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A systematic weighted-Hungarian-algorithm for optimization and postoptimal analysis of transportation problem

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Abstract

This paper first proposes an easy algorithm for optimizing the transportation problem. Then, optimization procedures of the algorithm are applied for sensitivity analysis and the parametric analysis. The efficient algorithm can be proceeded systematically and smoothly. Some numerical examples are given to demonstrate these procedures. The attractive features of the new algorithms include: (1) The algorithm for solving the optimal solution of the transportation problem is an easy weighted Hungarian algorithm that can be applied to transportation problem and assignment problem; (2) The algorithm conquer the obstacles, e.g. cycling and stalling etc., caused by degeneracy; (3) The algorithm is capable of solving the large scale transportation problem; (4) The algorithm is easily expended to identify the Type I, Type II and Type III sensitivity range perturbing one cost coefficient; (5) The algorithm can reoptimize by using the original optimal table when variation exceeding sensitivity range, needless to resolve from scratch; (6)The algorithm can perform the parametric analysis and determine the optimal value function perturbing one or multiple cost coefficients.

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I. Introduction

The transportation problem (TP) is a well-known model for practical applications in supply chains, production scheduling, sampling surveys etc. [2, 4, 17]. The problem is to find the shipping routes of a specific commodity that minimizes total transportation cost while satisfying the amounts of supply and demand. It usually assumes that there are m supply points ($A_1, A_2, \ldots, A_m$) and n demand points ($B_1, B_2, \ldots, B_n$) in the TP. Point A_i can supply s_i amounts, and point B_j demands d_j amounts. Further assume total supply equals total demand, i.e. satisfied the balanced condition $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. Denote c_{ij} as the unit transportation cost (without loss of generality, negative value is allowed) from A_i to B_j. Then the TP can be formulated as a linear programming problem with standard form as follows:

$$
\begin{aligned}
\min \; & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \\
\text{s.t.} \; & \sum_{j=1}^{n} x_{ij} = s_i \quad \text{for } i = 1, 2, \ldots, m, \\
& \sum_{i=1}^{m} x_{ij} = d_j \quad \text{for } j = 1, 2, \ldots, n, \\
& \forall x_{ij} \geq 0 \quad \text{for all } i \text{ and } j,
\end{aligned}
\tag{1}
$$

where x_{ij} is the decision variable and represents the shipping amounts from A_i to B_j. Denote matrix X^* as the optimal solution of (1). By the way,

Table 1

The transportation tableau

	B_1	$\cdots$	B_j	$\cdots$	B_n	
A_1	c_{11} x_{11}	$\cdots$	c_{1j} x_{1j}	$\cdots$	c_{1n} x_{1n}	s_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
A_i	c_{i1} x_{i1}	$\cdots$	c_{ij} x_{ij}	$\cdots$	c_{in} x_{in}	s_i
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
A_m	c_{m1} x_{m1}	$\cdots$	c_{mj} x_{mj}	$\cdots$	c_{mn} x_{mn}	s_m
	d_1	$\cdots$	d_j	$\cdots$	d_n	

the famous and wide-used assignment problem and semi-assignment problem are both special and inherently degenerate TPs.

TP is usually solved in a transportation tableau, as shown in Table 1. There are $m \times n$ cells in the interior of the table. In the bottom-right corner of these cells are decision variable x_{ij}'s, and cost coefficient c_{ij}'s are in the top-left corner of each cell. The m right-margin cells are the supply cells and n bottom-margin cells are the demand cells. The amounts of supply and demand are in their cells, respectively.

Suppose u_i and v_j are dual variables associate to row i and column j of table 1, respectively. The dual problem of (1) can be formulated as follows:

$$\begin{array}{lll} \max & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j & \\ \text{s.t.} & u_i + v_j \le c_{ij} & \text{for } i = 1,\ldots,m;\ j = 1,\ldots,n \\ & u_i, v_j \text{ unrestricted} & \text{for } i = 1,\ldots,m;\ j = 1,\ldots,n \end{array} \tag{2}$$

Many algorithms have proposed to determine the optimal solution of TP through (1) and/or (2). This paper proposes an easy algorithm, may be regarded as a kind of refined weighted Hungarian algorithm, that can optimize the transportation problem, and then, applies it for opstoptimal analysis, includes sensitivity analysis (SeA) and the parametric analysis (PA). The rest of this paper is organized as follows. In Section II, we present a literature review of TP's optimization, TP's SeA and TP's PA. A systematically refined weighted Hungarian algorithm for TP will be proposed and introduced in details in Section III. In Section IV, procedures for TP's SeA are illustrated. Then, procedures based on the proposed algorithm for TP's PA are introduced in Section V. Section VI is some helpful aspects and conclusion of the paper.

II. Related works in Literature

Hitchcock [3] is credited with the TP formulation. Now, TP has diverse types of models and rises in various applications, not confined to transportation planning. The most popular solution method for model (1) is transportation simplex method [4]. Transportation simplex method separates the decision variables of a feasible solution $X = [x_{ij}]$ into basic variables and nonbasic variables, and calls it basic solution. In a basic feasible solution, there will be $(m-1)\times(n-1)$ nonbasic variables with zero values, and $(m+n-1)$ basic variables with nonnegative values. Transportation simplex method searches for a basic optimal solution and is composed of two phases as follows:

Phase 1: Search for an initial basic feasible solution.

Phase 2: Determine optimal basic solution using the initial solution.

