Journal of Statistics & Management Systems
ISSN 0972-0510 (Print), ISSN 2169-0014 (Online)
Vol. 26 (2023), No. 8, pp. 1895–1904
DOI : 10.47974/JSMS-1085

Quadratic transmuted half logistic distribution (QTHLD)

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Abstract

In this paper, a proposed life time distribution is derived for analyzing data set in more efficient way. We construct a life time distribution using the quadratic transmuted technique of half logistic distribution and discuss some of its statistical characteristics such as the moment, graphs, and others statistical properties. We also employ a method of estimation known by the maximum likelihood estimate method (MLE) to estimate the parameters of the created distribution (QTHLD). Afterwards the optimal distribution among QTHLD and classical distribution for the data analysis has been evaluated with a real data set based on goodness of fit measures.

Subject Classification: *62Nxx, 62N99.*

Keywords: *Half logistic distribution, Quadratic transmuted half logistic distribution, Moments, MLE method, Goodness of fit measures.*

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1. Introduction

Although there are many distributions used to describe the data, the researchers were able to create new distributions that compared to traditional distribution and they are more flexible and they are more efficient. By combining two or more different existing distributions or giving them additional properties, these peculiar distributions are produced. This study presents a novel distribution of the interlocking half-logistic distributions (HLD) with quadratic transmuted method. Hence, we obtain a probability density function (pdf) formula of the QTHLD, and some of its properties were discussed. We used the MLE of the parameters of QTHLD and compare the results with half-logistic distribution through aropriate quality measures.

There are many researches and studies of the quadratic rank transmutation that were presented by several authors like Kareema and Maysaa which has presented the Cubic Transmuted of Weibull Distribution [10]. Aryal [6] has studied the transmuted log-logistic distribution. Merovci and Puka [8] has also discussed the transmuted Pareto distribution. Shaw and Buckley [5] have focused on the studies of alchemy for probability distribution. Balakrishnan [1] demonstrated the order statistics aspects of the half logistic distribution. Balakrishnan and Puthenpura [2] obtained the most accurate unbiased estimators of the distributions local on and scale parameter of the half logistic distribution, Adeyinka and Olapade studied Transmuted Half Logistic Distribution [11].

The organization of this paper is as follow: In the next section, we set some basic concepts related to the transmuted techniques, properties of half-logistic distribution, some basic formulas of binomial and exponential series, and rules of goodness of fits. In the third section, our main results are presented, and some figures, the MLE method equations mentioned in the section four, the section five consist the table of our numerical results and clarify the contents of this table, as well as the most important results that we reached in this paper, the conclusion are found in Section six.

2. Basic Concepts

A quadratic transmuted distribution of a random variable X is given by the formula in equation (1), [10]

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), \quad |\lambda| \leq 1 \tag{1}$$

While the pdf of that type of distribution is given by

$$f(x) = (1+\lambda)g(x) - 2\lambda G(x)g(x), \tag{2}$$

Moreover, some other fundamental properties like the pdf and the cumulative distribution function (cdf) given by,[9].

$$g(x) = \frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2}, 0 < x \langle \infty, \theta \rangle 0 \tag{3}$$

$$G(x) = \frac{1-e^{-\theta x}}{1+e^{-\theta x}}, 0 < x \langle \infty, \theta \rangle 0 \tag{4}$$

Moreover, measures of goodness-of-fit are used to determine the best distribution that can be used to fit some selected data set. The forms of these measures are given by in the equations (9)-(12),[12].

$$AIC = -2\hat{\ell} + 2k \tag{9}$$

$$BIC = -2\hat{\ell} + k\ln(m) \tag{10}$$

$$CAIC = -2\hat{\ell} + \frac{2km}{m-k-1} \tag{11}$$

$$HQIC = -2\hat{\ell} + 2k\ln(\ln(m)) \tag{12}$$

where $\hat{\ell}$ is log-likelihood function, m is sample size, while k represents the number of the parameters of the desired distribution.

