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Reduced bias estimation of variance for repeated measurements model

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Abstract

This paper discusses two important points; the first part describes the one-way repeated measurements model, which had two random effects in addition to the random error and two fixed factors. The model was also written as a set of matrices, which showed how the random effects and fixed factors interacted. The second part shows the maximum likelihood method for determining the variance components for a one-way repeated measures model. In this method, it was seen that the estimates of the variance combinations have some bias. The goal of this study is to use the Mean bias reduction method of the one-way repeated measurements model to reduce the bias in the estimators of the variance components.

Subject Classification: *62F10, 62G05.*

Keywords: *One-way repeated measurements model, Variance compounds, Mean bias reduction, Maximum likelihood method.*

1. Introduction

Data modelling has become an important matter for its applications in practical life to find the best solutions to life problems. In our study, the repeated

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measurements model was adopted as a linear/mixed model, meaning that the observation or the experimental unit is measured more than once by the same scales. This type of model has many advantages, as it reduces the variance of the error because all the experimental units participate in all the experimental treatments, so the variables of the experimental unit are fixed, so the variance of the error decreases. Also, repeated measures models are statistically strong designs regarding the tests' strength and the effect amount. Studying and analyzing the components of variance is one of the important topics in statistics because this topic has wide applications. The maximum likelihood method is one of the most common estimation methods because of its important properties, such as unbiased. When deriving this method for the effects of the one-way repeated measurements model, it showed its efficiency in the fixed effects, estimators, as getting an unbiased estimator. In contrast, in the Estimation of the random effects, an amount of bias appeared in the estimators of the variance components of the one-way repeated measurements model. This research aims to reduce the amount of bias through the Mean bias reduction method.

2. Description of the model

The One-Way repeated measurements model is defined as follows.

$$t_{ijk} = \xi + \varphi_j + \Psi_k + (\varphi\Psi)_{jk} + \omega_{1i(j)} + \omega_{2i(k)} + e_{ijk} \qquad (1)$$

Where:

$i = 1,2,\dots,n_1$ is a unit of experimentation index.

$j = 1,2,\dots,n_2$ is an indicator of the between units factor's levels (Group).

$k = 1,2,\dots,n_3$ is an indicator of the within unite factor's levels (Time).

t_{ijk} = is the measurement of a unit's response over time in a group.

e_{ijk} = is The random error.

ξ = is The overall mean, Table (1) classifies the effects of the math model (1) by type.

Table 1

Classifies The effects of the One-Way repeated measurements model.

The factor	The Type	The definition	The condition
$\boldsymbol{\varphi_i}$	Fixed	Is treatment between unite factor (group)	$\sum_{j=i}^{n_2} \varphi_i = 0$
$\boldsymbol{\Psi_k}$	Fixed	Is treatment within unite factor (Time)	$\sum_{k=1}^{n_3} \Psi_k = 0$
$\boldsymbol{(\varphi\Psi)_{ij}}$	Fixed	Is the effect (between X within) unites	$\sum_{j=1}^{n_2} \varphi_j\Psi_k = 0$ $\sum_{k=1}^{n_3} \varphi_j\Psi_k = 0$

Contd...

$\boldsymbol{\omega_{1i(j)}}$	Random	Is the random effect to unit i within Group (j)	$\omega_{1ij} \sim N(0, \sigma^2_{\omega_1})$
$\boldsymbol{\omega_{2i(k)}}$	Random	Is the random effect to unit i within time	$\omega_{2ij} \sim N(0, \sigma^2_{\omega_2})$
$\boldsymbol{e_{ijk}}$	Random	Is the random error	$e_{ijk} \sim N(0, \sigma^2_e)$

3. The model in matrix form

The one-way repeated measures model (1) can be written in its observations as follows:

$$\tau = [1_{n_1} \otimes 1_{n_2} \otimes 1_{n_3}]\xi + [1_{n_1} \otimes I_{n_2} \otimes 1_{n_3}]\varphi + [1_{n_1} \otimes 1_{n_2} \otimes I_{n_3}]\Psi + [1_{n_1} \otimes I_{n_2} \otimes I_{n_3}]\,(\varphi\Psi) + [I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}]\,\omega_1 + [I_{n_1} \otimes 1_{n_2} \otimes I_{n_3}]\,\omega_2 + [I_{n_1} \otimes I_{n_2} \otimes I_{n_3}]\,e \ . \tag{2}$$