Many heuristics are available to get an initial basic feasible solution. Well-known heuristics are Northwest corner rule, Least-cost method, Vogel's approximation method, and Russel's method, etc. For Phase 2, there are the stepping stone method and MODI method.

When basic variables equal to zero, it is called degeneracy occurred. Degeneracy is common in TP. Defects of transportation simplex method when degeneracy occur include [5, 8, 10, 11]:

[a] transportation simplex method is inefficient; or even worse, break down.

[b] sensitivity analysis gives unpractical sensitivity range or break down.

[c] shadow price may be wrong.

[d] re-optimization may fail.

[e] parameter analysis is inefficient or break down.

Number of degenerate basic variables are defined as degenerate degree. The larger degenerate degree makes the larger probability we meet the above obstacles.

Some dual simplex algorithms are proposed for model (2) [6, 18]. They need mass computation on matrix. The primal-dual algorithms [5, 9] are an alternative type of algorithm. They said these approaches don't have the degeneracy problem, whereas too many parameters in solution procedures make them complex. The other algorithms known as the weighted Hungarian algorithms [2, 18], based on the primal-dual algorithms, are given. Nevertheless, they lack systematic procedure to search lines with minimal total weight to cover all zero costs. There are still other optimization algorithms for TP [7, 11, 13] lacking the opstoptimal analysis skills.

After the optimal solution has been obtained and implemented, the opstoptimal analysis is starting up owing to the coefficients are often not precisely known. In SeA, this paper is interested in identifying one cost parameter is sensitive or insensitive through its sensitivity range. Three types of sensitivity ranges of TP are summarized as follows: [1, 10]

- Type I Sensitivity Range (Type I SR) , denoted as $[\dot{L}_{pq}, \dot{U}_{pq}]$ when perturbing c_{pq}, is the largest range by which the perturbed parameter can change so that the implemented optimal basic solution remains optimal, i.e. the optimal base unchanged.

- Type II Sensitivity Range (Type II SR) , denoted as $[\ddot{L}_{pq}, \ddot{U}_{pq}]$ when perturbing c_{pq}, is the largest range by which the perturbed parameter can change so that the implemented optimal shipping pattern of TP remains optimal, i.e. the optimal shipping pattern unchanged. The shipping pattern is composed of transportation routes with positive shipping amounts, exclude the degenerate basic variable in the optimal base. In AP, the shipping pattern is the assignment pattern.
- Type III Sensitivity Range (Type III SR) , denoted as $[\dddot{L}_{pq}, \dddot{U}_{pq}]$ when perturbing c_{pq} only, is the largest range by which the perturbed parameter can change so that every optimal solution remains optimal.

Usually, the parametric analysis is regarded as an extension of sensitivity analysis. In parametric analysis, we are interested in giving a direction along which the elements of cost matrix are perturbed, the optimal values of all possible perturbation amounts are determined. The optimal value is no doubt a function of the perturbation amount. Hence, we will focus on determining the optimal value function, composed of several linearity ranges with different slopes, which could give much information for decision making.

Suppose the obtained optimal solution is the implemented optimal solution of the practitioner. Some remarks concerning three types of sensitivity ranges of costs of TP are discussed below:

(1) If the implemented optimal basic solution is non-degenerate, the optimal basic solution is exactly the shipping pattern. Therefore, the Type I SR and Type II SR are identical. Otherwise, if the obtained optimal basic solution is degenerate, the basic optimal solution and the shipping pattern are different. At that time, the Type I SR is usually a subset of Type II SR. So it may be that perturbation amount has exceeded the Type I SR and the optimal basis has changed by switching location cell of degenerate basic variables, but the shipping pattern is unchanged.

(2) In reality, the decision-makers are only interested in the stability of transportation routing, i.e. shipping pattern, as a part of their optimal strategy [18]. In AP, stable means keeping the assignment pattern unchanged so that the labors work with the same familiar machines and the machines settle in their original locations, etc. So Type II SR is then more practical for application.

(3) If the implemented optimal basic solution (degenerate or non-degenerate) is the unique optimal shipping pattern, then the Type II SR and Type III SR are identical. When multiple optimal shipping patterns, the Type III SR is usually a subset of Type II SR.

(4) The optimal value function of parametric analysis gives us the other method to determine Type III SR. Suppose c_{pq} is the only perturbed element of cost matrix of the giving direction. Type III SR is identical to determine the largest range for which the closed line segment including origin of the optimal value function. The slope of the closed line segment remains unchanged. In case that it happens to be a breakpoint in the origin, Type III SR is defined as [0, 0]. The optimal shipping patterns corresponding to one open line segment, i.e., excluding the breakpoints on both ends, are the same. So the Type III SR excluding both ends is the largest range to keep the set of all optimal shipping patterns invariant. The set composed of these shipping patterns is a subset of shipping patterns corresponding to the breakpoints on its both ends.

(5) Practitioners seldom apply Type III SR since they usually implement only one optimal shipping pattern. In case of multiple optimal shipping patterns, nevertheless, Type III SR is the proper display when decision-makers emphasize the flexibility of optimal decision-making. Within the Type III SR, any optimal shipping pattern could be implemented.

Defects arising in postoptimal analysis of transportation simplex method when degeneracy occurs are the same as that for optimization procedure. Researches of the dual simplex algorithms, the primal-dual algorithms and the weighted Hungarian algorithms for postoptimal analysis are few.

III. Optimizing Procedure of The Proposed Algorithm

The proposed algorithm is a refined weighted Hungarian algorithm, the LCJ algorithm for short hereafter. It is another view of the maximal flow-minimal cut theorem for the network programming.