3. Derivations and Characteristics

3.1 *Quadratic Transmuted of Half Logistic Distribution*

In this part, we firstly show the construction of QTHLD by substituting equation (4) into equation (1) to we obtain the following transmuted cdf

$$F(x;\lambda,\theta) = (1+\lambda)\left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right) - \lambda\left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^2, x > 0, \ |\lambda| \le 1, \theta > 0 \tag{13}$$

For which λ is the transmuted parameter, and θ for shape parameters.

Also, we obtain the pdf of transmuted of QTHLD by substituting equation (4), and (3) into equation (2), which has the following pdf

$$f(x;\lambda,\theta) = \frac{2(1-\lambda+(1+3\lambda)e^{-\theta x})\theta e^{-\theta x}}{(1+e^{-\theta x})^3} 0 < x \langle \infty, \ |\lambda| \le 1, \theta \rangle 0 \tag{14}$$

We see that $f(x;\lambda,\theta)$ holds the properties of being pdf:

1. $f(x;\lambda,\theta) \geq 0, \forall x \geq 0$
2. $\int_{-\infty}^{\infty} f(x;\lambda,\theta)\,dx = 1$

3.2 *Characteristics of (QTHLD)*

In this part, we present through the cdf and pdf that we have obtained in equations (13), and (14), respectively, we may find other statistical characteristics of the QTHLD including survival function $S(x)$, hazard rank function $h(x)$, and the cumulative hazard rank function $H(x)$.

$$S(x;\lambda,\theta) = \frac{2(1-\lambda+(1+\lambda)e^{-\theta x})e^{-\theta x}}{(1+e^{-\theta x})^2} \quad 0 < x \langle \infty, \ |\lambda| \leq 1, \theta \rangle 0 \tag{15}$$

$$h(x;\lambda,\theta) = \frac{\theta[1-\lambda+(1+3\lambda)e^{-\theta x})]}{[1-\lambda+(1+\lambda)e^{-\theta x}](1+e^{-\theta x})} \tag{16}$$

$$H(x;\lambda,\theta) = -\ln(2) - \ln(1-\lambda+(1+\lambda)e^{-\theta x}) + \theta x + 2\ln(1+e^{-\theta x}) \tag{17}$$

Furthermore, we could derived other formulas of the QTHLD distribution like moments about the origin and the moments about the mean as we have stated in the following theorems:

Theorem 3.1: *The r^{th} moment of the random variable X from the QTHLD about the origin is provided by*

$$\mu_r' = E(x^r) = \frac{2}{\theta^r} \sum_{i=0}^{\infty} \binom{i+2}{i} (-1)^i \left[\frac{\Gamma(r+1)}{(1+i)^{r+1}} - \lambda \frac{\Gamma(r+1)}{(1+i)^{r+1}} + (1+3\lambda) \frac{\Gamma(r+1)}{(2+i)^{r+1}} \right] \tag{18}$$

Proof: By recalling equation(14),we have

$$E(x^r) = \int_0^{\infty} x^r f(x;\lambda,\theta)dx = \int_0^{\infty} x^r \left[2(1-\lambda+(1+3\lambda)e^{-\theta x})\theta e^{-\theta x}(1+e^{-\theta x})^{-3}\right]dx \tag{19}$$

$$= 2\theta \int_0^{\infty} x^r e^{-\theta x}(1+e^{-\theta x})^{-3}dx - 2\lambda\theta \int_0^{\infty} x^r e^{-\theta x}(1+e^{-\theta x})^{-3}dx$$
$$-2\theta(1+3\lambda)\int_0^{\infty} x^r e^{-2\theta x}(1+e^{-\theta x})^{-3}dx$$

By solving the integrals in (19) and using some techniques that depends on binomial and negative binomial expansions in (5), and (6), the proof is complete and we directly obtain the formula of r^{th} moment about the origin.

