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A new transmuted family of distributions: Properties and estimation with applications

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Abstract

In this paper, a new transmuted general family of distributions is introduced and studied. It offers an original alternative to the former transmuted family by revisiting its

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mathematical structure, opening a new perspective on modeling. We derive expressions for the moments, the incomplete moments, the moment generating function, entropy measures, and order statistics as main properties. Further, we introduce a bivariate extension of the new family. Some members of the family are introduced, involving Burr, Gompertz, Weibull and gamma distributions. We discuss the different methods of estimation of the model parameters and a simulation study is performed for one of the special models to check the asymptotic behavior of the estimates under all estimation methods. Further, the interest of the family is illustrated by fitting two real-life data sets to the mentioned sub-models.

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1. Introduction

The literature on lifetime distributions is rich with various continuous univariate distributions and is still growing rapidly. Several extensions of some well-known lifetime distributions have been developed during the last two decades for the modeling and analysis of many types of real-life data that have different random natures. This development is followed by many approaches for generating new families of distributions because a family contains a plethora of distributions, increasing the chance of modeling a large number of real-life data. Among these families, there are the exponential-G by Gupta et al. (1998), the Marshall-Olkin-G (MO-G) by Marshall and Olkin (1997), the transmuted distributions by Shaw and Buckley (2009), the beta-G by Eugene et al. (2002), the Kumaraswamy-G (Kw-G) by Cordeiro et al. (2011), the McDonald-G (Mc-G) by Alexander et al. (2012), the gamma-G (type 1) by Zografos and Balakrishnan (2009), the gamma-G (type 2) by Ristic and Balakrishnan (2012), the log-gamma-G by Amini et al. (2014), the logistic-G by Tahir et al. (2014), the exponentiated generalized-G by Cordeiro et al. (2013), the Transformed-Transformer (T-X) by Alzaatreh et al. (2013), the exponentiated (T-X) by Alzaghal et al. (2013), the Kumaraswamy odd log-logistic-G (KwOLL-G) by Alizadeh et al. (2015), the generalized transmuted-G by Nofal et al. (2017), the beta transmuted-H families by Afify et al. (2017), the cubic rank transmuted distributions by Granzotto et al. (2017), the trigonometric family by Chesneau et al. (2019), the extended odd family by Bakouch et al. (2019), the truncated Cauchy power class by Aldahlan et al. (2020), the transmuted odd Frechet-G class by Badr et al. (2020), the Marshall-Olkin exponentiated generalized G class by Yousof et al. (2020), the modified T-X class by Aslam et al. (2020), the modified odd Weibull class by Chesneau and El Achi (2020), the type II general inverse exponential class by Jamal et al. (2020) and the odd generalized gamma-G class by Nasir et al. (2020).

1.1 *Genesis of the family*

Here, we introduce a new family of distributions called the transmuted general (T-G) family of distributions. In order to motivate its interest, let us first present a simple case of the well-known T-X family, introduced by Alzaatreh et al. (2013). For a general baseline cumulative distribution function (cdf) of a continuous probability distribution denoted by $G(x)$, a new cdf is given by

$$F(x) = \int_0^{G(x)} p(t)dt, \qquad x \in \mathbb{R}, \tag{1.1}$$

where $p(t)$ denotes a probability density function (pdf) with support on [0, 1]. As noticed in Alizadeh et al. (2017), the famous transmuted distributions family with cdf given by $F(x) = (1+\lambda)G(x) - \lambda G^2(x)$, $\lambda \in [-1,1]$, Shaw and Buckley (2009), corresponds to the cdf given by (1.1) defined with the pdf: $p(t) = 1 + \lambda - 2\lambda t$. As a new observation, $p(t)$ can be expressed as a generalized mixture of two pdfs with support on [0, 1] :

$$p(t) = (1-\lambda)p_1(t) + \lambda p_2(t),$$

where $p_1(t) = 1$ and $p_2(t) = 2(1-t)$. This fact motivates the consideration of a new family of distributions derived to (1.1) and generalized mixture of two pdfs; for any pdfs $p_1(t)$ and $p_2(t)$ with support on [0, 1] and λ such that $\inf_{t \in [0,1]} [(1-\lambda)p_1(t) + \lambda p_2(t)] > 0$, the following function $F(x)$ is a cdf:

$$F(x) = \int_0^{G(x)} [(1-\lambda)p_1(t) + \lambda p_2(t)]dt, \qquad x \in \mathbb{R}.$$