Where I denoted the identity matrix, 1 denoted the vector one's, $\otimes$ be the Kronecker product operation of two matrices. To formulate our study model more appropriately, we used the matrix formula; first, we write model (1) in the form below:

$$t_{ij} = \alpha_j + 1_{n_3} b_{i(j)} + 1_{n_2} c_i + e_{ij} \tag{3}$$

where $t_{ij} = [t_{ij1,}, \dots, t_{ijn_3}]'$ is a vector of response, $\alpha_j = [\alpha_{j1,}, \dots, \alpha_{jn_3}]'$ is a vector of the fixed treatments ($\alpha_{ij} = \xi + \varphi_j + \Psi_k + (\varphi\Psi)_{jk}$), $b_{i(j)} = \omega_{1i(j)}$ is the random effect of unit i within a group (j), $c_i = [c_{i(1),}, \dots, c_{i(n_3)}]'$ is the random effect to unit i within time($\omega_{2i(k)}$) and $e_{ij} = [e_{ij1,}, \dots, e_{ijn_3}]'$ is a vector of random error. Let $б_{ij}$ represent the indicators of a group such that $б_{ij} = \begin{cases} 1 & \text{if unit i is from group j} \\ 0 & \text{o.w} \end{cases}$ j = 1,…, n_2 . By setting $б_i' = [\alpha_{i1,}, \dots, \alpha_{in_2}]'$, we can rewrite model (3) as

$$\tau_i = X_i\, \alpha + Z_i W + \varepsilon_i. \tag{4}$$

where $t_i' = [t_{i1}, \dots, t_{in_3}]$, $X_i = I_{n_3} \otimes б_i$, $\alpha' = [\alpha_{11,}, \dots, \alpha_{n_2 n_3}]$, $Z_i = 1_{n_3} \otimes 1_{n_2}$, W $= [\omega_{1i(j)}, \omega_{2i(k)}]$, $\varepsilon_i' = [\varepsilon_{i1}, \dots, \varepsilon_{in_3}]$ and $\alpha = \text{Vec}\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n_3} \\ \vdots & \cdots & \vdots \\ \alpha_{n_2 1} & \cdots & \alpha_{n_2 n_3} \end{bmatrix}$.The final formulation of the study model is an alternative form of the model(4) as shown in the relationship (5), let $\tau' = [\, t_1', \dots, t_{n_1}']$, $X' = [\, X_1', \dots, X_{n_1}']$, $W = [\omega_{1i(j)}, \omega_{2i(k)}, \varepsilon_i\,]$ and

$$Z = [z_1', \dots, z_{n_1}'] \text{ then } \tau = X\,\alpha + Z\,W. \tag{5}$$

4. Variance Components in a One-Way repeated measurements model

The variance matrix of τ is denoted as Σ, such that:

$$\Sigma_{\omega_1} = \text{var}\,([I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}]\,\omega_1) = [I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}]\, \sigma^2_{\omega_1} I_{n_1} [I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}]' = [I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}]\, \sigma^2_{\omega_1} [\, I_{n_1}' \otimes I_{n_2}' \otimes I_{n_3}']$$

$$= \sigma^2_{\omega_1} [I_{n_1} \otimes I_{n_2} \otimes J_{n_3}],$$

$\sum_{\omega_2} = \sigma^2_{\omega_2}[I_{n_1} \otimes J_{n_2} \otimes I_{n_3}]$ and $\sum_e = \sigma^2_e\ [I_{n_1} \otimes I_{n_2} \otimes I_{n_3}]$. where J denoted the matrix of ones. Then the variance matrix

$$\Sigma = \begin{bmatrix} \sum_{\omega_1} + \sum_{\omega_2} + \sum_e & 0 & \cdots & 0 \\ 0 & \sum_{\omega_1} + \sum_{\omega_2} + \sum_e & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sum_{\omega_1} + \sum_{\omega_2} + \sum_e \end{bmatrix} \quad (6)$$

