q_a \vartheta_a \overline{y}_a)(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax})}{(\sum_{a=1}^{g} q_a \vartheta_a)[\sum_{a=1}^{g} q_a \vartheta_a (s^2_{ax})^2] - (\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax})^2}$.

To determine the equation for the mean square error (MSE) of the suggested calibration estimator $\overline{y}_{st}$, we use Taylor linearization technique. Let us assume $q_a = 1 \ \forall \ k$ and

$$PAR^e = (\sum_{a=1}^{g} \vartheta_a \overline{Y}_a, \sum_{a=1}^{g} \vartheta_a S^2_{aX}, \sum_{a=1}^{g} \vartheta_a (S^2_{aX})^2, \sum_{a=1}^{g} \vartheta_a \overline{Y}_a S^2_{aX})$$

where $S^2_{aX} = \dfrac{1}{N_a - 1} \sum_{i=1}^{N_a} (x_{ai} - \overline{X}_a)^2$.

Therefore, we have

$$\left(\frac{\partial \overline{y}_{st}}{\partial \sum_{a=1}^{g} \vartheta_a \overline{y}_a} \right)_{PAR^e} = 1, \left(\frac{\partial \overline{y}_{st}}{\partial (\sum_{a=1}^{g} \vartheta_a s^{*2}_{ax})} \right)_{PAR^e} = \frac{\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \overline{Y}_a \right) - \left(\sum_{a=1}^{g} \vartheta_a \overline{Y}_a \right) \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \right)}{\left[\sum_{a=1}^{g} \vartheta_a \left(S^2_{aX} \right)^2 \right] - \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \right)^2} = \gamma_1,$$

$$\left[\frac{\partial \overline{y}_{st}}{\partial \left(\sum_{a=1}^{g} \vartheta_a s^2_{ax} \right)} \right]_{PAR^e} = \frac{\left(-\sum_{a=1}^{g} \vartheta_a S^2_{aX} \overline{Y}_a \right) + \left(\sum_{a=1}^{g} \vartheta_a \overline{Y}_a \right) \left(\sum_{a=1}^{g} \vartheta_a S^2_{kX} \right)}{\left[\sum_{a=1}^{g} \vartheta_a \left(S^2_{aX} \right)^2 \right] - \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \right)^2} - \frac{2 \left(-\sum_{a=1}^{g} \vartheta_a S^2_{aX} \right)^2 \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \overline{Y}_a \right) - \left(\sum_{a=1}^{g} \vartheta_a \overline{Y}_a \right) \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \right)}{\left[\left[\sum_{a=1}^{g} \vartheta_a (S^2_{aX})^2 \right] - \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \right)^2 \right]^2} = \gamma_2$$

and $\left(\dfrac{\partial \overline{y}_{st}}{\partial (\sum_{a=1}^{g} \vartheta_a \overline{y}_a s^2_{ax})} \right)_{PAR^e} = \left(\dfrac{\partial \overline{y}_{st}}{\partial (\sum_{a=1}^{g} \Omega_a \overline{y}_a s^2_{ax})} \right)_{PAR^e} = 0.$

Hence, the equation of the MSE of the suggested calibration estimator $\overline{y}_{st}$ is given as

$$\begin{aligned} MSE(\overline{y}_{st}) &= V(\sum_{a=1}^{g} \vartheta_a \overline{y}_a) + \gamma^2_1 V(\sum_{a=1}^{g} \vartheta_a s^{*2}_{ax}) + \gamma^2_2 V(\sum_{a=1}^{g} \vartheta_a s^2_{ax}) \\ &+ 2\gamma_1 Cov(\sum_{a=1}^{g} \vartheta_a \overline{y}_a, \sum_{a=1}^{g} \vartheta_a s^{*2}_{ax}) + 2\gamma_2 Cov(\sum_{a=1}^{g} \vartheta_a \overline{y}_a, \sum_{a=1}^{g} \vartheta_a s^2_{ax}) \\ &+ 2\gamma_1 \gamma_2 Cov(\sum_{a=1}^{g} \vartheta_a s^{*2}_{ax}, \sum_{a=1}^{g} \vartheta_a s^2_{ax}) \end{aligned} \quad (12)$$