Some developed theorems in literatures concerning the LCJ algorithm are listed below:

Theorem 1: *The optimal solution to a TP is invariant by addition or subtraction of a constant to any row and/or column of its cost matrix* [12]. *The resultant cost matrix is called a reduced cost matrix*.

Theorem 2: *Suppose all entries in the reduced cost matrix are greater than or equal to zero. If costs of positive allocation cells are all equal to zero, i.e. the total cost equals zero, then the objective function value is zero and clearly minimal, i.e. optimal* [12].

Theorem 3: *A complementary pair of primal and dual basic feasible solutions is optimal to their respective problem* [11]. *This is the famous Karush–Kuhn–Tucker conditions which are sufficient and necessary conditions for global optimality in linear programming problem, so as to TP.*

The LCJ algorithm will solve the TP in Table 1. Some terms used in LCJ algorithm are defined below:

(1) Define the cells without shipping amount x_{ij} as empty cells, otherwise, define them as allocated cells. Further, define empty cell with $c_{ij} = 0$ as zero-cost empty cell.

(2) Define the cells with $x_{ij} = 0$ as zero allocated cells. Define the cells with $x_{ij} > 0$ as positive allocated cells.

(3) Define the zero-cost empty cell which is unique in that row as row independent zero-cost empty cell. Define the zero-cost empty cell which is unique in that column as column independent zero-cost empty cell. Additionally, the row independent zero-cost empty cell or column independent zero-cost empty cell are both belong to independent zero-cost empty cell. The zero-cost empty cell which is not independent zero-cost empty cell is called dependent zero-cost empty cell.

(4) Define the cost matrix that both the minimum cost of each row and minimum cost of each column are zeroes as proper cost matrix. Hence, there are at least one zero cost at each row and each column.

(5) Augmented open path: an order sequence of cells with two endpoints lie in the right-marginal supply cells and bottom-marginal demand cell each, others lie in the interior cells with zero costs.

(6) Define $\hat{R}$ as the set composed of the rows where remaining supply amount are positive; $\hat{C}$ as the set composed of the columns where remaining demand amount are positive.

Besides, six procedures used in the LCJ Algorithm are defined below. These procedures consist of a series of simple steps to reach the optimal of TP.

1. *Cost Reduction Procedure*:

Subtract the minimum cost of each row from each cost of that row and get a new cost matrix. In the newly constructed cost matrix, subtract the minimum cost of each column from each cost of that column. Get the reduced cost matrix of the Cost Reduction Procedure and it will be a proper cost matrix. According to Theorem 1, the optimal solution of TPs with original cost matrix and the reduced cost matrix are identical.

2. *Positive Allocation Procedure*:

Allocate the smaller amount of the remaining supply of its row and the remaining demand of its column to the selected cell as shipping amount x_{ij}. Adjust the associated amounts of remaining supply and remaining demand by subtracting the allocated amount x_{ij}, respectively.

3. *Zero Allocation Procedure*:

Allocate zero shipping amounts, i.e. $x_{ij} = 0$, to empty cells of the row or column if the remaining supply or the remaining demand has used up. Hence, if the remaining supply and the remaining demand are exhausted in the same time after performing the positive allocation procedure, allocate $x_{ij} = 0$ to the empty cells of the row and the column both. Here is somewhat different from the transportation simplex method, and shows the LCJ Algorithm is not composed by a series of basic feasible solutions.

4. *Labeling Procedure*:

(1) Examine each row successively. Give label ✓ to rows in $\hat{R}$. These labels are now surely unmarked.

(2) Examine the rows successively from row 1 till the last row. If row p has an unmarked label and column q has no label, give label p to column q where $c_{pq} = 0$. Mark the label of row p, e.g., give the label a cycle, to denote the row p has been examined.

(3) Examine the columns successively from column 1 till the last column. If column q has an unmarked label and row k has no label, give label q to row k where $x_{kq} > 0$. Mark the label of column q after column q has been examined.

(4) Repeat (2) and (3) until further labeling is impossible.

5. *Cost Updating Procedure*:

(1) Select the smallest cost, denote as ρ, that its row is labeled and its column is unlabeled. Subtract ρ from every cost in each row labeled. In the resultant matrix, add ρ to every cost in each column labeled.

Or

(2) The costs of rows without labels and columns with labels are covered with horizontal and vertical lines, respectively. Let ρ denote the smallest uncovered cost. Add ρ to the cost in the covered rows and columns, and substract ρ from all the uncovered cost.

(3) Erase all labels and their marks. According to Theorem 1, the optimal solution is unchanged.

6. *Allocation Updating Procedure*:

(1) Suppose the chosen column is column g_1. Let r_1 be the label of column g_1, g_2 be the label of row r_1, r_2 be the label of column g_2, g_3 be the label of row r_2, r_3 be the label of column $g_3, \ldots, r_t$ be the label of column g_t, ✓ be the label of row r_t. Form an open path by following the labels in reverse order until reach the label ✓. Thus, cells (r_1, g_1), (r_1, g_2), (r_2, g_2), (r_2, g_3), (r_3, g_3), …, (r_t, g_t) compose the open path. The unit costs in the open path are equal to zero. Further define the augmented open path as the path composed by cells $(m+1, g_1)$, (r_1, g_1), (r_1, g_2), (r_2, g_2), (r_2, g_3), (r_3, g_3), …, (r_t, g_t), $(r_t, n+1)$. The $m+1$ row and $n+1$ column are row of remaining demand and column of remaining supply in the table, respectively.