5. Estimation of variance components by maximum likelihood

The maximum likelihood method is one of the common methods for estimating the parameters of mathematical models. This section discusses the maximum likelihood method for estimating the parameters of the linear model (5), where τ is the vector of $(N \times 1)$observations and N=$\sum n_1 n_2 n_3$, α refers to fixed effects, w refers to random effects, and X and Z are design matrices such that: $ZW = [\ Z_0, \ldots, Z_m\ ] \begin{bmatrix} w_0 \\ \vdots \\ w_m \end{bmatrix} = \sum_{i=0}^{m} z_i w_i$, where w_i is the vector for random effect with a_i levels and $w_0 = e$. w_i vector has the following properties :

$E(w_i) = 0$, Var $(w_i) = \sigma_i^2 I_{a_i}$, Cov $(w_i, w_h) = 0$, thus Var (w) = $\{\sigma_i^2 I_{a_i}\}_{i=0}^{m}$. Then $E(\tau) = X\alpha$ and Var $(\tau) = \Sigma$. The density function of $\tau \sim N(X\alpha, \Sigma)$ is

$f(\tau, \Sigma) = (2\pi)^{-\frac{1}{2}N}\ |\Sigma|^{-\frac{1}{2}} \exp(-\frac{1}{2}(\tau - X\alpha)'\ \Sigma^{-1}\ (\tau - X\alpha))$ (6)

by taking (log) for function (6) and letting $R = (\tau - X\alpha)$ then

$l(\alpha, \Sigma) = -\frac{1}{2}\left[\log|\sum| + R'\ \sum^{-1} R(\alpha) + N\log(2\pi)\right]$ where $|\sum|$ is a determinant of the variance matrix, $\Theta = (\sigma^2_{\omega_1}, \sigma^2_{\omega_2}, \sigma^2_e) = \Theta_r$ i.e Θ_r is γ^{th} component of the vector of variance Θ. By differentiating function (7) concerning the fixed effect α we obtain $\frac{\partial l(\alpha,\Sigma)}{\partial \alpha} = -\frac{1}{2}\ [(-X)'\ \Sigma^{-1}(\tau - X\alpha) + (\tau - X\alpha)'\Sigma^{-1}(-X)]$

$$= -\frac{1}{2}[-X'\Sigma^{-1}\tau + X'\Sigma X\alpha - \tau'\ \Sigma^{-1}X + X'\alpha'\Sigma^{-1}X] == [X'\Sigma^{-1}\tau - X'\Sigma^{-1}X\alpha] = 0$$

$$\text{Then } \hat{\alpha} = [X'\ \Sigma^{-1}X]^{-1}\ X'\Sigma^{-1}\tau \quad (7)$$

is the estimator of the fixed effect, to find the estimator of the random effect $\widehat{\Theta}$ = ($\widehat{\sigma^2_{\omega_1}}, \widehat{\sigma^2_{\omega_2}}, \widehat{\sigma^2_e}$), using the following facts: where $\frac{\partial \Sigma}{\partial \Theta_r} = \Sigma_r$, $\frac{\partial^2 \Sigma}{\partial \Theta_r \partial \Theta_s} = \Sigma_{rs}$, $I = \Sigma^{-1}\Sigma_r\Sigma^{-1}\Sigma_s$, $II = \Sigma^{-1}\Sigma_s\Sigma^{-1}\Sigma_r$ and $III = \Sigma_{rs}\Sigma^{-1}$

Then $\frac{\partial \log|\Sigma|}{\partial \Theta_r}$ = tr $(\Sigma^{-1}\Sigma_r)$, $\frac{\partial \Sigma^{-1}}{\partial \Theta_r} = -\Sigma^{-1}\Sigma_r\ \Sigma^{-1}$, $\frac{\partial^2 \log|\Sigma|}{\partial \Theta_r \partial \Theta_s} = tr(I - III)$

and $\frac{\partial^2 \Sigma^{-1}}{\partial \Theta_r \partial \Theta_s} = I\,\Sigma^{-1} + II\Sigma^{-1} - III$