where $V(\sum_{a=1}^{g} \vartheta_a \overline{y}_a) = \sum_{a=1}^{g} \vartheta^2_a (\frac{1}{n_a} - \frac{1}{N_a}) S^2_{aY}$,

$$V\left(\sum_{a=1}^{g} \vartheta_a s^2_{ax} \right) = \sum_{a=1}^{g} \vartheta^2_a \left(\frac{1}{n_a} - \frac{1}{N_a} \right) S^4_{aX} (\lambda_{04_a} - 1),$$

$$V\left(\sum_{a=1}^{g} \vartheta_a s^{*2}_{ax} \right) = \sum_{a=1}^{g} \vartheta^2_a \left(\frac{1}{m_a} - \frac{1}{N_a} \right) S^4_{aX} (\lambda_{04_a} - 1),$$

$$Cov(\sum_{a=1}^{g}\vartheta_a\overline{y}_a,\sum_{a=1}^{g}\vartheta_a s^2{}_{ax})=\sum_{a=1}^{g}\vartheta^2{}_a(\frac{1}{n_a}-\frac{1}{N_a})\overline{Y}_a S^2{}_{aX}C_{aY}\lambda_{12_a},$$

$$Cov(\sum_{a=1}^{g}\vartheta_a\overline{y}_a,\sum_{a=1}^{g}\vartheta_a s^{*2}{}_{ax})=\sum_{a=1}^{g}\vartheta^2{}_a(\frac{1}{m_a}-\frac{1}{N_a})\overline{Y}_a S^2{}_{aX}C_{aY}\lambda_{12_a},$$

$$Cov\left(\sum_{a=1}^{g}\vartheta_a s^{*2}_{ax},\sum_{a=1}^{g}\vartheta_a s^2_{ax}\right)=\sum_{a=1}^{g}\vartheta_a^2\left(\frac{1}{m_a}-\frac{1}{N_a}\right)S^4{}_{aX}(\lambda_{04_a}-1),\ \lambda_{rs_a}=\frac{\mu_{rs_a}}{\mu_{20_a}^{r/2}\mu_{02_a}^{s/2}},$$

$$\mu_{rs_a}=\frac{1}{N_a-1}\sum_{i=1}^{N_a}(y_{ai}-\overline{Y}_a)^r(x_{ai}-\overline{X}_a)^s \text{ and } C_{aY}=\frac{S_{aY}}{\overline{Y}_a}.$$

3.2 *Use of Information on Two Auxiliary Variables*

Under stratified double sampling, the knowledge about more than one auxiliary variable can be used to create more effective calibration weights, allowing for a more precise estimation of the population mean $\overline{Y}$. Recently, Ozgul (2019) and Chaudhary and Dutta (2021) have utilized the data on two auxiliary variables to get modified calibration weights while evaluating the population mean in stratified double sampling. We now suggest an improved calibration estimator of the population mean $\overline{Y}$ under stratified double sampling by making the use of two auxiliary variables *X*and *Z* as

$$\overline{y}^{*}_{st}=\sum_{a=1}^{g}w^{*}_a\overline{y}_a \tag{13}$$

subject to the calibration constraints.

$$\sum_{a=1}^{g}w^{*}{}_a s^2{}_{ax}=\sum_{a=1}^{g}\vartheta_a s^{*2}{}_{ax} \tag{14}$$

$$\sum_{a=1}^{g}w^{*}{}_a s^2{}_{az}=\sum_{a=1}^{g}\vartheta_a s^{*2}{}_{az} \tag{15}$$

$$\sum_{a=1}^{g}w^{*}{}_a=\sum_{a=1}^{g}\vartheta_a \tag{16}$$

where $s^{*2}{}_{az}=\frac{1}{m_a}\sum_i^{m_a}(z_{ai}-\overline{z}^{*}_a)^2$, $s^2{}_{az}=\frac{1}{n_a}\sum_i^{n_a}(z_{ai}-\overline{z}_a)^2$, $\overline{z}^{*}_a=\frac{1}{m_a}\sum_i^{m_a}z_{ai}$ and $\overline{z}_a=\frac{1}{n_a}\sum_i^{n_a}z_{ai}$. w^{*}_a is the calibrated weight for the a^{th} stratum.

