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Restricted estimators of repeated measurements model for unbalanced data

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Abstract

The model of one-way repeated measurements for imbalanced data, as the number of explanations are not the same at some levels, is the subject of this research. We used two methods to estimate the model parameters: the restricted maximum likelihood method and maximum likelihood approach, and we considered some of their characteristics. The Experiment's practical research goal is to determine how the experimental elements and their interactions affect date palm productivity. The Experiment included the factor of irrigation method with three treatments: drip irrigation, basin irrigation, and the regular tidal method, while the irrigation interval factor included three treatments: 3, 6 and 9 days.

Subject Classification: *46L53, 46N30.*

Keywords: *Maximum likelihood method, Restricted maximum likelihood method one-way repeated measurements model.*

1. Introduction

Various disciplines used analyzing repeated measurements widely in health,and life sciences, biomedical research, epidemiology, and others [3] [6]. Different variables, repeated measurements of a single variable, or repeated measurements(RM) of a group of variables could represent these reactions. Unbalanced and balanced data are the two types of data. The balance data repeated measurements design, mean that every level shares the same

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observation numbers while the imbalanced data mean that some of the observations are not the same in at least one levels [6]. Many researchers have investigated the repeated measurements model(RMM). Al-Mouel. (2004) [2], by equated estimators, and calculated multivariate (RMM). Al-Mouel, and Wang (2004)[1] examined one-way multivariate (RM) in the variance model with sphericity tests. They used RM examination of the variance model and focused on the asymptotic sphericity test Falhy expansion (2006)[5], employed the Bayesian approach for the estimation of the regionalized variable covariance functions that exist in the spatial covariance structures in the mixed linear models. He presented a Bayes invariant quadratic unbiased estimator. The combination study of (RM) and survival time has frequently utilized maximum likelihood estimates. However, the asymptotic features of the maximum likelihood estimators are not theoretically justified [7-8].

This work estimates the parameters of the model by the use of the maximum likelihood estimate and restricted maximum likelihood estimate and take the practical side about the effect of irrigation interval on moisture and salt distribution of data palm orchards soil under different irrigation systems.

2- The Model Of One-Way Repeated Measurements

We consider the one-way (RMM) for unbalanced data(where the number of explanations is not the same at each level).The model's mathematical expression is presented and its components' probability distributions are discussed. The parameter of the model is estimators in various methods, and some properties of the estimators.

2.1 *The Model Mathematical*

The one-way (RMM) to imbalanced data is,

$$v_{\ell\mathcal{L}\varkappa} = \vartheta + \xi_{\mathcal{L}} + \zeta_{\varkappa} + (\xi\zeta)_{\mathcal{L}\varkappa} + \varsigma_{\ell(\mathcal{L})} + e_{\ell\mathcal{L}\varkappa}\,, \qquad (2.1.1)$$

As that: $\ell = 1,2,\dots,n_{\mathcal{L}}$ is an index to the units in experiments in group $\mathcal{L}$,

$\varkappa = 1,2,\dots,p$ refers to the level index within-units factor (Time),

$\mathcal{L} = 1,2,\dots,q$ stands for a level index between-units factor (Group),

$v_{\ell\mathcal{L}\varkappa}$ is the measuring response at time $\varkappa$ for unit ℓ in group $\mathcal{L}$, ϑ seems to be the general mean , and $\boldsymbol{\zeta_{\varkappa}}$ refers to the added effect to time $\varkappa$ while $\boldsymbol{\xi_{\mathcal{L}}}$ to the treatment group $\mathcal{L}$. $(\xi\zeta)_{\mathcal{L}\varkappa}$ refers to the group $\mathcal{L}$ × time $\varkappa$ (Interaction) ,$e_{\ell\mathcal{L}\varkappa}$ seems to be the random error time $\varkappa$ to unit ℓ in group $\mathcal{L}$, $\varsigma_{\ell(\mathcal{L})}$ stands to the random effect because of the experimental unit ℓ in treatment group $\mathcal{L}$, below the resulting sentences for the added effect (added parameter)