By differentiating (log) for function (6) concerning the random effect Θ_r we obtain

$$\Gamma_\Theta = \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_r} = \nabla\left[-\tfrac{1}{2}\log|\Sigma|\right] + \nabla\left[-\tfrac{1}{2}(\tau-\alpha X')\,\Sigma^{-1}(\tau - X\alpha)\right]$$

$$= \tfrac{1}{2}\left[R'\,\Sigma^{-1}(\,\Theta)\,\Sigma_r\,\Sigma^{-1}(\,\Theta)\,R - t_r(\Sigma^{-1}(\,\Theta)\Sigma_r)\right] = 0 \quad (8)$$

i.e
$$\frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_r} = \begin{bmatrix} \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_0} \\ \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_1} \\ \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[R'\,\Sigma^{-1}(\,\Theta)\,\Sigma_0\,\Sigma^{-1}(\,\Theta)\,R - t_r(\Sigma^{-1}(\,\Theta)\Sigma_0)] \\ \frac{1}{2}[R'\,\Sigma^{-1}(\,\Theta)\,\Sigma_1\,\Sigma^{-1}(\,\Theta)\,R - t_r(\Sigma^{-1}(\,\Theta)\Sigma_1)] \\ \frac{1}{2}[R'\,\Sigma^{-1}(\,\Theta)\,\Sigma_2\,\Sigma^{-1}(\,\Theta)\,R - t_r(\Sigma^{-1}(\,\Theta)\Sigma_2)] \end{bmatrix} \quad (9)$$

To find the value of the estimator $\widehat{\Theta}$, which represents the root of the equation (*9*) according to the Newton-Raphson method, depending on the second-order approximation of the Taylor series.

So
$$\Gamma_{\Theta\Theta} = \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_r\partial\Theta_s} = -\,\tfrac{1}{2}\,[\text{ tr (II - I) } + R'\text{ (I+II-III) }\Sigma^{-1}\text{R}] \quad (10)$$

i.e
$$\frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_r\partial\Theta_s} = \begin{bmatrix} \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_0\partial\Theta_0} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_0\partial\Theta_1} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_0\partial\Theta_2} \\ \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_1\partial\Theta_0} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_1\partial\Theta_1} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_1\partial\Theta_2} \\ \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_2\partial\Theta_0} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_2\partial\Theta_1} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_2\partial 2} \end{bmatrix} = \begin{bmatrix} \Gamma_{00} & \Gamma_{01} & \Gamma_{02} \\ \Gamma_{10} & \Gamma_{11} & \Gamma_{12} \\ \Gamma_{20} & \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$
such that :

$$\Gamma_{00} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_0 - \varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_0) + R'(\varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_0 + \varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_0 - \Sigma_{00}\varSigma^{-1})\Sigma^{-1}R]$$

$$\Gamma_{01} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_0 - \varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_1) + R'\,(\varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_1 + \varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_0 - \Sigma_{01}\varSigma^{-1})\Sigma^{-1}\,R]$$

$$\Gamma_{02} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_0 - \varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_2) + R'\,(\varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_2 + \varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_0 - \Sigma_{02}\varSigma^{-1})\,\Sigma^{-1}\,R\,]$$

$$\Gamma_{10} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_1 - \varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_0) + R'(\varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_0 + \varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_1 - \Sigma_{10}\varSigma^{-1})\Sigma^{-1}R]$$

$$\Gamma_{11} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_1 - \varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_1) + R'\,(\varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_1 + \varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_1 - \Sigma_{11}\varSigma^{-1})\Sigma^{-1}\,R]$$

$$\Gamma_{12} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_1 - \varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_2) + R'\,(\varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_2 + \varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_1 - \Sigma_{12}\varSigma^{-1})\,\Sigma^{-1}\,R\,]$$

$$\Gamma_{20} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_2 - \varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_0) + R'(\varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_0 + \varSigma^{-1}\Sigma_0\varSigma^{-1}\Sigma_2 - \Sigma_{20}\varSigma^{-1})\Sigma^{-1}R]$$

$$\Gamma_{21} = -\frac{1}{2}\,[\text{ tr }(\varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_2 - \varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_1) + R'(\varSigma^{-1}\Sigma_2\varSigma^{-1}\Sigma_1 + \varSigma^{-1}\Sigma_1\varSigma^{-1}\Sigma_2 - \Sigma_{21}\varSigma^{-1})\Sigma^{-1}R]$$

$$\Gamma_{22} = -\frac{1}{2}\,[\,\mathrm{tr}\,(\mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_2 - \mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_2) + R'(\mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_2 + \mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_2 - \Sigma_{22}\mathit{\Sigma}^{-1})\,\Sigma^{-1}R\,]$$