Now, we defined the Lagrange function based on chi-square distance measure and written calibration conditions as

$$\Delta_2 = \sum_{a=1}^{g} \frac{(w_a^* - \vartheta_a)^2}{q_a \vartheta_a} - 2\lambda^*_1 \left(\sum_{a=1}^{g} w_a^* s_{ax}{}^2 - \sum_{a=1}^{g} \vartheta_a s_{ax}{}^{*2}\right) -$$
$$2\lambda^*_2 \left(\sum_{a=1}^{g} w_a^* s_{az}{}^2 - \sum_{a=1}^{g} \vartheta_a s_{az}{}^{*2}\right) - 2\lambda^*_3 \left(\sum_{a=1}^{g} w_a^* - \sum_{a=1}^{g} \vartheta_a\right) \quad (17)$$

Differentiating equation (17) w.r.t w_a^* and then equalizing the derivatives to zero,

$$w_a^* = \vartheta_a + q_a \vartheta_a \left(\lambda^*_1 \sum_{a=1}^{g} s^2_{ax} + \lambda^*_2 \sum_{a=1}^{g} s^2_{az} + \lambda^*_3\right) \quad (18)$$

From equations (14), (15), (16) and (18), we get the following system of equations:

$$\begin{bmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \end{bmatrix} \begin{bmatrix} \sum_{a=1}^{g} q_a \vartheta_a s^4_{ax} & \sum_{a=1}^{g} q_a \vartheta_a s_{ax}{}^2 s^2_{az} & \sum_{k=1}^{g} q_a \vartheta_a s^2_{ax} \\ \sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s_{az}{}^2 & \sum_{a=1}^{g} q_a \vartheta_a s^4_{ax} & \sum_{k=1}^{g} q_a \vartheta_a s^2_{az} \\ \sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} & \sum_{a=1}^{g} q_a \vartheta_a s^2_{az} & \sum_{a=1}^{g} q_a \vartheta_a \end{bmatrix} = \begin{bmatrix} \sum_{a=1}^{g} \vartheta_a s^{*2}_{ax} - \sum_{a=1}^{g} \vartheta_a s^2_{ax} \\ \sum_{a=1}^{g} \vartheta_a s^{*2}_{az} - \sum_{a=1}^{g} \vartheta_a s^2_{az} \\ 0 \end{bmatrix}$$

(19)

By resolving the equation (19), we obtain the values of λ_1^*, λ_2^* and λ_3^* as

$$\lambda_1^* = \frac{\left[\left(\sum_{a=1}^{g} \vartheta_a s^{*2}_{ax} - \sum_{a=1}^{g} \vartheta_a s^2_{ax}\right)\right]\left[\left(\sum_{a=1}^{g} q_a \vartheta_a . \sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)^2\right] +}{D}$$
$$\frac{\left[\left(\sum_{a=1}^{g} \vartheta_a s^{*2}_{az} - \sum_{a=1}^{g} \vartheta_a s^2_{az}\right)\right]\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} . \sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az} s^2_{ax}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a\right)\right]}{D}$$

(20)

$$\lambda_2^* = \frac{\left[\left(\sum_{a=1}^{g} \vartheta_a s^{*2}_{ax} - \sum_{a=1}^{g} \vartheta_a s^2_{ax}\right)\right]\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} . \sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\right] +}{D}$$
$$\frac{\left[\left(\sum_{a=1}^{g} \vartheta_a s^{*2}_{az} - \sum_{a=1}^{g} \vartheta_a s^2_{az}\right)\right]\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{ax} . \sum_{a=1}^{g} q_a \vartheta_a\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax}\right)^2\right]}{D}$$

(21)

$$\lambda_3^* = \frac{\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)\right]\left[\sum_{a=1}^{g} \vartheta_a \left(s^{*2}_{ax} - s^2_{ax}\right)\right] + \left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax}\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{ax}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)\right]\left[\sum_{a=1}^{g} \vartheta_a \left(s^{*2}_{az} - s^2_{az}\right)\right]}{D} \tag{22}$$

where

$$D = \left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{ax}\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)^2\right] -$$

$$\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left(q_a \Omega_a\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s_{ax}{}^2\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)\right] +$$

$$\left(\sum_{a=1}^{g} q_a \vartheta_a s_{ax}{}^2\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax}\right)\right].$$

