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Cubic transmuted half-logistic distribution (CTHLD)

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Abstract

This study is concerned with a generalization of a half-logistic distribution by using a cubic transmuted technique. This combination is denoted by (CTHLD). Several properties like moments, survival function, hazard function, and moments have been shown. Estimation of the obtained parameters model is found by the maximum likelihood estimate (MLE). The goodness-of-fit measures have implemented to data set to test the best distribution that fit the desired data.

Subject Classification: *62Nxx, 62N99.*

Keywords: *Half-logistic distribution, Transmuted technique, Statistical inference, Estimation methods, Goodness of-fit measures.*

1. Introduction

Ten years ago, transmutation of baseline distributions has been increasingly popular. Many researchers have examined many transmuted distributions, including the exponential, Weibull, gamma, Pareto, and

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normal distributions, among others. The researchers were able to develop novel distributions that when compared to classical distributions are more accurate in describing the data. These unordinary distributions are generated by adding additional traits to the existing distributions. We developed half-logistic distributions (HLD) using the cubic transmuted approach in this paper. As a result, we derived the probability density function (pdf) formula for the CTHLD, and some of its characteristics. We used the MLE to estimate CTHLD's parameters and contrasted the outcomes with half-logistic distributions using the appropriate quality metrics. Several authors have proposed a new technique for creating families of distribution, Show and Bukley [15] used the rank transmutation map. The ranking quadratic transformation map's cumulative distribution function, in their estimation, is given by:

$$F_T(x) = (1+\lambda)F(x) - \lambda F^2(x), \qquad \lambda \in [-1,1] \tag{1}$$

It is clear that function in equation (1.1) satisfies the properties of being a cumulative distribution function (cdf). Also, we can easily notice that cdf above turns into the original one when . Adeyinka and Olapade [1] created a transmuted half-logistic distribution using the quadratic transmuted HLD of Equation (1.1). Aryal [4] has discussed the transmuted log-logistic distribution. Olapade [11] has studied the characterizations of the half-logistic distribution. Numerous authors, including Kareema and Maysaa [10] who have presented the Cubic rank transmutation, have conducted numerous investigations and explorations in this area. Granzotto, Louzada and Balakrishnan [6] have focused on the studies cubic rank transmuted distributions. In particular, the general formula of transmuted technique which is studied in [15] and [3], has the cdf of the cubic degree, as shown below:

$$F_T(x) = \lambda F^3(x) - 2\lambda F^2(x) + (1+\lambda)F(x)\lambda \in [-1,1] \tag{2}$$

This approach was used by Rahman,Al- Zahran and Shahbas [12] have studied Cubic Transmuted Weibull distribution. Rahman,Al-Zahran and Shahbas [13] have studied Cubic Transmuted pareto distribution. And Adeyinka [2] has Lesson-related logistical distribution of type I and its applications.

The following is how this essay is organized: The following section introduces some fundamental ideas about transmuted approaches, half-logistic distribution characteristics, and some definitions. Formula of exponential and binomial series, as well as goodness of fit guidelines, the section three consist some properties of CTHLD and some figures, the parameters estimation by MLE method found in section four, key findings are given in the fifth section, The conclusion is mentioned in the sixth section.

2. Background

A random variable X from HLD has a pdf, and cdf, respectively as defined in equations (3), and (4), [16].

$$f(x) = \frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2}, 0 < x \langle \infty, \theta \rangle 0 \tag{3}$$

$$F(x) = \frac{1-e^{-\theta x}}{1+e^{-\theta x}}, 0 < x \langle \infty, \theta \rangle 0 \tag{4}$$

By recalling the formula of the cdf in equation (2), we can obtain the pdf formula of cubic transmuted distribution which is showin in (5).

$$f_T(x) = 3\lambda F^2(x) f(x) - 4\lambda F(x) f(x) + (1+\lambda) f(x), \tag{5}$$

Another notion is related to the binomial expansion series that is

$$(c+d)^n = \sum_{i=0}^{n} \binom{n}{i} d^i c^{n-i} \tag{6}$$

Also, the negative binomial expansion is another important expansion series, which is given by [7,8]

$$(1-d)^{-s} = \sum_{x=0}^{\infty} \binom{x+s-1}{x} d^x \tag{7}$$

Additionally, we refer to the gamma pdf, which is defined by [9]

$$g(x) = \frac{1}{\Gamma_{\propto} \beta^{\propto}} (x)^{\alpha-1} e^{-\frac{x}{\beta}}, x > 0 ; \propto, \beta > 0 \tag{8}$$