In particular, let $r = 1$, we obtain the 1st moment. Then

$$E(x) = \frac{2}{\theta}\sum_{i=0}^{\infty}\binom{i+2}{i}(-1)^i\left[\frac{\Gamma(2)}{(1+i)^2} - \lambda\frac{\Gamma(2)}{(1+i)^2} + (1+3\lambda)\frac{\Gamma(2)}{(2+i)^2}\right] \tag{20}$$

And to find the variance, we require $E(x^2)$

$$E(x^2) = \frac{2}{\theta^2}\sum_{i=0}^{\infty}\binom{i+2}{i}(-1)^i\left[\frac{\Gamma(3)}{(1+i)^3} - \lambda\frac{\Gamma(3)}{(1+i)^3} + (1+3\lambda)\frac{\Gamma(3)}{(2+i)^3}\right] \tag{21}$$

Theorem 3.2: *The r^{th} moment about the mean of the random variable Xfrom QTHLD is provided by*

$$\begin{aligned}\mu_r &= E(x-\mu)^r \\ &= \frac{2}{\theta^k}\sum_{k=0}^{r}\sum_{i=0}^{\infty}\binom{i+2}{i}(-1)^{i+r-k}\binom{r}{k}\mu^{r-k}\left[\frac{\Gamma(k+1)}{(1+i)^{k+1}} - \lambda\frac{\Gamma(k+1)}{(1+i)^{k+1}} + (1+3\lambda)\frac{\Gamma(k+1)}{(2+i)^{k+1}}\right]\end{aligned} \tag{22}$$

Proof:

$$\begin{aligned}E(x-\mu)^r &= \int_0^{\infty}(x-\mu)^r f(x;\lambda,\theta)dx \\ &= \int_0^{\infty}(x-\mu)^r[2(1-\lambda+(1+3\lambda)e^{-\theta x})\theta e^{-\theta x}(1+e^{-\theta x})^{-3}]dx\end{aligned} \tag{23}$$

By using the binomial expansion formula, we obtain that

$$(x-\mu)^r = \sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\mu^{r-k}x^k \tag{24}$$

By substituting the formula of equation (24) into equation (23), we obtain that

$$E(x-\mu)^r = \sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\mu^{r-k}\int_0^{\infty}x^k[2(1-\lambda+(1+3\lambda)e^{-\theta x})\theta e^{-\theta x}(1+e^{-\theta x})^{-3}]dx$$

$$= \sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\mu^{r-k}\mu'_k \tag{25}$$

where, μ'_k represents the kth moment about the origin of the QTHLD, and so we obtain that the r^{th} moment about the mean of the QTHLD is as given in equation (22) you will go down a step. On the other hand, Figures 1,2,3 display plots of cdf, pdf and survival function of the QTHLD for various settings.

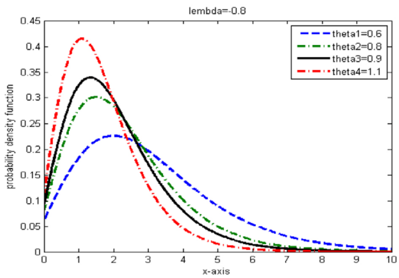

Figure 1

QTHLD pdf with different values of parameter θ

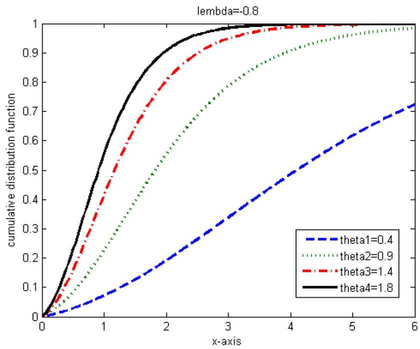

Figure 2

QTHLD cdf with different values of parameter θ

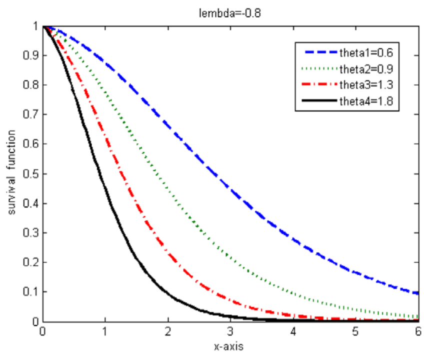

Figure 3

QTHLD survival function with different values of parameter θ

4. Maximum Likelihood Estimation of QTHLD

In this part, we focus on the estimation of the model parameters for the QTHLD which is produced by the MLE. If $X_1, X_2, \ldots, X_n$ are random sample from QTHLD, then the likelihood function for such distribution is obtain by:

$$L(\mathrm{v}) = \prod_{i=1}^{n} f(x_i; \lambda, \theta) \tag{26}$$

By substituting the pdf in (14) into (26), we obtain the likelihood function, that is

$$L(\lambda,\theta,\underline{x})=\prod_{i=1}^{n}2(1-\lambda+(1+3\lambda)e^{-\theta x_i})\theta e^{-\theta x_i}(1+e^{-\theta x_i})^{-3} \tag{27}$$

$$=2^n\theta^n e^{-\theta\sum_{i=0}^{\infty}x_i}\prod_{i=1}^{n}(1-\lambda+(1+3\lambda)e^{-\theta x_i})\prod_{i=1}^{n}(1+e^{-\theta x_i})^{-3} \tag{28}$$

By taking the natural logarithm of the likelihood function in (28), we have

$$\ell=\ln L=n\ln 2+n\ln\theta-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\ln(1-\lambda+(1+3\lambda)e^{-\theta x_i})$$
$$-3\sum_{i=1}^{\infty}\ln(1+e^{-\theta x_i}) \tag{29}$$

By differentiating ℓ partially with respect to $\theta,\lambda,$ respectively, and equalize the derivatives to zero, we obtain the following forms in (30), and (31)

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{(1+3\lambda)(-x_ie^{-\theta x_i})}{1-\lambda+(1+3\lambda)e^{-\theta x_i}}-3\sum_{i=1}^{n}\frac{-x_ie^{-\theta x_i}}{(1+e^{-\theta x_i})}=0 \tag{30}$$

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{n}\frac{-1+3e^{-\theta x_i}}{1-\lambda+(1+3\lambda)e^{-\theta x_i}}=0 \tag{31}$$

Certainly, the nonlinear equations above can be solved numerically in order to find the estimates of λ and θ. In fact, one of the best numerical methods related to MLE is the Newton-Raphson method.

5. Application

The information represents the duration of 72 Guinea pigs' survival contaminated dangerous bacterial tubercle. The data set was gathered through the efforts of Usman, Hag and Talib (2017) [9].

The date set can be listed as seen below:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93,0.96, 1.00, 1.00, 1.02, 1.05, 1.07, 0.7, 0.08, 1.08, 1.08, 1.09,1.12, 1.13, 1.15, 1.16, 1.20, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34,1.36, 1.39, 1.44, 1.46, 1.53,1.59, 1.60, 1.63, 1.63, 1.68, 1.71,1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22,2.302.31,2.40, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27,3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 1 displays the model performances.

Information criterion of Akaike (AIC), Akaike information criterion (CAIC) and other are employed in order to achieve the best vision, compare the old half logistic distribution to the QTHLD. To estimate parameters of the QTHLD, we use Matlab softeware to find the solution of non-linear equations in equations (30), (31). The estimated parameter values are provided in the following table 1.

Table 1

Results of goodness-of fit of QTHLD, and HLD distributions

Model	MLE	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
HLD	$\hat{\theta} = 4.838$	-449.193	900.3862	902.6628	900.4433	901.2925
QTHLD	$\hat{\theta} = 1.173$ $\hat{\lambda} = 0.569$	-132.868	269.7377	274.2910	269.9116	271.5504

Table 1 shows the values of estimated parameters, log-likelihood function and the four distinct goodness-of-fit. As a result, from Table 1 we can say that the log-likelihood, AIC, BIC, CAIC, and HQIC of the QTHLD are superior to those of the HLD in respect of these criteria. That means the transmuted distribution is more benefit than the classical distribution to analyze lifetime data

6. Conclusion

In this paper, we generated a new distribution by the quadratic transformation method, and obtained a transformed semi-logistical distribution (QTHLD). We perform the obtained distributions on a real data set. A comparison of the data set analysis between the classical distribution and the QTHL distribution will be achieved. By looking at the values of the quality standards for each of the two mentioned distributions, we find that the generalization of the half logistical distribution is better than the classical distribution.