Substituting the values of λ_1^*, λ_2^* and λ_3^* from equations (20),(21) and (22) into the equation (18) and then putting the weight so obtained in equation (13), then calibrated estimator becomes

$$\overline{y}^*_{st} = \sum_{a=1}^{g} \vartheta_a \overline{y_a} + \beta_1 \sum_{a=1}^{g} \vartheta_a \left(s^{*2}_{ax} - s^2_{ax}\right) + \beta_2 \sum_{a=1}^{g} \vartheta_a \left(s^{*2}_{az} - s^2_{az}\right) \tag{23}$$

where $\beta_1 = \frac{R}{D}$, $\beta_2 = \frac{T}{D}$,

$$R = \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} \overline{y_a}\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)^2\right] +$$

$$\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az} \overline{y_a}\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a\right)\right] +$$

$$\left(\sum_{a=1}^{g} q_a \vartheta_a \overline{y_a}\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right) - \left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax}\right)\right]$$

and

$$R = \left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} \overline{y_a}\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right) - \left(\sum_{a=1}^{g} q_a \Omega_a s^2_{ax} s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \Omega_a\right)\right] +$$

$$\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az} \overline{y}_a\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{ax}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a\right)-\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax}\right)^2\right]+$$

$$\left(\sum_{a=1}^{g} q_a \vartheta_a \overline{y}_a\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{ax}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a\right)-\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax}\right)^2\right]+$$

$$\left(\sum_{a=1}^{g} q_a \vartheta_a \overline{y}_a\right)\left[\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax} s^2_{az}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{ax}\right)-\left(\sum_{a=1}^{g} q_a \vartheta_a s^4_{ax}\right)\left(\sum_{a=1}^{g} q_a \vartheta_a s^2_{az}\right)\right].$$

Again, by using Taylor linearization technique, the expression for MSE of suggested calibration estimator $\overline{y}^*_{st}$ may be obtained. Here, we assume $q_a = 1 \ \forall \ a$. Let us further assume

$$PAR^* = \begin{pmatrix} \sum_{a=1}^{g} \vartheta_a \overline{Y}_a, \sum_{a=1}^{g} \vartheta_a S^2_{aX}, \sum_{a=1}^{g} \vartheta_a S^2_{aZ}, \\ \sum_{a=1}^{g} \vartheta_a (S^2_{aX})^2, \sum_{a=1}^{g} \theta_a (S^2_{aZ})^2, \sum_{a=1}^{g} \vartheta_a \overline{Y}_a S^2_{aX}, \sum_{a=1}^{g} \vartheta_a \overline{Y}_a S^2_{aZ}, \sum_{a=1}^{g} \vartheta_a S^2_{aX} S^2_{aZ} \end{pmatrix}$$

where $S^2_{aZ} = \frac{1}{N_a - 1}\sum_{i=1}^{N_a}(z_{ai} - \overline{Z}_a)^2$.

Now, we have

$$\left(\frac{\partial \overline{y}^*_{st}}{\partial \sum_{a=1}^{g} \vartheta_a \overline{y}_a}\right)_{PAR^*} = 1,$$

$$\left(\frac{\partial \overline{y}^*_{st}}{\partial \left(\sum_{a=1}^{g} \vartheta_a s^{'2}_{ax}\right)}\right)_{PAR^*} = \begin{bmatrix} \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \overline{Y}_a\right)\left(\sum_{a=1}^{g} \vartheta_a S^4_{aZ}\right)-\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} \overline{Y}_a\right)\left(\sum_{a=1}^{g} \vartheta_k S^2_{aZ}\right)^2 + \\ \left(\sum_{a=1}^{g} \vartheta_a S^2_{aZ} \overline{Y}_a\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aZ}\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX}\right)-\left(\sum_{a=1}^{g} \vartheta_a S^2_{aZ} \overline{Y}_a\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} S^2_{az}\right)+ \\ \left(\sum_{a=1}^{g} \vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} S^2_{aZ}\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aZ}\right)-\left(\sum_{a=1}^{g} \vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g} \vartheta_a S^4_{aZ}\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX}\right) \end{bmatrix}$$