$$\left.\begin{array}{c}\sum_{\mathcal{L}=1}^{q}\xi_{\mathcal{L}} = 0\,, \sum_{\varkappa=1}^{p}\zeta_{\varkappa} = 0\,, \sum_{\mathcal{L}=1}^{q}(\xi\zeta)_{\mathcal{L}\varkappa} = 0\,, \forall\,\varkappa = 1,2,\dots,p\,, \\ \sum_{\varkappa=1}^{p}(\xi\zeta)_{\mathcal{L}\varkappa} = 0\,, \forall\mathcal{L} = 1,2,\dots,q\,, with\ N = \sum_{\mathcal{L}} n_{\mathcal{L}}\ , and\ \sum_{\mathcal{L}=1}^{q} n_{\mathcal{L}}\xi_{\mathcal{L}} = 0\,.\end{array}\right\} \qquad (2.1.2)$$

As that the $\varsigma_{\ell(\mathcal{L})}{}'s$ and $e_{\ell\mathcal{L}\varkappa}{}'s$ are identical, and independent with

$$e_{\ell\mathcal{L}\varkappa} \sim i.i.d\ N(0,\sigma_e^2)\,, \varsigma_{\ell(\mathcal{L})} \sim i.i.d\ N\big(0,\sigma_\varsigma^2\big)\,, \qquad (2.1.3)$$

In the following manner we display the sum of square terms

$$\left.\begin{aligned} SS_{\xi} &= P\sum_{\mathcal{L}=1}^{q} n_{\mathcal{L}}(\bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{...})^2 , \\ SS_{\zeta} &= N\sum_{\varkappa=1}^{p}(\bar{\upsilon}_{..\varkappa} - \bar{\upsilon}_{...})^2 , \\ SS_{\varsigma} &= p\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}(\bar{\upsilon}_{\ell\mathcal{L}.} - \bar{\upsilon}_{.\mathcal{L}.})^2 , \\ SS_{\xi\zeta} &= \sum_{\mathcal{L}=1}^{q}\sum_{\varkappa=1}^{p} n_{\mathcal{L}}(\bar{\upsilon}_{.\mathcal{L}\varkappa} - \bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{..\varkappa} + \bar{\upsilon}_{...})^2 , \\ SS_{e} &= \sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa} - \bar{\upsilon}_{.\mathcal{L}\varkappa} - \bar{\upsilon}_{\ell\mathcal{L}.} + \bar{\upsilon}_{.\mathcal{L}.})^2 , \end{aligned}\right\} \tag{2.1.4}$$

See Table (1)and

$$\left.\begin{aligned} &\bar{\upsilon}_{...} = \frac{1}{\mathrm{Np}}\sum_{\mathcal{L}=1}^{\mathrm{q}}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{\mathrm{p}} \upsilon_{\ell\mathcal{L}\varkappa} \text{ is the overall mean ,} \\ &\bar{\upsilon}_{.\mathcal{L}.} = \frac{1}{n_{\mathcal{L}}\mathrm{p}}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{\mathrm{p}} \upsilon_{\ell\mathcal{L}\varkappa} , \forall\mathcal{L} = 1,2,\dots,q \text{ , is the mean for group } \mathcal{L} , \\ &\bar{\upsilon}_{.\mathcal{L}\varkappa} = \frac{1}{n_{\mathcal{L}}}\sum_{\ell=1}^{n_{\mathcal{L}}} \upsilon_{\ell\mathcal{L}\varkappa} , \forall\mathcal{L} = 1,2,\dots,q \text{ refers to the mean in the e group } \mathcal{L} \text{ at time } \varkappa , \\ &\bar{\upsilon}_{..\varkappa} = \frac{1}{\mathrm{N}}\sum_{\mathcal{L}=1}^{\mathrm{q}}\sum_{\ell=1}^{n_{\mathcal{L}}} \upsilon_{\ell\mathcal{L}\varkappa} \text{ is the mean for time } \varkappa , \\ &\bar{\upsilon}_{\ell\mathcal{L}.} = \frac{1}{\mathrm{p}}\sum_{\varkappa=1}^{\mathrm{p}} \upsilon_{\ell\mathcal{L}\varkappa} \text{ , is a means for the } \ell^{\text{th}} \text{ subject in group } \mathcal{L} , \end{aligned}\right\}$$

$$\tag{2.1.5}$$

2.2 *Estimation*

We take two methods to find the estimators of the model parameters, and some of characteristics.