Where
$$\int_0^{\infty} \frac{1}{\Gamma_{\propto} \beta^{\propto}} (x)^{\alpha-1} \mathrm{e}^{-\frac{x}{\beta}} dx = 1 \tag{9}$$

Also, we need to recall to the goodness-of-fit measures to determine which distribution is more suit data set, [5].

$$AIC = -2\hat{\ell} + 2h \tag{10}$$

$$BIC = -2\hat{\ell} + h\ln(n) \tag{11}$$

$$CAIC = -2\hat{\ell} + \frac{2hn}{n-h-1} \tag{12}$$

$$HQIC = -2\hat{\ell} + 2h\ln(\ln(n)) \tag{13}$$

A log-likelihood function is represented by $\hat{\ell}$, n is the sample size, the number of the parameters of distributions is represented by h.

3. Derivations and Properties

3.1 *Cubic Transmuted Half-Logistic Distribution*

On this page, we firstly show the construction of CTHLD by substituting equation (4) into equation (2), which provides the following transmuted cdf

$$F_T(x;\lambda,\theta) = \frac{(1-e^{-\theta x})((1+e^{-\theta x})^2+4\lambda e^{-2\theta x})}{(1+e^{-\theta x})^3}, x>0, \theta>0, \lambda\in[-1,1] \tag{14}$$

Where is the transmuted parameter, θ is the shape parameters. Also, we obtain the pdf of transmuted of CTHLD by substituting equation (3), and (4) into equation (5), which has the following pdf

$$\begin{aligned} f_T(x;\lambda,\theta) = 6\lambda\theta e^{-\theta x}\frac{(1-e^{-\theta x})^2}{(1+e^{-\theta x})^4} - 8\lambda\theta e^{-\theta x}\frac{(1-e^{-\theta x})}{(1+e^{-\theta x})^3} \\ + \frac{2\lambda\theta e^{-\theta x}}{(1+e^{-\theta x})^2} + \frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2} 0 < x\langle\infty, \theta\rangle 0, \end{aligned} \tag{15}$$

We see that the function in equation (15) holds the properties of being pdf:

1. $f_T(x;\lambda,\theta) \geq 0, \forall x \geq 0$
2. $\int_{-\infty}^{\infty} f_T(x;\lambda,\theta)dx = 1$

3.2 *Statistical Properties of (CTHLD)*

This section provides through the cdf and pdf in equations (14), and (15) other statistical characteristics of the CTHLD including the reliability function $S_T(x)$, hazard rank function $h_T(x)$, and the cumulative hazard function $H_T(x)$.

$$S_T(x;\lambda,\theta)=\frac{(1+e^{-\theta x})^3-(1-e^{-\theta x})((1+e^{-\theta x})^2+4\lambda e^{-2\theta x})}{(1+e^{-\theta x})^3} \tag{16}$$

$$h_T(x;\lambda,\theta)=\frac{2\theta e^{-\theta x}(1+e^{-\theta x})\left[\frac{3\lambda(1-e^{-\theta x})^2}{(1+e^{-\theta x})^2}-4\lambda\frac{(1-e^{-\theta x})}{(1+e^{-\theta x})}+\lambda+1\right]}{(1+e^{-\theta x})^3-(1-e^{-\theta x})((1+e^{-\theta x})^2+4\lambda e^{-2\theta x})} \tag{17}$$

$$H_T(x;\lambda,\theta)=-\ln[(1+e^{-\theta x})^3 -(1-e^{-\theta x})((1+e^{-\theta x})^2+4\lambda e^{-2\theta x})]+3\ln(1+e^{-\theta x}) \tag{18}$$

Also, we have derived the formulas for moments about the origin and the moments about the mean of the CTHLD, respectively given by Figures 1,2,3 display plots of $f_T(x;\lambda,\theta)$, $F_T(x;\lambda,\theta)$ and $S_T(x;\lambda,\theta)$ of the CTHLD for various settings.