$$\begin{bmatrix} \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX}\right)\left(\sum_{a=1}^{g} \vartheta_a S^4_{aZ}\right)-\left(\sum_{a=1}^{g} \vartheta_a S^4_{aX}\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{az}\right)^2 - \\ \left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} S^2_{aZ}\right)^2 + 2\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX} S^2_{aZ}\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX}\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aZ}\right)- \\ \left(\sum_{a=1}^{g} \vartheta_a S^4_{aZ}\right)\left(\sum_{a=1}^{g} \vartheta_a S^2_{aX}\right)^2 \end{bmatrix}^{-1} = \gamma^*_1,$$

$$\left(\frac{\partial \overline{y}^{*}_{st}}{\partial\left(\sum_{a=1}^{g}\vartheta_a s^{*2}_{az}\right)}\right)_{PAR^{*}} = \begin{bmatrix} -\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)+\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aX}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)^2 - 2\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)+\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)- \\ \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)+ \\ 2\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right) \end{bmatrix} \begin{bmatrix} \left(\sum_{a=1}^{g}\vartheta_a S^4_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)-\left(\sum_{a=1}^{g}\vartheta_a S^4_{aX}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)^2-\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)^2+2\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{az}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)-\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)^2 \end{bmatrix}^{-1} +$$

$$\begin{bmatrix} 2\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)- \\ 2\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right) \end{bmatrix} \begin{bmatrix} \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)-\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aX}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)^2+\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{az}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)-\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)+ \\ \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)- \\ \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right) \end{bmatrix}$$

$$\begin{bmatrix} \left(\sum_{a=1}^{g}\vartheta_a S^4_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)-\left(\sum_{a=1}^{g}\vartheta_a S^4_{aX}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)^2-\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)^2+2\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)-\left(\sum_{a=1}^{g}\vartheta_a S^4_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)^2 \end{bmatrix}^{-2} \left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)=\gamma_2^{*}$$

$$\left(\frac{\partial \overline{y}^{*}_{st}}{\partial\left(\sum_{a=1}^{g}\vartheta_a s^{*2}_{az}\right)}\right)_{PAR^{*}} = \begin{bmatrix} \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right)-\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aX}\right)+ \\ \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aX}\right)-\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)^2+\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S^2_{aX} S^2_{aZ}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aX}\right)-\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S^4_{aX}\right)\left(\sum_{a=1}^{g}\vartheta_a S^2_{aZ}\right) \end{bmatrix},$$

$$\left[\begin{array}{l}\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^4\right)-\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)^2- \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)^2+2\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)-\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)^2\end{array}\right]^{-1}=\gamma_3^*,$$

$$\left(\frac{\partial \overline{y}_{st}^*}{\partial\left(\sum_{a=1}^{g}\vartheta_a s_{az}^2\right)}\right)_{PAR^*}=\left[\begin{array}{l}-2\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aX}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)-\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aX}^2\right)- \\ \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)+\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)^2-\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aX}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)+2\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)\end{array}\right]$$

$$\left[\begin{array}{l}\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^4\right)-\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)^2- \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)^2+2\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)-\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)^2\end{array}\right]^{-1}\left[\begin{array}{l}-2\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)+ \\ 2\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)\end{array}\right]$$

$$\left[\begin{array}{l}\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aX}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)- \\ \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aX}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)+\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aX}^2\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)-\left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)^2+ \\ \left(\sum_{a=1}^{a}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)- \\ \left(\sum_{a=1}^{g}\vartheta_a \overline{Y}_a\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)\end{array}\right]\left[\begin{array}{l}\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^4\right)-\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^4\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)^2-\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right)^2+2\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2 S_{aZ}^2\right) \\ \left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^2\right)-\left(\sum_{a=1}^{g}\vartheta_a S_{aZ}^4\right)\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)^2\end{array}\right]^{-2}$$

$$\left(\sum_{a=1}^{g}\vartheta_a S_{aX}^2\right)=\gamma_4^*,$$

$$\left[\frac{\partial \overline{y}^{*}_{st}}{\partial\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^4\right)}\right]_{PAR^*}=0,\left[\frac{\partial \overline{y}^{*}_{st}}{\partial\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a s_{ax}^2\right)}\right]_{PAR^*}=0,\left[\frac{\partial \overline{y}^{*}_{st}}{\partial\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a s_{az}^2\right)}\right]_{PAR^*}=0,$$

$$\left[\frac{\partial \overline{y}^{*}_{st}}{\partial\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^2 s_{az}^2\right)}\right]_{PAR^*}=0 \text{ and } \left[\frac{\partial \overline{y}^{*}_{st}}{\partial\left(\sum_{a=1}^{g}\vartheta_a s_{az}^4\right)}\right]_{PAR^*}=0.$$