(2) Start from one endpoint cell of the augmented open path to the other endpoint, give cells alternatively with a minus sign (–) or plus sign (+). The start endpoint cell will be given a minus sign (–). Take the *smallest* amount (it may be shipping amount, supply amount or demand amount) in the cells with minus sign (–) as θ. For each cell with a plus sign (+), increase the amount by θ. For each cell with a minus sign (–), decrease the amount by θ. Then, the total amount of allocation increases θ.

Outline the solution steps of the proposed LCJ algorithm for optimization as follows:

Step 1: Construct the given transportation problem as Table 1 which satisfies the balanced condition with m sources and n destinations. Interior of the table are initially $m \times n$ empty cells.

Step 2: Perform the Cost Reduction Procedure.

Step 3: Examine each row successively for row independent zero-cost empty cell. If any, select the cell and directly perform the Positive Allocation Procedure and then the Zero Allocation Procedure.

Step 4: Examine each column successively for column independent zero-cost empty cell. If any, select the cell and directly perform the Positive Allocation Procedure and then the Zero Allocation Procedure.

Step 5: Repeat steps 3 and 4 successively until further allocation is impossible. The resultant table possess none independent zero-cost empty cell.

Step 6: Check each cell if it is a zero-cost empty cell. If any, select the cell and directly perform the Positive Allocation Procedure and then the Zero Allocation Procedure. Repeat step 6 until all $m \times n$ cells are checked.

Step 7: One of the two cases below will occur:

(a) Every cell has its allocation amount x_{ij}, the optimal solution is obtained and terminates the algorithm.

(b) There are some empty cells left, and none with zero cost. Find $\hat{R}$ and $\hat{C}$. Go to step 8.

Step 8: Perform the Labeling Procedure. If no column in $\hat{C}$ receives a label, go to step 10. Otherwise, choose the leftmost column in $\hat{C}$ which receives a label and go to step 9.

Step 9: Perform the Allocation Updating Procedure. Then, perform the Zero Allocation Procedure. Update $\hat{R}$ and $\hat{C}$. Clear all labels with their marks and go to step 7.

Step 10: Perform the Cost Updating Procedure. Go to step 6.

In the LCJ algorithm, step 1 is elementary for solving. Step 2 results a new TP without changing the optimal solution. Step 3, 4, 5 and 6 try to get the largest total allocated amounts to zero-cost cells. Each time we finish step 6, costs of positive allocation cells are all equal to zero. When there is no empty cell left in step 7 (a), the final table has the largest total allocated amounts to zero-cost cells, i.e., the optimal solution is obtained by Theorem 2. Theoretically speaking, in the final table the values of x_{ij}'s

Table 2

The original TP tableau for Example 1

4	5	5	10	6	8	3
4	6	4	5	3	3	1
6	5	7	10	5	9	4
1	2	1	1	1	2	

surely compose a primal feasible solution of (1), and it can always obtain $u_i = 0$, $v_j = 0$ for all i and j, as a dual feasible solution of (2). Additionally, the two solutions are complementary pair. By Theorem 3, the values of x_{ij}'s are an optimal solution of (1). In step 7 (b), we know the current table is not optimal. Labeling procedure in step 8, analogous to Ford & Fulkerson [9] and Hadley [5], gives us the message if the total amounts in the current table are the largest allocation allowed. When the total amounts in the current table are largest, we go to step 10 to use Cost Updating Procedure which change the cost matrix for more zero-cost cells. When the total amounts in the current table are not largest, we go to step 9 to use Allocation Updating Procedure to increase the total allocated amounts. Cost Updating Procedure imply a new dual feasible solution is given. Allocation Updating Procedure imply an increase of flow if we treat the TP as a maximal flow problem.

In fact, the LCJ algorithm, from step 7 till get the optimal solution, maintain the dual feasibility while improve to reach the primal feasibility and complementary slackness theorem. Finally, the Karush–Kuhn–Tucker conditions hold and thus the optimal solution is obtained.

Example 1: Solve the balanced TP shown in Table 2 with the LCJ algorithm.

Now that the transportation problem is a balanced one with initially $3{\times}6 = 18$ empty cells. We directly go to step 2 to perform the Cost Reduction Procedure, and get cost elements c_{ij} in the Table 3. The reduced cost matrix is a proper cost matrix.

We perform steps 3 and find there are no row independent zero-cost empty cell in every row of Table 3. Then perform step 4 by allocating column independent zero-cost empty cell in every column of Table 3. Follow step 5

Table 3

Result table of executing step 2, 3, 4 and 5, repeatedly

0 1	1 0	0 1	4 0	2 0	4	1
1 0	3 0	0 0	0 1	0 0	0 0	0
1 0	0 2	1 0	3 0	0 1	4	1
0	0	0	0	0	2	

to repeat steps 3 and 4 and get Table 3. There is no independent zero-cost empty cell in this table.

Follow step 6, we find there is no zero-cost empty cell in Table 3. But, there are some empty cells left and none with zero cost. Find $\hat{R}$ ={1,3}, Ĉ = {6} and perform the Labeling Procedure, we get the labels - the circled number in the left and upper of Table 4. In the table, none element of Ĉ receive a label, so we go to step 10.