Where $\mathrm{E}\left(\frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_r\partial\Theta_s}\right) = \begin{bmatrix} 0 & \mathrm{E}_{01} & \mathrm{E}_{02} \\ \mathrm{E}_{10} & 0 & \mathrm{E}_{12} \\ \mathrm{E}_{20} & \mathrm{E}_{21} & 0 \end{bmatrix}$ then

$$\begin{aligned}
\mathrm{E}_{01} &= -\frac{1}{2}\,\mathrm{tr}\,(\mathit{\Sigma}^{-1}\Sigma_1\mathit{\Sigma}^{-1}\Sigma_0 - \mathit{\Sigma}^{-1}\Sigma_0\mathit{\Sigma}^{-1}\Sigma_1),\\
\mathrm{E}_{02} &= -\frac{1}{2}\,\mathrm{tr}\,(\mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_0 - \mathit{\Sigma}^{-1}\Sigma_0\mathit{\Sigma}^{-1}\Sigma_2)\\
\mathrm{E}_{10} &= -\frac{1}{2}\,\mathrm{tr}\,(\mathit{\Sigma}^{-1}\Sigma_0\mathit{\Sigma}^{-1}\Sigma_1 - \mathit{\Sigma}^{-1}\Sigma_1\mathit{\Sigma}^{-1}\Sigma_0),\\
\mathrm{E}_{12} &= -\frac{1}{2}\,\mathrm{tr}\,(\mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_1 - \mathit{\Sigma}^{-1}\Sigma_1\mathit{\Sigma}^{-1}\Sigma_2)\\
\mathrm{E}_{20} &= -\frac{1}{2}\,\mathrm{tr}\,(\mathit{\Sigma}^{-1}\Sigma_0\mathit{\Sigma}^{-1}\Sigma_2 - \mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_0),\\
\mathrm{E}_{21} &= -\frac{1}{2}\,\mathrm{tr}\,(\mathit{\Sigma}^{-1}\Sigma_1\mathit{\Sigma}^{-1}\Sigma_2 - \mathit{\Sigma}^{-1}\Sigma_2\mathit{\Sigma}^{-1}\Sigma_1)
\end{aligned}$$

Then $\Gamma_{\widehat{\Theta}_{k+1}} \approx \Gamma_{\widehat{\Theta}_k} + \Gamma_{\widehat{\Theta}_k\widehat{\Theta}_k}(\widehat{\Theta}_{k+1} - \widehat{\Theta}_k)$, estimator vector $\widehat{\Theta}$ by Newton Raphson's method is :

$$\widehat{\Theta}_{k+1} = \widehat{\Theta}_k - \Gamma_{\widehat{\Theta}_k\widehat{\Theta}_k}^{-1}\,\Gamma_{\widehat{\Theta}_k} \tag{11}$$

where $\Theta_{\mathrm{in}} \in \mathbb{R}^r$ is initial values, While $\| \widehat{\Theta}_{k+1} - \widehat{\Theta}_k \| < \varepsilon$, then the convergence is achieved.

6. The Bias in the Estimation of variance components

The disparity between the actual value of a variance component and its estimated value is what is meant by the term "bias" in the context of estimating variance components. In the context of linear models, having a bias in estimating the variance components can lead to inaccurate conclusions about the relationships between variables and wrong inferences about the precision of the estimated parameters. This can occur if an inaccurate model structure was used, the sample size was too small, or other factors contributed to the measurement error. The estimator $\widehat{\Theta}$ of variance components, Θ is unbiased if $\mathrm{E}(\widehat{\Theta}) = \Theta$ Whereas $\mathrm{E}(\widehat{\Theta}) = \Theta + \delta$ then $\widehat{\Theta}$ is biased estimator, and δ is amount of bias. From equation (11) :