A new family of distributions arises by keeping $p_1(t) = 1$ for the sake of simplicity but choosing a new pdf for $p_2(t)$, with a different nature. In this paper, we focus our attention on the decreasing convex pdf given by $p_2(t) = 2/(1+t)^2$. The corresponding cdf is given by

$$F(x) = \int_0^{G(x)} \left[(1-\lambda) + \lambda \frac{2}{(1+t)^2} \right] dt = \left[1 - \lambda + \frac{2\lambda}{1+G(x)} \right] G(x),\ \lambda \in [-1,1], x \in \mathbb{R}. \tag{1.2}$$

An equivalent expression of (1.2) is given by

$$F(x) = G(x) + \lambda \frac{G(x)[1-G(x)]}{1+G(x)}, \qquad x \in \mathbb{R}.$$

The T-G family is defined by this cdf. The following are the alpha motivations. When $\lambda = 0$, $F(x)$ becomes the baseline cdf. When $\lambda = 1$, we get $F(x) = 2G(x)/(1+G(x))$, which corresponds to the cdf of the M-G family introduced by Kumar et al. (2017). It has proved to be an interesting alternative to several well-established models. Also, it corresponds to the already mentioned MO-G family with the parameter $1/2$. When $\lambda = -1$, we have $F(x) = 2G^2(x)/(1+G(x))$, which is the cdf of the maximum of two independent random variables, one with the cdf $G(x)$ is the other with the cdf of the M-G family. All the intermediate values for $\lambda \in [-1,1]$ are unexplored, which motivated the consideration of the T-G family for new perspectives of modelling.

1.2 *Aim and organization of the paper*

The aim of this paper is to explore the mathematical and practical aspects of the T-G family, and to show its importance and its flexibility using applications to real-life data sets. In particular, we give expressions for the moments, incomplete moments, generating function, mean deviation and residual and reversed residual life moments. Stochastic ordering results and the expression of the reliability parameter are discussed. General results for order statistics are derived. Entropy measures (Rényi and Mathai-Hanboldand entropy measures) are calculated. Some members of the family are introduced, involving Burr, Gompertz, Weibull and gamma distributions. A bivariate extension of the new family is also introduced. We provide simulation results of different estimation methods (maximum likelihood, least square, minimum spacing, weighted least square and Cramér-von Mises) for one of the special models to check the asymptotic behavior of the estimates under those methods. Further, applications on two different real-life data sets are given, with fair comparisons with recent flexible distributions.

This paper is unfolded as follows. Section 2 is devoted to the mathematical properties of the T-G family. Special models are introduced in Section 3. An extension to the bivariate case is explored in Section 4. Section 5 presents estimation methods for the parameters of the T-G family. Simulation results and applications to real-life data sets are given in Section 6. Section 7 presents some conclusions.

2. Mathematical properties

Hereafter, we say that a random variable follows a distribution from the T-G family if it has the cdf given by (1.2). Such a random variable will be denoted by X.

2.1 *Shape of the pdf*

Let $g(x)$ be the baseline pdf. Then, the pdf of X is given by

$$f(x)=\left[1-\lambda+\frac{2\lambda}{(1+G(x))^2}\right]g(x), \qquad \lambda\in[-1,1], x\in\mathbb{R}.$$

An equivalent expression is given by

$$f(x)=\frac{\left\{(1+\lambda)+(1-\lambda)G(x)[2+G(x)]\right\}g(x)}{(1+G(x))^2}, \qquad \lambda\in[-1,1], x\in\mathbb{R}.$$

The critical points of $f(x)$ are the roots of the equation:

$$\frac{d}{dx}\log(f(x))=\frac{g'(x)}{g(x)}-2\frac{g(x)}{1+G(x)}+\frac{2g(x)(\lambda-1)[1+G(x)]}{1+\lambda-(1-\lambda)G(x)[2+G(x)]}=0.$$

If x_0 is a root, its nature depends on the sign of $\nu(x)=d^2\log(f(x))/dx$ taking with $x=x_0$; if $\nu(x_0)<0$, it is a local maximum, if $\nu(x_0)>0$, it is a local minimum and if $\nu(x_0)=0$, it is a point of inflection.

2.2 *Expansions for the pdf and the cdf*

By using the generalized binomial expansion, we can show that the pdf of X has the expansion

$$f(x)=\sum_{j=0}^{\infty}a_j h_{j+1}(x), \tag{2.1}$$

where $a_0=1+\lambda$, $a_j=2\lambda(-1)^j$ for $j\geq 1$ and $h_{j+1}(x)=(j+1)G^j(x)g(x)$ for $j\geq 0$.

The cdf of X can be expressed as

$$F(x)=\sum_{j=0}^{\infty}a_j H_{j+1}(x), \tag{2.2}$$

where $H_{j+1}(x)=G^{j+1}(x)$ for $j\geq 0$.