Ultimately, the expression for the MSE of suggested calibration estimator $\overline{y}^{*}_{st}$ is stated as follows:

$$MSE\left(\overline{y}^{*}_{st}\right)=V\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a\right)+\gamma_1^{*}V\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^{*2}\right)+\gamma_2^{*}V\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^2\right)+\gamma_3^{*}V\left(\sum_{a=1}^{g}\vartheta_a s_{az}^{*2}\right)+$$
$$\gamma_4^{*}V\left(\sum_{a=1}^{g}\vartheta_a s_{az}^2\right)+2\gamma_1^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a,\sum_{a=1}^{g}\vartheta_a s_{ax}^{*2}\right)+2\gamma_2^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a,\sum_{a=1}^{g}\vartheta_a s_{ax}^2\right)+$$
$$2\gamma_3^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a,\sum_{a=1}^{g}\vartheta_a s_{az}^{*2}\right)+2\gamma_4^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a,\sum_{a=1}^{g}\vartheta_a s_{az}^2\right)+2\gamma_1^{*}\gamma_2^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^{*2},\sum_{a=1}^{g}\vartheta_a s_{ax}^2\right)+$$
$$2\gamma_1^{*}\gamma_3^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^{*2},\sum_{a=1}^{g}\vartheta_a s_{az}^{*2}\right)+2\gamma_1^{*}\gamma_4^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^{*2},\sum_{a=1}^{g}\vartheta_a s_{az}^2\right)+2\gamma_2^{*}\gamma_3^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^2,\sum_{a=1}^{g}\vartheta_a s_{az}^{*2}\right)+$$
$$2\gamma_2^{*}\gamma_4^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^2,\sum_{a=1}^{g}\vartheta_a s_{az}^2\right)+2\gamma_3^{*}\gamma_4^{*}Cov\left(\sum_{a=1}^{g}\vartheta_a s_{az}^{*2},\sum_{a=1}^{g}\vartheta_a s_{az}^2\right)$$

(24)

where $V(\sum_{a=1}^{g}\vartheta_a s^2{}_{az})=\sum_{a=1}^{g}\vartheta^2{}_a\left(\frac{1}{n_a}-\frac{1}{N_a}\right)S^4{}_{aZ}(\lambda_{004_a}-1),$

$$V(\sum_{a=1}^{g}\vartheta_a s^{*2}{}_{az})=\sum_{a=1}^{g}\vartheta^2{}_a\left(\frac{1}{m_a}-\frac{1}{N_a}\right)S^4{}_{aZ}(\lambda_{004_a}-1),$$

$$Cov\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a,\sum_{a=1}^{g}\vartheta_a s^2{}_{az}\right)=\sum_{a=1}^{g}\vartheta^2{}_a\left(\frac{1}{n_a}-\frac{1}{N_a}\right)\overline{Y}_a S^2{}_{aZ}C_{aY}\lambda_{102_a},$$

$$Cov\left(\sum_{a=1}^{g}\vartheta_a \overline{y}_a,\sum_{a=1}^{g}\vartheta_a s^{*2}{}_{az}\right)=\sum_{a=1}^{g}\vartheta^2{}_a\left(\frac{1}{m_a}-\frac{1}{N_a}\right)\overline{Y}_a S^2{}_{aZ}C_{aY}\lambda_{102_a},$$

$$Cov\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^2,\sum_{a=1}^{g}\vartheta_a s_{az}^2\right)=\sum_{a=1}^{g}\vartheta_a^2\left(\frac{1}{n_a}-\frac{1}{N_a}\right)S_{aX}^2 S^2{}_{aZ}(\lambda_{022_a}-1),$$

$$Cov\left(\sum_{a=1}^{g}\vartheta_a s_{ax}^{*2}, \sum_{a=1}^{g}\vartheta_a s_{az}^{2}\right)=\sum_{a=1}^{g}\vartheta_a^2\left(\frac{1}{m_a}-\frac{1}{N_a}\right)S_{aX}^2 S^2{}_{aZ}(\lambda_{022_a}-1),$$