In Table 4, we perform the Cost Updating Procedure and get ρ = 3. The new costs are as that in Table 5. Again, we perform steps 6, but cannot

Table 4

Rusult table of the Labeling Procedure

	①	③	①		③		
✓	0 1	1 0	0 1	4 0	2 0	4	1
	1 0	3 0	0 0	0 1	0 0	0 0	0
✓	1 0	0 2	1 0	3 0	0 1	4	1
	0	0	0	0	0	2	

Table 5

The table with an augmented open path

	①	③	①	③	③	②	
✓	$\boxed{0}$ 1	$\boxed{1}$ 0	$\boxed{0}$ 1	$\boxed{1}$ 0	$\boxed{2}$ 0	$\boxed{1}$	1
④	$\boxed{4}$ 0	$\boxed{6}$ 0	$\boxed{3}$ 0	$\boxed{0}$ $-\theta$ 1	$\boxed{3}$ 0	$\boxed{0}$ $+\theta$ 0	0
✓	$\boxed{1}$ 0	$\boxed{0}$ 2	$\boxed{1}$ 0	$\boxed{0}$ $+\theta$ 0	$\boxed{0}$ 1	$\boxed{1}$	$-\theta$ 1
	0	0	0	0	0	$-\theta$ 2	

allocate any amount to any cell. Find $\hat{R}$ ={1, 3}, $\hat{C}$ ={6} and perform the Labeling Procedure in Table 5 and get the labels Table 5.

Column 6, an element of $\hat{C}$, receives a label. We perform step 9 and get the augmented open path composed by cells (4, 6), (2, 6), (2, 4), (3, 4) and (3, 7). The path is denoted by dash line in Table 5. Through the path we get $\theta = \min(2, 1, 1) = 1$ and get the updated x_{ij}, s_i and d_j in Table 6.

There are some empty cells left, and none with zero cost. Go to step 8. Find $\hat{R}$ ={1}, $\hat{C}$ ={6} and perform the Labeling Procedure, we get the labels

Table 6

Updated allocation amounts

	①		①				
✓	$\boxed{0}$ 1	$\boxed{1}$ 0	$\boxed{0}$ 1	$\boxed{1}$ 0	$\boxed{2}$ 0	$\boxed{1}$	1
	$\boxed{4}$ 0	$\boxed{6}$ 0	$\boxed{3}$ 0	$\boxed{0}$ 0	$\boxed{3}$ 0	$\boxed{0}$ 1	0
	$\boxed{1}$ 0	$\boxed{0}$ 2	$\boxed{1}$ 0	$\boxed{0}$ 1	$\boxed{0}$ 1	$\boxed{1}$ 0	0
	0	0	0	0	0	1	

Table 7

The optimal solution for Example 1

0	0	0	0	1	0	
1	0	1	0	0	1	0
5	6	4	0	3	0	
0	0	0	0	0	1	0
2	0	2	0	0	1	
0	2	0	1	1	0	0
0	0	0	0	0	0	

in Table 6. In the table, none element of $\hat{C}$ receive a label, so we go to step 10.

In Table 6, we perform the Cost Updating Procedure and step 6. Every cell has its allocation amount x_{ij} in Table 7, so we terminate the algorithm.

So, the optimal solution is $X^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$ and optimal value is $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} = 45$. Additionally, matrix $C^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 6 & 4 & 0 & 3 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$ is the optimal marginal cost of transportation simplex method.

IV. Procedures of Sensitivity Analysis

The LCJ Algorithm generates an optimal solution with no more than (m+n–1) positive x_{ij} values. Thus, we get a nondegenerate or degenerate basic optimal solution. The following sensitivity analysis (SeA) procedure, rooted in the LCJ algorithm, determines the Type II SR when one cost perturbed.

Denote Δ_{pq} as the variation of c_{pq} and $[\ddot{L}_{pq}, \ddot{U}_{pq}]$ as Δ_{pq}'s Type II SR. Further, $[c_{ij}]$ is the original cost matrix, and $X^* = [x_{ij}^*]$ is the obtained optimal solution matrix. Matrix $[c_{ij}^*]$ is the cost matrix of the optimal table, moreover it is the initial and updated cost matrix in this section.

In addition to terms in Section III, some more terms are defined below.

(1) Define the cell in the initial and updated table with $x_{ij}^* \times c_{ij}^* \neq 0$ as unsatisfied cell. This kind of cells unsatisfy the complementary slackness theorem of the Karush–Kuhn–Tucker condition.

(2) Define $\check{R}$ and $\check{C}$ as the set composed of the rows and columns of unsatisfied cells, respectively.

Likewise, some more procedures are defined below.

1. *Unsatisfied Cell Checking Procedure*:

Examine the $m \times n$ cells successively. Identify if the cell is an unsatisfied cell or not.

2. *Labeling Procedure for SeA*:

(1) Examine each row successively. If row i is an element of Ř, give label to row i.

(2) Examine the rows successively from row 1 till the last row. If row p has an unmarked label and column q has no label, give label p to column q where $c_{pq} = 0$. Mark the label of row p, e.g., give the label a cycle, after row p has examined.

(3) Examine the columns successively from column 1 till the last column. If column q has an unmarked label and row k has no label, give label q to the associated row where (k, q) is a positive allocated cell (i.e. $x_{kq} > 0$). Mark the label of column q after column q has examined.

(4) Repeat (2) and (3) until further labeling is impossible.

3. *Bound Determining Procedure*:

Using conditions of all costs ≥ 0 and the supposed condition $\Delta_{ij} > \ddot{U}_{pq}$ (or $\Delta_{ij} < \ddot{L}_{pq}$), we have some inequalities. Derive intersection of these inequalities, the upper (or lower) bound of Δ_{ij} can be updated.