References

[1] Balakrishnan N. Order Statistics from the half logistic distribution. *Journal of Statistical Computation and Simulation*. 20(4): 287-309 (1985).

[2] Balakrishnan N., Puthenpura, S. Best linear unbiased estimators of location and scale parameters of the half logistic distribution. *Journal of Statistical Computation and Simulation*, 25, 193-204 (1986).

[3] Hogg, R. V., Craig, A. T. Introduction to mathematical statistics.(5 edition). Englewood Hills, New Jersey (1995).

[4] Ivady, P. A note on a gamma function inequality. *J. Math. Inequal*, 3(2), 227-236 (2009).

[5] Shaw, W. T, Buckley, I. R. Alchemy of ProbabilityDistributions: Beyond Gram-Charlier and Cornish -FisherExpansions, and Skewed-kurtotic Normal Distribution from aRank Transmutation Map. arxivpreprint arxiv: 0901.0434 (2009).

[6] Aryal, G.R. Transmuted log-logistic distribution. *Journal of Statistics Alications, Probability An International Journal*, 1, 11-20 (2013).

[7] Rinne, H. The Hazard Rate. Theory and Inference, With sulementary MATLAB-Programs, Justus-Liebig University (2014).

[8] Merovci, F., Puka, L. Transmuted Pareto Distribution.Probstat. 7, 1-11 (2014).

[9] Usman, R. M, Haq, M. A, Talib, J. KumaraswamyHalf-Logistic Distribution: Properties and Alications. *Journal of Statistics Alications and Probability*. 3, 597-609 (2017).

[10] Kareema A.A, Maysaa H.M, The Cubic Transmutea weibull Distribution, 862-876 (2017).

[11] Adeyinka F. S, Olapade A. K, A Study on Transmuted Half Logistic Distribution Properties and Alication, 2472-3487 (2019).

[12] Badmus, N. I., Amusa, S. O., Ajiboye, Y.O. Weibull-Extended Pranav Distribution: Alication To lifetime Data Sets. *Unilag Journal Of Mathematics And Alications*, 1(1), 104-120 (2021).

[13] M.R. Farahani, S. Jafari, S.A. Mohiuddine, M. Cancan. Intuitionistic Fuzzy Stability of Generalized Additive Set-Valued Functional Equation via Fixed Point Method. Mathematical Statistician and Engineering Applications, 71(3s3), 142-154. (2022). https://www.philstat.org.ph/index.php/MSEA/article/view/355.

[14] R.F. Kadam, K.M.M. Al-Abrahemee. Neuro-fuzzy system for solving fuzzy singular perturbation problems. *Journal of Interdisciplinary Mathematics*. 25(5), 1509-1524 (2022). https://doi.org/10.1080/09720502.2022.2079229.

[15] D.E. Abdulrasool, A.N. Alkiffai. Population growth equation by fuzzy common integral transforms. *Journal of Interdisciplinary Mathematics.* 25(6), (2022) 1909-1918. https://doi.org/10.1080/09720502.2022.2095960.

[16] S. Ediz,İ. Çiftçi,M. Cancan,M.R.Farahani, On k-total distance degrees and k-total Wiener polarity index. *Journal of Information and Optimization Sciences*. 42(7), 1469-1477 (2021). https://doi.org/10.1080/02522667.2021.1896652.

[17] K. Mahesh Krishna, P. Sam Johnson. New identity on Parseval p-approximate Schauder frames and applications. *Journal of Interdisciplinary Mathematics*. 24(7), 1751-1760 (2021). https://doi.org/10.1080/09720502.2021.1891698.

[18] M.B. Kadhem, M.R. Nasif. Quadratic non-polynomial spline approximation for non-linear Volterra-Fredholm integral equations of the second kind. *Journal of Interdisciplinary Mathematics*. 25(5), 1383-1390 (2022), https://doi.org/10.1080/09720502.2022.2046333.

Received February, 2023

Revised April, 2023