2.2.1 MLM

The joint $\upsilon_{\ell\mathcal{L}\varkappa}$ density functions have the following form:

$$f\left(\upsilon\backslash\vartheta , \xi_{\mathcal{L}} , \zeta_{\varkappa} , (\xi\zeta)_{\mathcal{L}\varkappa} , \varsigma_{\ell(\mathcal{L})}, \sigma_{e}^2, \sigma_{\varsigma}^2\right) = \left(2\pi(\sigma_e^2 + \sigma_\varsigma^2)\right)^{\frac{-Np}{2}} exp\left[\frac{-(\upsilon_{\ell\mathcal{L}\varkappa}-\vartheta-\xi_{\mathcal{L}}-\zeta_{\varkappa}-(\xi\zeta)_{\mathcal{L}\varkappa})^2}{2(\sigma_e^2+\sigma_\varsigma^2)}\right],$$

Following is a derivation of the likelihood function for (2.1.1):

$$L\left(\upsilon\backslash\vartheta , \xi_{\mathcal{L}} , \zeta_{\varkappa} , (\xi\zeta)_{\mathcal{L}\varkappa} , \varsigma_{\ell(\mathcal{L})}, \sigma_{e}^2, \sigma_{\varsigma}^2\right) \propto \left(2\pi(\sigma_e^2 + \sigma_\varsigma^2)\right)^{\frac{-Np}{2}} exp\left[\frac{-\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa}-\vartheta-\xi_{\mathcal{L}}-\zeta_{\varkappa}-(\xi\zeta)_{\mathcal{L}\varkappa})^2}{2(\sigma_e^2+\sigma_\varsigma^2)}\right],$$

then, by taking the logarithm of both sides, we get

$$\ln(\mathrm{L}) \propto \frac{-Np}{2}\ln(2\pi) - \frac{Np}{2}\ln\left(\sigma_e^2 + \sigma_\varsigma^2\right) - \left[\frac{\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa}-\vartheta-\xi_{\mathcal{L}}-\zeta_{\varkappa}-(\xi\zeta)_{\mathcal{L}\varkappa})^2}{2(\sigma_e^2+\sigma_\varsigma^2)}\right], \tag{2.1.6}$$

The result is obtained by differentiating (2.1.6) with respect to $\vartheta , \xi_{\mathcal{L}} , \zeta_{\varkappa} , (\xi\zeta)_{\mathcal{L}\varkappa} , \sigma_{\varsigma}^2$ respectively and setting the derivative to zero we have

$$\frac{\partial\, ln(L)}{\partial\vartheta} = -2\left[-\frac{\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa} - \vartheta - \xi_{\mathcal{L}} - \zeta_{\varkappa} - (\xi\zeta)_{\mathcal{L}\varkappa})}{2(\sigma_e^2 + \sigma_\varsigma^2)}\right] = 0$$

$$\Longrightarrow \hat{\vartheta} = \frac{1}{Np}\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa}) \Longrightarrow \hat{\vartheta} = \bar{\upsilon}_{...}$$

$$\frac{\partial\, ln(L)}{\partial \xi_{\mathcal{L}}} = -2q\left[-\frac{\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa} - \vartheta - \xi_{\mathcal{L}} - \zeta_{\varkappa} - (\xi\zeta)_{\mathcal{L}\varkappa})}{2(\,\sigma_e^2 + \sigma_\varsigma^2)}\right] = 0 \implies \hat{\xi}_{\mathcal{L}} = \bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{...}$$

$$\frac{\partial\, ln(L)}{\partial \zeta_{\varkappa}} = -2p\left[-\frac{\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}(\upsilon_{\ell\mathcal{L}\varkappa} - \vartheta - \xi_{\mathcal{L}} - \zeta_{\varkappa} - (\xi\zeta)_{\mathcal{L}\varkappa})}{2(\,\sigma_e^2 + \sigma_\varsigma^2)}\right] = 0 \Rightarrow \hat{\zeta}_{\varkappa} = \bar{\upsilon}_{..\varkappa} - \bar{\upsilon}_{...}$$