Outline the steps of the proposed LCJ algorithm for sensitivity range that keeps shipping pattern unchanged as follows:

Step 1s: Suppose (p, q) is the perturbed cell in the obtained optimal solution X^*. The initial bounds $[\ddot{L}_{pq}, \ddot{U}_{pq}] = [0,0]$. Update $c^*_{pq} \leftarrow c^*_{pq} + \Delta_{pq}$.

Step 2s: To compute (update) the upper bound of Δ_{pq}, we suppose $\Delta_{pq} > \ddot{U}_{pq}$; yet regard it as $\Delta_{pq} = \ddot{U}_{pq} + \varepsilon$ will be more precise in the procedure, where $\varepsilon \to 0^+$. Thus, ε is treated as a positive number smaller than any positive number. (To compute (update) the lower bound, we suppose $\Delta_{pq} < \ddot{L}_{pq}$; yet regard it as $\Delta_{pq} = \ddot{L}_{pq} - \varepsilon$ will be more precise.)

Step 3s: Perform the Cost Reduction Procedure.

Step 4s: Perform the Unsatisfied Cell Checking Procedure, therefore sets $\check{R}$ and $\check{C}$ are identified.

If Ř is a null set (denote as ϕ), go to step 5s; otherwise, go to step 6s.

Step 5s: Perform the Bound Determining Procedure, and get the updated $\ddot{U}_{pq}(\text{or}\,\ddot{L}_{pq})$ of Δ_{pq}. If the current $\ddot{U}_{pq} \to \infty$ (or $\ddot{L}_{pq} \to -\infty$), we stop. Otherwise, go to step 2s.

Step 6s: Perform the Labeling Procedure for SeA. If none element of $\check{C}$ receive a column label, go to step 7s. Otherwise, stop the algorithm and the correct $\ddot{U}_{pq}(\text{or}\,\ddot{L}_{pq})$ is obtained.

Step 7s: Perform the Cost Updating Procedure and clear all labels with their marks. Go to step 4s.

In addition, we introduce two theorems that can speed the efficiency of determining $[\ddot{L}_{pq}, \ddot{U}_{pq}]$ [10].

Theorem 4: $\ddot{U}_{pq} \to +\infty$ *if* $x^*_{pq} = 0$.

Theorem 5: $\ddot{L}_{pq} \to -\infty$ *if* x^*_{pq} *is the unique positive shopping cell in row p or column q.*

When the variation exceeds $[\ddot{L}_{pq}, \ddot{U}_{pq}]$, *the current optimal shipping pattern no more optimal. The LCJ algorithm has the advantage of reoptimizing the nonoptimal table taking it as the initial table. Thus, the reoptimization procedure needs only few iterations, needless to resolve from scratch. This reoptimization technique will be applied and introduced in Example* 5 *in the next section for parametric analysis.*

Table 8

The initial table for Example 2

0	0	0	0	1	0
1	0	1	0	0	1
5	6+Δ	4	0	3	0
0	0	0	0	0	1
2	0	2	0	0	1
0	2	0	1	1	0

Table 9

Reduced costs, labels and lines

	②		③	③		
	0 / 1	0 / 0	0 / 1	0 / 0	1 / 0	0 / 1
✓	−Δ−1 / 0	0 / 0	−Δ−2 / 0	−Δ−6 / 0	−Δ−3 / 0	−Δ−6 / 1
②	2 / 0	0 / 2	2 / 0	0 / 1	0 / 1	1 / 0

Example 2: Follow Example 1. Suppose cell (2,2) is the perturbed cell, Δ_{22} is the variation (Δ for short hereafter) and we want to determine $[\ddot{L}_{22}, \ddot{U}_{22}]$. In the beginning, let $[\ddot{L}_{22}, \ddot{U}_{22}] = [0,0]$ and change c^*_{22} in Table 7 to 6 + Δ showing in Table 8.

Since $x^*_{22} = 0$, we have $\ddot{U}_{22} \to +\infty$ by Theorem 4. To compute the lower bound, we let Δ smaller than the current $\ddot{L}_{22}$, i.e. Δ < 0. Perform the Cost Reduction Procedure, we get the Table 8 unchanged. Since there is no unsatisfied cell, we find $\check{R} = \phi$ and $\check{C} = \phi$. To perform the Bound Determining Procedure, we let $\Delta < 0$, $6 + \Delta \geq 0$, and get $-6 \leq \Delta < 0$. Updated $\ddot{L}_{22} = -6$ and go to step 2s.

Suppose $\Delta < -6$, we perform the Cost Reduction Procedure and get cost entries in Table 9. Cell (2, 6) is the only unsatisfied cell, so $\check{R} = \{2\}$ and $\check{C} = \{6\}$. Perform the Labeling Procedure for SeA gets the labels in Table 9. $\check{C}$ receives no label. Perform the Cost Updating Procedure.

Update the cost entries as that in Table 10. Since none cells where $c_{ij} \times x_{ij} \neq 0$, we find $\check{R} = \phi$ and $\check{C} = \phi$.