$$Cov\left(\sum_{a=1}^{g}\vartheta_a s_{az}^{*2}, \sum_{a=1}^{g}\vartheta_a s_{az}^{2}\right)=\sum_{a=1}^{g}\vartheta_a^2\left(\frac{1}{m_a}-\frac{1}{N_a}\right)S^4{}_{aZ}(\lambda_{004_a}-1),$$

$$S_{aZ}^2=\frac{1}{N_a-1}\sum_{i=1}^{N_a}(z_{ai}-\overline{Z}_a)^2,$$

$$\lambda_{rst_a}=\frac{\mu_{rst_a}}{\mu_{200_a}^{r/2}\mu_{020_a}^{s/2}\mu_{002_a}^{t/2}} \text{ and } \mu_{rst_a}=\frac{1}{N_a-1}\sum_{i=1}^{N_a}(y_{ai}-\overline{Y}_a)^r(x_{ai}-\overline{X}_a)^s(z_{ai}-\overline{Z}_a)^t.$$

4. Empirical Study

In this section, an actual data set has been utilized under empirical investigation to examine the peculiarities of the suggested calibration estimators. Additionally, a simulation evaluation has also been done to support the theoretical findings.

4.1 *Actual Data Set*

The actual data set provided by Särndal et al. (1992) has been used by us for illustrating the theoretical findings. The population data C0124 represent the 124 countries that make up the five continents of Africa, North Central America, Asia (non-Soviet section), Europe (non-Soviet part), and South America. Here, =P83 stands for 1983 population (in millions, or MM), =P80 for 1980 population (also in MM), and = GNP for 1982 gross national product (in tens of millions of US dollars). Table 1 displays the population specifics:

Table 1

Population parameters for the study and auxiliary variables

Strat. No.	N_a	n_a^*	n_a	$\overline{Y}_a$	$\overline{X}_a$	$\overline{Z}_a$	C_{aY}	C_{aX}	C_{aZ}	ρ_{YX_a}	ρ_{YZ_a}	ρ_{XZ_a}
1	38	27	16	13.04	11.88	1029.16	1.26	1.26	1.86	0.999	0.743	0.743
2	14	10	6	27.35	26.19	2567.57	2.2	2.23	3.04	0.999	0.969	0.973
3	11	8	4	23.14	21.88	5028.82	1.51	1.49	1.51	0.999	0.976	0.977
4	33	23	14	79.65	75.24	7533.94	2.63	2.66	2.74	0.999	0.294	0.300
5	24	17	10	20.28	20.1	16134.92	0.93	0.93	1.21	0.999	0.901	0.905

Here, ρ_{YX_a}, ρ_{YZ_a} and ρ_{XZ_a} are respectively the correlation coefficients between Y and X; Y and Z; X and Z. $C_{aX} = \frac{S_{aX}}{\overline{X}_a}$ & $C_{aZ} = \frac{S_{aZ}}{\overline{Z}_a}$.

Table 2 portrays the variance/MSE of the estimators $\overline{y}_{usual}$, $\overline{y}_{st}$ and $\overline{y}^*_{st}$. The percentage relative efficiency (PRE) of the suggested calibration estimators $\overline{y}_{st}$ & $\overline{y}^*_{st}$ w.r.t the $\overline{y}_{usual}$ is also computed. Moreover, we have computed the range of gain using the formulae given below:

$$\%\,\text{Gain}(\overline{y}^*_{st} \text{ or } \overline{y}_{st}) = \frac{\text{Precision}(\overline{y}^*_{st} \text{ or } \overline{y}_{st}) - \text{Precision}(\overline{y}_{usual})}{\text{Precision}(\overline{y}_{usual})} \times 100;$$

$$\text{Precision} = \frac{1}{\text{Variance / MSE}}.$$

Table 2

Variance/MSE, PRE, Precision and Gain

Estimator	Variance/MSE	PRE	Precision	Gain
$\overline{y}_{usual}$	144.612	100	0.00691	___
$\overline{y}_{st}$	79.516	181.865	0.01257	81.91
$\overline{y}^*_{st}$	73.272	197.363	0.01364	97.39