$$\frac{\partial\, ln(L)}{\partial (\xi\zeta)_{\mathcal{L}\varkappa}} = -2qp\left[-\frac{\sum_{\ell=1}^{n_{\mathcal{L}}}(\upsilon_{\ell\mathcal{L}\varkappa} - \vartheta - \xi_{\mathcal{L}} - \zeta_{\varkappa} - (\xi\zeta)_{\mathcal{L}\varkappa})}{2(\sigma_e^2 + \sigma_\varsigma^2)}\right] = 0 \Rightarrow \widehat{(\xi\zeta)}_{\mathcal{L}\varkappa} = \bar{\upsilon}_{.\mathcal{L}\varkappa} - \bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{..\varkappa} + \bar{\upsilon}_{...}$$

$since\ \sigma_e^2 = \frac{SS_e}{(p-1)(N-q)}$, then $\hat{\sigma}_e^2 = \frac{\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa} - \bar{\upsilon}_{.\mathcal{L}\varkappa} - \bar{\upsilon}_{\ell\mathcal{L}.} + \bar{\upsilon}_{.\mathcal{L}.})^2}{(p-1)(N-q)}$, $\therefore\ \hat{\sigma}_e^2 = MSe$.

$$\frac{\partial\, ln(L)}{\partial \sigma_\varsigma^2} = \frac{-Np}{2}\left(\frac{1}{\sigma_\varsigma^2+\sigma_e^2}\right) + \frac{2(\sigma_\varsigma^2+\sigma_e^2)(0) + 2\sum_{\mathcal{L}=1}^{q}\sum_{\ell=1}^{n_{\mathcal{L}}}\sum_{\varkappa=1}^{p}(\upsilon_{\ell\mathcal{L}\varkappa} - \vartheta - \xi_{\mathcal{L}} - \zeta_{\varkappa} - (\xi\zeta)_{\mathcal{L}\varkappa})^2}{\left(2(\sigma_\varsigma^2+\sigma_e^2)\right)^2} = 0$$

$$\hat{\sigma}_\varsigma^2 = \frac{1}{Np(p-1)} MS\varsigma + \frac{((p-1)(N-q) - Np)}{Np} MSe$$

2.2.2 Restricted Maximum Likelihood Method (RMLM)

The restricted maximum likelihood estimators of $\vartheta, \xi_{\mathcal{L}}, \zeta_{\varkappa}, (\xi\zeta)_{\mathcal{L}\varkappa}, \varsigma_{\ell(\mathcal{L})}, \sigma_e^2, and\ \sigma_\varsigma^2$ are determined by the parameter values, which maximize the function:

$$L \propto (\sigma_e^2)^{\frac{-(p-1)(N-q)}{2}}(\sigma_e^2 + p\sigma_\varsigma^2)^{\frac{-(N-q)}{2}}\, exp\left\{-\frac{1}{2}\left[\frac{SS_e}{\sigma_e^2} + \frac{SS_\varsigma}{(\sigma_e^2+p\sigma_\varsigma^2)} + \frac{Np(\bar{\upsilon}_{...}-\vartheta)^2}{(\sigma_e^2+p\sigma_\varsigma^2)} + \frac{Np(\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{...}-\xi_{\mathcal{L}})^2}{(\sigma_e^2+p\sigma_\varsigma^2)} + \frac{Np(\bar{\upsilon}_{..\varkappa}-\bar{\upsilon}_{...}-\zeta_{\varkappa})^2}{(\sigma_e^2+p\sigma_\varsigma^2)} + \frac{Np(\bar{\upsilon}_{.\mathcal{L}\varkappa}+\bar{\upsilon}_{...}-\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{..\varkappa}-(\xi\zeta)_{\mathcal{L}\varkappa})^2}{(\sigma_e^2+p\sigma_\varsigma^2)}\right]\right\}, \qquad (2.2.1)$$