Table 10

New cost entries after Cost Updating Procedure

0 / 1	−Δ−6 / 0	0 / 1	−Δ−6 / 0	−Δ−5 / 0	0 / 1
5 / 0	0 / 0	4 / 0	−Δ−6 / 0	−Δ−3 / 0	0 / 1
Δ+8 / 0	0 / 2	Δ+8 / 0	0 / 1	0 / 1	Δ+7 / 0

Table 11

Final table of Example 2

	①	②	①	③	③	③
⑥	0 1	$-\Delta-6$ 0	0 1	1 0	2 0	0 1
⑥	5 0	0 0	4 0	1 0	4 0	0 1
✓	1 0	$-\Delta-7$ 2	1 0	0 1	0 1	0 0

Table 12

$[\ddot{L}_{pq}, \ddot{U}_{pq}]$ of all c_{ij}

$(-\infty,+2]$	$[-1,+\infty)$	$(-\infty, 2]$	$[-1,+\infty)$	$[-2,+\infty)$	$[-2, +1]$
$[-5,+\infty)$	$[-7,+\infty)$	$[-4,+\infty)$	$[-1,+\infty)$	$[-4,+\infty)$	$(-\infty,+1]$
$[-2,+\infty)$	$(-\infty, 1]$	$[-2,+\infty)$	$(-\infty,+1]$	$(-\infty,+2]$	$[-1,+\infty)$

To perform the Bound Determining Procedure, we let $\Delta<-6$, $-\Delta-6\geq 0$, $-\Delta-5\geq 0$, $-\Delta-6\geq 0$, $-\Delta-3\geq 0$, $\Delta+8\geq 0$, $\Delta+7\geq 0$, we have $-7 \leq \Delta<-6$. Updated $\ddot{L}_{22} = -7$ and go to step 2s.

Suppose $\Delta<-7$, we perform the Cost Reduction Procedure and get cost entries in Table 11. Perform the Unsatisfied cell Checking Procedure, we find cell (3, 2) is the only unsatisfied cell, so $\check{R}=\{3\}$ and $\check{C}=\{2\}$. Perform the Labeling Procedure for SeA gets the labels in Table 11. Column 2 of $\check{C}$ receive a column label, we stop the algorithm.

All the $[\ddot{L}_{pq}, \ddot{U}_{pq}]$ are listed in Table 12. Besides, there is only one shipping pattern of Example 1 so the Type II SR and Type III SR are identical.

V. Procedures of Parametric Cost Analysis

The parametric analysis studies how the effects of one or multiple c_{ij}'s change over a range affect the optimal values. The optimal value function is composed of several linearity ranges with different slopes. Denote δ as the variation, then the optimal value is a function of δ, denotes as $f(\delta)$. Further, suppose ${}_0\ddot{P} = \ddot{P}_0 = 0$. Denote ${}_3\ddot{P}$, ${}_2\ddot{P}$, ${}_1\ddot{P}$, $\ddot{P}_1$, $\ddot{P}_2$, and $\ddot{P}_3\ldots$, where $-\infty<\ldots \leq {}_3\ddot{P} \leq {}_2\ddot{P} \leq {}_1\ddot{P} \leq {}_0\ddot{P} = \ddot{P}_0 = 0 \leq \ddot{P}_1 \leq \ddot{P}_2 \leq \ddot{P}_3 \leq \ldots < \infty$, as the

δ value at which $f(\delta)$ are breakpoints. But at ${}_0\dddot{P} = \dddot{P}_0 = 0,$ it may or may not be a breakpoint.

In addition to the terms in procedures in Section III and IV, one more procedure is defined below.

1. *Breakpoint Determining Procedure for PA*:

Using conditions of all cost ≥ 0 and the supposed condition $\delta > \dddot{P}_k$ (or $\delta < {}_k\dddot{P}$), we have some inequalities. Derive intersection of these inequalities, we have the upper bound as the $\dddot{P}_{k+1}$ (or the lower bound as the ${}_{k+1}\dddot{P}$).

Step 1p: Update $[c_{ij}^*] \leftarrow [c_{ij}^*] + \delta \times [d_{ij}]$ in the optimal table, where $[d_{ij}]$ denotes the cost direction matrix. We get the new cost matrix by updating one or many elements of the cost matrix in optimal solution obtained by the LCJ algorithm. Let $k = 0$, ${}_0\dddot{P} = \dddot{P}_0 = 0,$ and ${}_0X^* = \dddot{X}_0^* = X^*$.

Step 2p: To compute the $\dddot{P}_{k+1}$, we suppose $\delta > \dddot{P}_k$, i.e. $\delta = \dddot{P}_k + \varepsilon$. To compute the ${}_{k+1}\dddot{P}$, we suppose $\delta < {}_k\dddot{P}$, i.e. $\delta = {}_k\dddot{P} - \varepsilon$.

Step 3p: Perform the Cost Reduction Procedure.

Step 4p: Perform the Unsatisfied Cell Checking Procedure. Return the shopping amounts x_{pq} of unsatisfied cells back to s_p supply amount and d_q demand amount. These cells become empty cells. Find $\hat{R}$ and $\hat{C}$.

Step 5p: If $\hat{R} = \phi$, go to step 6p; otherwise, go to step 7p.

Step 6p: Let $k \leftarrow k + 1$. Perform the Breakpoint Determining Procedure for PA, and get the updated ${}_k\dddot{P}$ or $\dddot{P}_k$ of δ. We also get the optimal solution ${}_k\dddot{X}^*$ or $\dddot{X}_k^*$. Compute the corresponding optimal value of the current interval $[{}_k\dddot{P}, {}_{k-1}\dddot{P}]$ or $[\dddot{P}_{k-1}, \dddot{P}_k]$. If ${}_k\dddot{P} \to -\infty$ or $\dddot{P}_k \to \infty$, we go to step 10p. Otherwise, go to step 2p.

Step 7p: Perform the Labeling Procedure. If none of the element of $\hat{C}$ receive a label, go to step 8p. Otherwise, choose the leftmost column of $\hat{C}$ which receives a label and go to step 9p.