$$E\,(\widehat{\Theta}_{k+1})) = \begin{bmatrix} E\left(\widehat{\sigma_e^2}\right) \\ E\left(\widehat{\sigma_{\omega_1}^2}\right) \\ E\left(\widehat{\sigma_{\omega_2}^2}\right) \end{bmatrix} - \left[E \begin{bmatrix} \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_0\partial\Theta_0} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_0\partial\Theta_1} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_0\partial\Theta_2} \\ \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_1\partial\Theta_0} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_1\partial\Theta_1} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_1\partial\Theta_2} \\ \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_2\partial\Theta_0} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_2\partial\Theta_1} & \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_2\partial\Theta 2} \end{bmatrix} \right]^{-1} E \begin{bmatrix} \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_0} \\ \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_1} \\ \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{\sigma_e^2} \\ \widehat{\sigma_{\omega_1}^2} \\ \widehat{\sigma_{\omega_2}^2} \end{bmatrix} - \frac{1}{N} \begin{bmatrix} \widehat{\sigma_e^2} \\ \widehat{\sigma_{\omega_1}^2} \\ \widehat{\sigma_{\omega_2}^2} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} t_r(\Sigma^{-1}(\Theta)\Sigma_0) \\ t_r(\Sigma^{-1}(\Theta)\Sigma_1) \\ t_r(\Sigma^{-1}(\Theta)\Sigma_2) \end{bmatrix}$$

Then $E\left(\widehat{\Theta}_{k+1}\right) = \widehat{\Theta}_k + \delta$

$$\text{Where } \delta = -\left(\frac{1}{N} \begin{bmatrix} \widehat{\sigma_e^2} \\ \widehat{\sigma_{\omega_1}^2} \\ \widehat{\sigma_{\omega_2}^2} \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} t_r(\Sigma^{-1}(\Theta)\Sigma_0) \\ t_r(\Sigma^{-1}(\Theta)\Sigma_1) \\ t_r(\Sigma^{-1}(\Theta)\Sigma_2) \end{bmatrix} \right)$$

is the amount of bias.

7. Mean bias reduction in Estimation for one-way repeated measurements model

Bias reduction refers to reducing systematic error or bias in a statistical model. Bias in a model can lead to incorrect or misleading results, so reducing bias is an important step in improving the model's accuracy. It is possible to reduce the bias in the $\widehat{\Theta}$ estimator by solving the equation: $\Gamma_{\widetilde{\Theta}} = \Gamma_{\Theta} + A = 0$ Where $\Gamma_{\Theta} = \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_r}$ is score function, and $A = \frac{1}{2}\,\mathrm{tr}\,[\,\underline{E}^{-1}\,\{P_t + Q_t\}]$ such that: $\underline{E} = \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}$, $P_t = E\,[\,\Gamma_{\Theta}\,\grave{\Gamma}_{\Theta}\,\Gamma_{t\Theta}\,]$ and $Q_t = E\,[\,-\Gamma_{\Theta\Theta}\,\Gamma_{t\Theta}\,]$ for t = 1,2,3. In our model

$$\{\,Pt + Qt\} = \frac{1}{2}\,\mathrm{tr}\left[\Sigma^{-1} \begin{bmatrix} \frac{\partial^2\Sigma}{\partial\Theta_0\partial\Theta_0} & \frac{\partial^2\Sigma}{\partial\Theta_0\partial\Theta_1} & \frac{\partial^2\Sigma}{\partial\Theta_0\partial\Theta_2} \\ \frac{\partial^2\Sigma}{\partial\Theta_1\partial\Theta_0} & \frac{\partial^2\Sigma}{\partial\Theta_1\partial\Theta_1} & \frac{\partial^2\Sigma}{\partial\Theta_1\partial\Theta_2} \\ \frac{\partial^2\Sigma}{\partial\Theta_2\partial\Theta_0} & \frac{\partial^2\Sigma}{\partial\Theta_2\partial\Theta_1} & \frac{\partial^2\Sigma}{\partial\Theta_2\partial\Theta 2} \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \frac{\partial\Sigma}{\partial\Theta_0} \\ \frac{\partial\Sigma}{\partial\Theta_1} \\ \frac{\partial\Sigma}{\partial\Theta_2} \end{bmatrix} \right]$$

$$\Gamma = \begin{bmatrix} \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_0} \\ \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_1} \\ \frac{\partial l(\alpha,\Sigma)}{\partial\,\Theta_2} \end{bmatrix} +$$