4.2 *Simulation Analysis*

An empirical analysis has been done to assess the effectiveness of the suggested calibration estimators $\overline{y}_{st}$ and $\overline{y}^*_{st}$ over the usual estimator $\overline{y}_{usual}$. We have considered a population of 5000 units which consists of six strata

Table 3

Particulars of Population

Strat. No.	Distribution for Study Variable Y	Strat. Size (N_a)	ρ_{YZ_a}	ρ_{YX_a}	ρ_{XZ_a}
1	$N(\mu = 6, \sigma = 4)$	600	0.6471	0.8889	0.9244
2	$N(\mu = 8, \sigma = 5)$	500	0.5896	0.8355	0.9365
3	$N(\mu = 12, \sigma = 6)$	300	0.6710	0.8928	0.9330
4	$N(\mu = 16, \sigma = 7)$	700	0.5371	0.8036	0.9337
5	$N(\mu = 22, \sigma = 8)$	400	0.5464	0.8095	0.9340
6	$N(\mu = 28, \sigma = 9)$	200	0.6640	0.8995	0.9240

with respective sizes 600, 500, 300, 700, 400 and 200. The population data for the study variable Y have been developed from the $N(\mu,\sigma)$. The population data for the auxiliary variables X and Z have been produced on assumption of specific correlation coefficients [See Reddy et al. (2010)]. Table 3 provides information on the considered population.

Now, using the SRSWOR technique, we choose a 1st phase sample consisting of 30% units of all strata. Thereafter, again using the SRSWOR technique for each stratum, a 2nd phase sample consisting of 20% units is chosen from the 1st phase sample. We repeat process of selecting the 1st phase and 2nd phase samples 5000 times. Finally, we compute the approximate MSE (AMSE) of estimators $\overline{y}_{usual}$, $\overline{y}_{st}$ and $\overline{y}^{*}_{st}$. The formulae of computing the AMSE are given below:

$$AMSE(\overline{y}_{usual}) = \frac{1}{5000}\sum_{K=1}^{5000}(\overline{y}_{usual} - \overline{Y})^2;\ k = 1,2,...,5000.$$

$$AMSE(\overline{y}_{st}) = \frac{1}{5000}\sum_{K=1}^{5000}(\overline{y}_{st} - \overline{Y})^2;\ k = 1,2,...,5000.$$

$$AMSE(\overline{y}^{*}_{st}) = \frac{1}{5000}\sum_{K=1}^{5000}(\overline{y}^{*}_{st} - \overline{Y})^2;\ k = 1,2,...,5000.$$

Table 4 depicts the AMSE of the estimators $\overline{y}_{usual}$, $\overline{y}_{st}$ and $\overline{y}^{*}_{st}$. The PRE of the proposed estimators $\overline{y}_{st}$ & $\overline{y}^{*}_{st}$ with respect to $\overline{y}_{usual}$ is provided. Additionally, the precision in terms of gain is also displayed.

Table 4

AMSE, PRE, Precision and Gain

Estimator	AMSE	PRE	Precision	Gain
$\overline{y}_{usual}$	0.208062	100.00	4.806	____
$\overline{y}_{st}$	0.046757	444.96	21.41	346.04
$\overline{y}^{*}_{st}$	0.038510	540.27	25.96	440.95

5. Concluding Remarks

In the present study, we have defined new sets of constraints and introduced some improved calibration estimators of the population mean utilizing the data on one/two auxiliary variables in stratified double sampling design. First, we have discussed the calibration estimator on utilizing the data on single auxiliary variable and then the data on two auxiliary variables has been utilized to deduce improved calibration

estimator. The characteristics of the suggested calibration estimators have been analyzed in detail. The Taylor linearization technique has been used in deriving the expressions for the MSE of the suggested calibration estimators. We have conducted a quantitative evaluation using both actual data and artificially created data sets. Similar trend of conclusion has been seen under both real data analysis and simulation analysis. The Table 2 and Table 4 demonstrate that the suggested calibration estimators $\overline{y}_{st}$ and $\overline{y}^{*}_{st}$ perform well as compared to the usual estimator $\overline{y}_{usual}$. This gives out that the suggested calibration estimator $\overline{y}^{*}_{st}$ would be the best choice among the rest.

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