Table 13

The initial table for Example 3

δ	δ	2δ	δ	$1+2\delta$	δ	
1	0	1	0	0	1	0
5	6	$4-2\delta$	0	3	0	
0	0	0	0	0	1	0
2	0	$2-\delta$	0	0	1	
0	2	0	1	1	0	0
0	0	0	0	0	0	

Step 8p: Perform the Cost Updating Procedure. Then, perform the Positive Allocation Procedure and follow the Zero Allocation Procedure. Clear all labels with their marks. Update $\hat{R}$ and $\hat{C}$. Go to step 5p.

Step 9p: Perform the Allocation Updating Procedure. Then, perform the Zero Allocation Procedure. Clear all labels with their marks and update $\hat{R}$ and $\hat{C}$. Go to step 5p.

Step 10p: Stop the algorithm and combine the intervals that correspond to the same optimal value.

Example 3: Follow Example 1. Suppose both the supply point A_1 and the demand point B_3 move to other areas, hence all the costs concerning A_1 and B_3 may change. Let the cost direction matrix $[d_{ij}] = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$. Update $[c^*_{ij}] \leftarrow [c^*_{ij}] + \delta \times [d_{ij}]$, we obtain Table 13.

Table 14

Results of parametric analysis of the LCJ algorithm

δ	**Optimal solution**	**Optimal value $f(\delta)$**
$-\infty < \delta \le -2$	${}_3\dddot{X}^*$	$47 + 5\delta$
$-2 \le \delta \le -1$	${}_2\dddot{X}^*$	$45 + 4\delta$
$-1 \le \delta \le 0$	${}_1\dddot{X}^*$	$45 + 4\delta$
$\delta = 0$	${}_0X^* = \dddot{X}^*_0$	$45 + 4\delta$

Contd..

$0 \le \delta \le 1$	$\dddot{X}_1^*$	$45 + 4\delta$
$1 \le \delta \le 2$	$\dddot{X}_2^*$	$47 + 2\delta$
$2 \le g < \infty$	$\dddot{X}_3^*$	$49 + \delta$

Table 15

Final results of PA of example 3

δ	**Optimal value $f(\delta)$**
$-\infty < \delta \le -2$	$47 + 5\delta$
$-2 \le \delta \le 1$	$45 + 4\delta$
$1 \le \delta \le 2$	$47 + 2\delta$
$2 \le \delta < \infty$	$49 + \delta$

Follow the proposed procedure, we get ${}_3\dddot{P} = -\infty$, ${}_2\dddot{P} = -2$, ${}_1\dddot{P} = -1$, $\dddot{P}_1 = 1$, $\dddot{P}_2 = 2$, and $\dddot{P}_3 = \infty$. Additionly, we find ${}_3\dddot{X}^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{bmatrix}$,

${}_2\dddot{X}^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$, ${}_1\dddot{X}^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$, ${}_0X^* = \dddot{X}_0^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$,

$\dddot{X}_1^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$, $\dddot{X}_2^* = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ and $\dddot{X}_3^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$.

Combine the adjacent segments with the same optimal value in Table 14, we get Table 15 which shows the final results.

Table 16

Final results of PA of Example 4

δ	**Optimal value $f(\delta)$**
$-\infty < \delta \le -2$	$48 + 2\delta$
$-2 \le \delta \le -1$	$46 + \delta$
$-1 \le \delta < \infty$	45

Example 4: Let the cost direction matrix $[d_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ in Example 3. We get the optimal value function in Table 16. Table 16 shows $[\ddot{L}_{12}, \ddot{U}_{12}] = [-1,\infty)$ since the range include $\delta = 0$.

Example 5: Let $\Delta = -8$ in Example 2. We find it is not in the range $[\ddot{L}_{22}, \ddot{U}_{22}] = [-7,\infty)$, so the optimal shipping pattern is no more optimal. To reoptimize it, we let $\Delta = -8$ in Table 12. Then, execute step 3p, step 4p in this section, and go to step 8 in Section III following the procedures of the LCJ algorithm. The new optimal solution can be obtained.

VI. Discussion

In this section, we discuss some other helpful procedures that may speed up or give flexibility to the LCJ algorithm.

(1) The preprocessing of the TP [18] to satisfy the Hesse-Woolsey-Stern condition [18] will speed up the LCJ algorithm in Section III. The Hesse-Woolsey-Stern condition is a necessary condition and can be stated as follows: Suppose $[c_{ij}]$ is the reduced cost matrix,

$$\sum_{j|c_{ij}=0} b_j \geq a_i \text{ for } i = 1, 2, \ldots, m \text{ and } \sum_{i|c_{ij}=0} a_i \geq b_j \text{ for } j = 1, 2, \ldots, n.$$

(2) Compute the penalty of each independent zero-cost empty cell, the penalty is the sum of the smallest entry of its row and its column excluding the picked cell itself. The allocated cells in step 3, 4 and 5 in the LCJ algorithm is changed to select the cell with the largest penalty.

(3) Any optimal solution could be applied to execute the sensitivity analysis and parametric analysis, not limited to the optimum obtained by the LCJ algorithm.

VII. Conclusion

In this paper, the LCJ algorithm was developed to find the optimal solution of TP, determine sensitivity range, and perform parametric analysis. It is found that the LCJ method is better than transportation simplex method specially in degenerate problems. Besides, the LCJ algorithm is able to perform the sensitivity analysis and parametric analysis. The algorithm could also be completely applied to assignment problem, semi-assignment problem etc. The LCJ algorithm performing the SeA and PA for the supply amounts and demand amounts will be research directions of future.

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