We know that

$L(\upsilon \backslash \vartheta, \xi_{\mathcal{L}}, \zeta_{\varkappa}, (\xi\zeta)_{\mathcal{L}\varkappa}, \varsigma_{\ell(\mathcal{L})}, \sigma_e^2, \sigma_\varsigma^2) = L(\bar{\upsilon}_{...} \backslash \vartheta, \xi_{\mathcal{L}}, \zeta_{\varkappa}, (\xi\zeta)_{\mathcal{L}\varkappa}, \varsigma_{\ell(\mathcal{L})}) = L(SS_e, SS_\varsigma \backslash\, \sigma_e^2, \sigma_\varsigma^2)$, (2.2.2) Then, we have

$$L(\bar{\upsilon}_{...} \backslash \vartheta, \xi_{\mathcal{L}}, \zeta_{\varkappa}, (\xi\zeta)_{\mathcal{L}\varkappa}, \varsigma_{\ell(\mathcal{L})}) = (\sigma_e^2 + p\sigma_\varsigma^2)^{\frac{-(N-q)}{2}} \exp\left\{-\frac{1}{2}\left[\frac{Np(\bar{\upsilon}_{...}-\vartheta)^2}{(\sigma_e^2+p\sigma_\varsigma^2)} + \frac{Np(\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{...}-\xi_{\mathcal{L}})^2}{(\sigma_e^2+p\sigma_\varsigma^2)} + \frac{Np(\bar{\upsilon}_{..\varkappa}-\bar{\upsilon}_{...}-\zeta_{\varkappa})^2}{(\sigma_e^2+p\sigma_\varsigma^2)} + \frac{Np(\bar{\upsilon}_{.\mathcal{L}\varkappa}+\bar{\upsilon}_{...}-\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{..\varkappa}-(\xi\zeta)_{\mathcal{L}\varkappa})^2}{(\sigma_e^2+p\sigma_\varsigma^2)}\right]\right\} \qquad (2.2.3)$$

and

$$L(SS_e, SS_\varsigma \backslash\, \sigma_e^2, \sigma_\varsigma^2) = (\sigma_e^2)^{\frac{-(p-1)(N-q)}{2}}(\sigma_e^2 + p\sigma_\varsigma^2)^{\frac{-(N-q)}{2}} \exp\left\{-\frac{1}{2}\left[\frac{SS_e}{\sigma_e^2} + \frac{SS_\varsigma}{(\sigma_e^2+p\sigma_\varsigma^2)}\right]\right\}, \qquad (2.2.4)$$

Then, by taking the logarithm of both sides of equation (2.2.3), we get

$$\ln\left(L\left(\bar{\upsilon}_{...}\backslash\vartheta\,,\xi_{\mathcal{L}}\,,\zeta_{\varkappa}\,,(\xi\zeta)_{\mathcal{L}\varkappa}\,,\varsigma_{\ell(\mathcal{L})}\right)\right)=$$

$$\frac{-(n-q)}{2}ln\left(\sigma_e^2+p\sigma_\varsigma^2\right)-\frac{1}{2}\left[\begin{array}{c}\frac{Np(\bar{\upsilon}_{...}-\vartheta)^2}{(\sigma_e^2+p\sigma_\varsigma^2)}+\frac{Np(\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{...}-\xi_{\mathcal{L}})^2}{(\sigma_e^2+p\sigma_\varsigma^2)}+\frac{Np(\bar{\upsilon}_{..\varkappa}-\bar{\upsilon}_{...}-\zeta_{\varkappa})^2}{(\sigma_e^2+p\sigma_\varsigma^2)}\\ +\frac{Np(\bar{\upsilon}_{.\mathcal{L}\varkappa}+\bar{\upsilon}_{...}-\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{..\varkappa}-(\xi\zeta)_{\mathcal{L}\varkappa})^2}{(\sigma_e^2+p\sigma_\varsigma^2)}\end{array}\right] \quad (2.2.5)$$