$$\mu'_r=E(x^r)=\frac{2}{\theta^r}\left[3\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{2}{j}\binom{k+3}{k}(-1)^{j+k}\frac{\Gamma(r+1)}{(j+k+1)^{r+1}} -4\lambda\sum_{h=0}^{\infty}\binom{h+2}{h}(-1)^h\left(\frac{\Gamma(r+1)}{(1+h)^{r+1}}-\frac{\Gamma(r+1)}{(2+h)^{r+1}}\right)+\sum_{i=0}^{\infty}\binom{i+1}{i}(-1)^i\frac{\Gamma(r+1)}{(1+i)^{r+1}}(\lambda+1)\right] \tag{19}$$

$$\mu_r=E(x-\mu)^r=\frac{2}{\theta^n}\sum_{n=0}^{r}\binom{r}{n}(-1)^{r-n}\mu^{r-n}\left[3\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{2}{j}\binom{k+3}{k}(-1)^{j+k}\frac{\Gamma(n+1)}{(j+k+1)^{n+1}} -4\lambda\sum_{h=0}^{\infty}\binom{h+2}{h}(-1)^h\left(\frac{\Gamma(n+1)}{(1+h)^{n+1}}-\frac{\Gamma(n+1)}{(2+h)^{n+1}}\right)+\sum_{i=0}^{\infty}\binom{i+1}{i}(-1)^i\frac{\Gamma(n+1)}{(1+i)^{n+1}}(\lambda+1)\right] \tag{20}$$

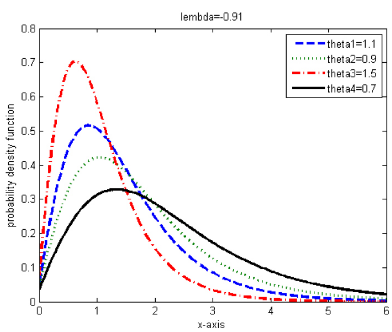

Figure 1

The $f_T(x;\lambda,\theta)$ of (CTHLD) with fixed parameter λ

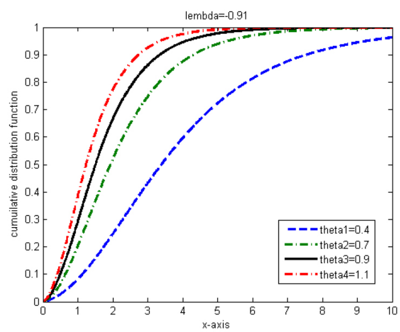

Figure 2

The $F_T(x;\lambda,\theta)$ of the (CTHLD) with fixed parameter λ

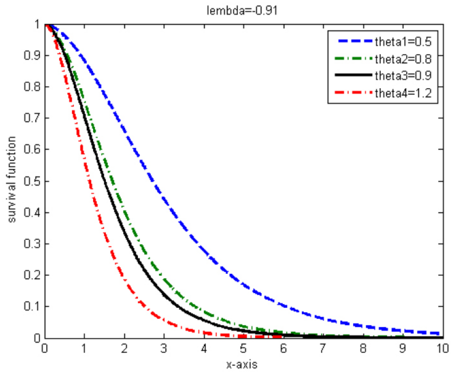

Figure 3

The $S_T(x;\lambda,\theta)$ of the (CTHL) with fixed parameter λ

4. Parameters Estimation of CTHLD

In this section, we show the estimation of the model parameters for the CTHLD which is produced by the MLE. If $X_1, X_2, \ldots, X_n$ are random samples from CTHLD, then the likelihood function for such distribution is obtain by:

$$L(\lambda,\theta,\underline{x}) = \prod_{i=1}^{n} f_T(x_i;\lambda,\theta) \tag{21}$$

By simplifying the pdf in equation (15), we get a more appropriate form as follows

$$f_T(x_i;\lambda,\theta) = 2\theta e^{-\theta x_i}(1+e^{-\theta x_i})^{-2}[(8\lambda e^{-2\theta x_i} - 4\lambda e^{-\theta x_i})(1+e^{-\theta x_i})^{-2}+1]$$

then we substitute it into equation (21), we obtain the likelihood function, that is

$$\begin{aligned} L(\lambda,\theta,\underline{x}) &= \prod_{i=1}^{n} 2\theta e^{-\theta x_i}(1+e^{-\theta x_i})^{-2}[(8\lambda e^{-2\theta x_i} - 4\lambda e^{-\theta x_i})(1+e^{-\theta x_i})^{-2}+1] \\ &= 2^n \theta^n e^{-\theta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(1+e^{-\theta x_i})^{-2} \prod_{i=1}^{n}[(8\lambda e^{-2\theta x_i} - 4\lambda e^{-\theta x_i})(1+e^{-\theta x_i})^{-2}+1] \end{aligned} \tag{22}$$