$$\frac{1}{2}\,\mathrm{tr}\left[\begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}^{-1}\left[\frac{1}{2}\,\mathrm{tr}\left[\Sigma^{-1}\begin{bmatrix} \frac{\partial^2 \Sigma}{\partial\Theta_0\partial\Theta_0} & \frac{\partial^2 \Sigma}{\partial\Theta_0\partial\Theta_1} & \frac{\partial^2 \Sigma}{\partial\Theta_0\partial\Theta_2} \\ \frac{\partial^2 \Sigma}{\partial\Theta_1\partial\Theta_0} & \frac{\partial^2 \Sigma}{\partial\Theta_1\partial\Theta_1} & \frac{\partial^2 \Sigma}{\partial\Theta_1\partial\Theta_2} \\ \frac{\partial^2 \Sigma}{\partial\Theta_2\partial\Theta_0} & \frac{\partial^2 \Sigma}{\partial\Theta_2\partial\Theta_1} & \frac{\partial^2 \Sigma}{\partial\Theta_2\partial 2} \end{bmatrix}\Sigma^{-1}\begin{bmatrix} \frac{\partial \Sigma}{\partial\Theta_0} \\ \frac{\partial \Sigma}{\partial\Theta_1} \\ \frac{\partial \Sigma}{\partial\Theta_2} \end{bmatrix}\right]\right]\right] \tag{12}$$

The mean bias reduction estimator $\widetilde{\Theta} = (\widetilde{\sigma^2_{\omega_1}}, \widetilde{\sigma^2_{\omega_2}}, \widetilde{\sigma^2_e})$ for the vector of variance Θ is found by solving equation (12) by Newton Raphson method as follows:

$$\Gamma_{\widetilde{\Theta}} = \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_r\partial\Theta_s} + \frac{1}{2}\Big(\mathrm{tr}\,\Big\{\frac{\partial E^{-1}}{\partial\Theta_r\partial\Theta_s\partial\Theta_q}\cdot\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s}\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\Big\} + \mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}$$
$$(-\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r}\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s}\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\} + \mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}\frac{\partial^3\Sigma}{\partial\Theta_r\partial\Theta_s\partial\Theta_q}\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\}$$
$$+\,\mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s} - \Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r}\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\} + \mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s}$$
$$\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s})\}\Big) = \Gamma_{\Theta\Theta} + \frac{1}{2}\,\mathcal{T} \tag{13}$$

Where $\Gamma_{\Theta\Theta} = \frac{\partial^2 l(\alpha,\Sigma)}{\partial\Theta_r\partial\Theta_s}$ $\mathcal{T} = \mathrm{tr}\,\{\frac{\partial E^{-1}}{\partial\Theta_r\partial\Theta_s\partial\Theta_q}\cdot\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s}\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\}$ $+\,\mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}\,(-\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r}\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s}\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\} + \mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}\frac{\partial^3\Sigma}{\partial\Theta_r\partial\Theta_s\partial\Theta_q}$ $\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\} + \mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s} - \Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r}\Sigma^{-1}\frac{\partial\,\Sigma}{\partial\,\Theta_r})\} + \mathrm{tr}\{E^{-1}.\frac{1}{2}\,\mathrm{tr}\,(\Sigma^{-1}$ $\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s}\Sigma^{-1}\frac{\partial^2\Sigma}{\partial\Theta_r\partial\Theta_s})\}$

The estimator vector $\widetilde{\Theta}$ by Mean bias reduction method for the one-way repeated measurements model is: $\widetilde{\Theta}_{k+1} = \widetilde{\Theta}_k + \Gamma_{\widetilde{\Theta}}^{-1}\,\Gamma$, $E(\widetilde{\Theta}_{k+1}) = E(\widetilde{\Theta}_k) + E(\Gamma_{\widetilde{\Theta}})^{-1}\,E(\Gamma)$ δ^* is the bias amount of the estimator $\widetilde{\Theta}$ where:

$$\delta^* = \frac{1}{N}\begin{bmatrix} \tilde{\sigma}^2_e \\ \tilde{\sigma}^2_{\omega_1} \\ \tilde{\sigma}^2_{\omega_2} \end{bmatrix} + E(\Gamma_{\widetilde{\Theta}}^{-1}\,\Gamma) \tag{19}$$

8. Conclusions

Through this study, it was observed that after deriving the estimator of the fixed effects of the one-way repeated measurements model using the maximum likelihood method, it was found that the estimator ($\hat{\alpha}$) is unbiased, while in the case of deriving the estimator of the variance for the study model, it was found that the estimator is biased. A new estimator was developed specifically for this scenario to cut down on this bias. Put another way, the variance estimator derived using the mean bias reduction approach has a lower degree of bias than the variance estimate obtained from the traditional maximum likelihood method.

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