The derivative is then equaled to zero by equating (2.2.5) with regard to parameters. This gives us

$$\frac{\partial\ln(L)}{\partial\vartheta}=-\frac{1}{2}\left[2Np\frac{(\bar{\upsilon}_{...}-\vartheta)}{(\sigma_e^2+p\sigma_\varsigma^2)}\right]=0\;\Rightarrow\;\hat{\vartheta}=\bar{\upsilon}_{...}\,,$$

$$\frac{\partial\ln(L)}{\partial\xi_{\mathcal{L}}}=-\frac{1}{2}\left[\frac{2Np(\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{...}-\xi_{\mathcal{L}})}{(\sigma_e^2+p\sigma_\varsigma^2)}\right]=0\;\Rightarrow\;\hat{\xi}_{\mathcal{L}}=\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{...}\,,$$

$$\frac{\partial\ln(L)}{\partial\zeta_{\varkappa}}=-\frac{1}{2}\left[\frac{2Np(\bar{\upsilon}_{..\varkappa}-\bar{\upsilon}_{...}-\zeta_{\varkappa})}{(\sigma_e^2+p\sigma_\varsigma^2)}\right]=0\;\Rightarrow\;\hat{\zeta}_{\varkappa}=\bar{\upsilon}_{..\varkappa}-\bar{\upsilon}_{...}\,,$$

$$\frac{\partial\,ln(L)}{\partial(\xi\zeta)_{\mathcal{L}\varkappa}}=\frac{-1}{2}\left[\frac{2Np(\bar{\upsilon}_{.\mathcal{L}\varkappa}+\bar{\upsilon}_{...}-\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{..\varkappa}-(\xi\zeta)_{\mathcal{L}\varkappa})}{(\sigma_e^2+p\sigma_\varsigma^2)}\right]=0\Rightarrow\left(\widehat{\xi\zeta}\right)_{\mathcal{L}\varkappa}=\bar{\upsilon}_{.\mathcal{L}\varkappa}+\bar{\upsilon}_{...}-\bar{\upsilon}_{.\mathcal{L}.}-\bar{\upsilon}_{..\varkappa}\,,$$

$$\ln\left(L\left(SS_e,SS_\varsigma\backslash\,\sigma_e^2,\sigma_\varsigma^2\right)\right)=\frac{-(p-1)(n-q)}{2}\ln(\sigma_e^2)-\frac{(n-q)}{2}\ln\left(\sigma_e^2+p\sigma_\varsigma^2\right)-\frac{1}{2}\left[\frac{SS_e}{\sigma_e^2}+\frac{SS_\varsigma}{(\sigma_e^2+p\sigma_\varsigma^2)}\right], \quad (2.2.6)$$

let $\gamma=\sigma_e^2+p\sigma_\varsigma^2$ then $\ln\left(L\left(SS_e,SS_\varsigma\backslash\,\gamma\right)\right)=\frac{-(p-1)(N-q)}{2}\ln(\sigma_e^2)-\frac{(N-q)}{2}\ln(\gamma)-\frac{1}{2}\left[\frac{SS_e}{\sigma_e^2}+\frac{SS_\varsigma}{\gamma}\right]$ (2.2.7)

subject to the restrictions $\gamma\geq\;\sigma_e^2>0$. The derivative is then equaled to zero by equating (2.2.7). Then

$$\frac{\partial\,ln(L)}{\partial\sigma_e^2}=\frac{-(p-1)(N-q)}{2\sigma_e^2}-\frac{SS_e}{2\sigma_e^4}=0\;\Rightarrow\hat{\sigma}_e^2=\frac{SS_e}{(p-1)(N-q)}=MSe$$

$$\frac{\partial\,ln(L)}{\partial\gamma}=-\frac{(N-q)}{2\gamma}-\frac{SS_\varsigma}{2\gamma^2}=0\Rightarrow\hat{\gamma}=\frac{SS_\varsigma}{(N-q)}\;,then$$

$$\hat{\sigma}_e^2+p\hat{\sigma}_\varsigma^2=\frac{SS_\varsigma}{(N-q)}\;\Rightarrow\hat{\sigma}_\varsigma^2=\frac{1}{p}\left[\frac{SS_\varsigma}{(N-q)}-\frac{SSe}{(p-1)(N-q)}\right]=\frac{1}{p}[MS\varsigma-MSe]$$