The log-likelihood function is given as

$$\begin{aligned} \ell = \ln L = n\ln 2 + n\ln\theta - \theta\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}\ln(1+e^{-\theta x_i}) \\ + \sum_{i=1}^{n}\ln[(8\lambda e^{-2\theta x_i} - 4\lambda e^{-\theta x_i})(1+e^{-\theta x_i})^{-2}+1] \end{aligned}$$

By differentiating ℓ partially for θ and λ, equalize the derivatives to the number 0, we obtain the following forms in (24), and (25)

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + 2\sum_{i=0}^{\infty} \frac{x_i e^{-\theta x_i}}{(1+e^{-\theta x_i})}$$

$$+\sum_{i=1}^{\infty} \frac{2x_i e^{-\theta x_i}(8\lambda e^{-2\theta x_i} - 4\lambda e^{-\theta x_i})(1+e^{-\theta x_i})^{-3} + (1+e^{-\theta x_i})^{-2}(-16\lambda x_i e^{-2\theta x_i} + 4\lambda x_i e^{-\theta x_i})}{[(8\lambda e^{-2\theta x_i} - 4\lambda e^{-\theta x_i})(1+e^{-\theta x_i})^{-2}+1]} \quad (24)$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{\infty} \frac{(1+e^{-\theta x_i})^{-2}(8e^{-2\theta x_i} - 4e^{-\theta x_i})}{(8\lambda e^{-2\theta x_i} - 4\lambda e^{-\theta x_i})(1+e^{-\theta x_i})^{-2}+1} = 0 \quad (25)$$

Certainly, the nonlinear equations above can be solved numerically in order to find the estimates of and . In fact, one of the best numerical methods related to MLE is the Newton-Raphson method.

5. Application

The data shows the duration of 72 Guinea pigs survival contaminated dangerous bacterial tubercle. Usman, Hag and Talib (2017) [16] collected the data set

Following is a list of the set date:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93,0.96, 1.00, 1.00, 1.02, 1.05, 1.07, 0.7, 0.08, 1.08, 1.08, 1.09,1.12, 1.13, 1.15, 1.16, 1.20, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34,1.36, 1.39, 1.44, 1.46, 1.53,1.59, 1.60, 1.63, 1.63, 1.68, 1.71,1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22,2.302.31,2.40, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27,3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 1 shows the model performances from where the estimated parameter values, the log-likelihood and goodness-of-fit criteria (the Akaike Information criterion (AIC), the Corrected Akaike Information criterion (CAIC), the Bayesian Information criterion (BIC), and (HQIC)) shown in table 2. In order to achieve the best distribution, we compare the half-logistic distribution and (CTHLD). These results are created using MatLab.

Table 1

Values of estimated parameters of CTHLD, and HLD

Distribution	Estimated Parameters
HLD	$\hat{\theta} = 4.713$
CTHLD	$\hat{\theta} = 1.14$, $\hat{\lambda} = 0.8$

Table 2

Results of Log-Likelihood and goodness-of fit of HLD, and CT HLD

Distribution	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
HLD	-435.4542	872.9084	875.1851	872.9656	873.8148
CTHLD	-124.0124	252.0249	256.5782	252.1988	253.8376

As a result of Table 2, we can conclude that the CTHLD's log-likelihood, and (AIC, BIC, CAIC, and HQIC) are superior to those of the HLD in terms of these criteria. This means that when analyzing lifetime data, the transmuted distribution is more advantageous than the classical distribution.

6. Conclusion

In this article, the cubic transmuted half-logistic distribution has been proposed. In addition, the pdf, the cdf, survival function, hazard function, and others were found, as well as some of these functions were drawn. In order to estimate the parameters of each of the classical distribution and the transmuted distribution, we used the maximum likelihood method. The distributions were compared using quality criterions, where we concluded that the transmuted distribution is better in suiting the data analysis than the classical distribution.

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