3. The practical side of the study

In this section, we check the formulas and theories reached on the theoretical side that have been taken from a realistic experiment from the Palm Research Center at Basrah University during the two growing seasons of 2019-2020 [3]. The objective of the Experiment is to study the effect of the experimental factors and their interactions on the production of the date palm cultivar El Halawi. The Experiment included three levels of a unit factor, 3 days, 6 days, and 9 days of the irrigation interval; also, there are five levels between unit factor, fruit length, fruit weight, fruit size, dry weight, and total sugars, and three levels of within unit factor, drip irrigation, basin irrigation, and the regular tidal method. The sum squares of the effects between units factors, within units factors, and the effects of the interaction between the unit factors of group and time are obtained using the mathematical formula of the model study (2.1.1) and applied to the Experiment as follows:

Table 1
The sum squares table for the data of Experiment

$SS_{\xi_{\mathcal{L}}}$	$SS_{\zeta_{\varkappa}}$	$SS_{(\xi\zeta)_{\mathcal{L}\varkappa}}$	SS_{ς}	SSe
61916.134	2.972	2.103	74.106	52.188

The results of the sum squares for the model are shown in Table (1) above, and it is clear from this Table that the irrigation intervals significantly impact productivity.

Table 2
Estimates of variance components of the model(2.1.1) for the Experiment

Method of estimation	**Random effects σ_{ς}^2**	**Experiment error σ_e^2**
ANOVA	2.374	1.450
MLE	1.564	1.243
REML	2.374	1.450

The Maximum likelihood estimator of σ_{ς}^2 is different from those produced by the ANOVA method, and RMLM.

The estimation values for parameters in (2.1.1) for Experiment are the same in the three methods and are in Table (3).

Table 3
Estimation values for the parameters of (2.1.1) for the Experiment

$\hat{\vartheta}$	$\hat{\xi}_{\mathcal{L}}$	$\hat{\zeta}_{\varkappa}$	$\widehat{(\xi\zeta)}_{\mathcal{L}\varkappa}$
34.155	136.619	68.309	273.237

The estimation values for the model's parameters are shown in Table (3) above using the least-squares approach, MLM, and RMLM. We can see from these estimator values that they are encouraging statistically.

4. Conclusions

In this paper, we came to these conclusions: The maximum likelihood estimators of parameters of the repeated measurements model are $\hat{\vartheta} = \bar{\upsilon}_{\ldots}$, $\hat{\xi}_{\mathcal{L}} = \bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{\ldots}$, $\hat{\zeta}_{\varkappa} = \bar{\upsilon}_{..\varkappa} - \bar{\upsilon}_{\ldots}$ $\widehat{(\xi\zeta)}_{\mathcal{L}\varkappa} = \bar{\upsilon}_{.\mathcal{L}\varkappa} - \bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{..\varkappa} + \bar{\upsilon}_{\ldots}$ and $\hat{\sigma}_e^2 = MSe$, and $\hat{\sigma}_{\delta}^2 = \frac{1}{Np(p-1)} MS\varsigma + \frac{((p-1)(N-q)-np)}{Np} MSe$. The restricted maximum likelihood estimators of parameters of the repeated measurements model are $\hat{\vartheta} = \bar{\upsilon}_{\ldots}$, $\hat{\xi}_{\mathcal{L}} = \bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{\ldots}$, $\hat{\zeta}_{\varkappa} = \bar{\upsilon}_{..\varkappa} - \bar{\upsilon}_{\ldots}$ $\widehat{(\xi\zeta)}_{\mathcal{L}\varkappa} = \bar{\upsilon}_{.\mathcal{L}\varkappa} - \bar{\upsilon}_{.\mathcal{L}.} - \bar{\upsilon}_{..\varkappa} + \bar{\upsilon}_{\ldots}$, $\hat{\sigma}_e^2 =$

MSe, and $\hat{\sigma}_{\varsigma}^{2} = \frac{1}{p}[MS\varsigma - MSe]$. The studied irrigation methods and irrigation interval affect improving productive qualities.

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