It is worth noting that $h_{j+1}(x)$ is the pdf of the exp-G distribution with power parameter $j+1$, and $H_{j+1}(x)$ is the associated cdf. Therefore, (2.1) (resp. (2.2)) reveals that the pdf (resp. cdf) of a distribution from the T-G family is a linear combination of pdfs (resp. cdfs) of the exp-G distribution. Hence, some properties of the exp-G distribution can be used to determine some mathematical properties of the T-G family. The reference Gupta et al. (1998) contains technical information on the exp-G distribution.

2.3 *Quantile function*

For $\lambda \in [-1,1)$, the quantile function of X is given by

$$F^{-1}(x) = G^{-1}\left[\frac{1-x+\lambda-\sqrt{x^2+x(2-6\lambda)+(1+\lambda)^2}}{2(\lambda-1)}\right], \qquad x \in (0,1). \tag{2.3}$$

For $\lambda = 1$, we have

$$F^{-1}(x) = G^{-1}\left[\frac{x}{2-x}\right], \qquad x \in (0,1). \tag{2.4}$$

As a result, for any random variable U following the uniform $U(0,1)$ distribution, the random variable X given by $X = F^{-1}(U)$ follows a distribution of the T-G family.

2.4 *Hazard rate function*

The hazard rate function (hrf) of X is given by

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\{(1+\lambda)+(1-\lambda)G(x)[2+G(x)]\}g(x)}{[1+G(x)][1-\lambda G(x)-(1-\lambda)G^2(x)]}, \quad \lambda \in [-1,1], x \in \mathbb{R}.$$

The critical points of $h(x)$ are obtained from the equation:

$$\frac{d}{dx}\log(h(x)) = \frac{g'(x)}{g(x)} - \frac{g(x)}{1+G(x)} + \frac{[\lambda-2(\lambda-1)G(x)]g(x)}{1-\lambda G(x)+(\lambda-1)G^2(x)} + \frac{2(1-\lambda)[1+G(x)]g(x)}{1+\lambda-(1-\lambda)G(x)[2+G(x)]} = 0.$$

If x_0 is a root, the nature of x_0 depends on the sign of $\omega(x) = d^2\log(h(x))/dx$ taking with $x = x_0$; if $\omega(x_0) < 0$, it is a local maximum, if $\omega(x_0) > 0$, it is a local minimum and if $\omega(x_0) = 0$, it is a point of inflection.

2.5 *Moments, moment generating function and mean deviations*

Using the expansion (2.1), the r^{th} moment of X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{j=0}^{\infty} a_j \int_{-\infty}^{\infty} x^r h_{j+1}(x)dx. \tag{2.5}$$

Similarly, denoting by 1_A the indicator random variable on A, the r^{th} incomplete moment of X can be obtained as

$$\mu^r(x) = E(X^r 1_{\{X \le x\}}) = \sum_{j=0}^{\infty} a_j T_{j,r}(x), \tag{2.6}$$

where $T_{j,r}(x) = \int_{-\infty}^{x} t^r h_{j+1}(t)\,dt.$

The moment generating function of X is given by

$$M(t) = E(e^{tX}) = \sum_{j=0}^{\infty} a_j M_j(t),$$

where $M_j(t) = \int_{-\infty}^{\infty} e^{tx} h_{j+1}(x)dx.$

The mean deviations of X about the mean and median, respectively, can be put as

$$D_\mu = E(|X - \mu|) = 2\mu F(\mu) - 2\mu^1(\mu) \tag{2.7}$$

and

$$D_M = E(|X - M|) = \mu - 2\mu^1(M), \tag{2.8}$$

where $\mu = E(X)$ is given by (2.5), $F(\mu)$ is easily calculated from (1.2) and $\mu^1(M)$ is obtained from (2.6) with $M = F^{-1}(\frac{1}{2}),$ using (2.3) or (2.4).

2.6 *Residual and reversed residual life moments*

Using the binomial expansion, the r^{th} residual life moments of X can be obtained as

$$\begin{aligned} M_r(t) &= E((X-t)^r \mid X > t) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} t^{r-j} E(X^j \mid X > t) \\ &= \frac{1}{1-F(t)} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} t^{r-j} [E(X^j) - \mu^j(t)], \end{aligned}$$

where $E(X^j)$ is given by (2.5) and $\mu^j(t)$ is obtained from (2.6).

Similarly, the r^{th} reversed residual life moments of X can be obtained as

$$m_r(t) = E((t-X)^r \mid X \le t) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j t^{r-j} E(X^j \mid X \le t) = \frac{1}{F(t)} \sum_{j=0}^{r} \binom{r}{j} (-1)^j t^{r-j} \mu^j(t).$$

2.7 *Stochastic ordering*

This subsection is devoted to the stochastic ordering property of the new family. Stochastic ordering is a common concept to show the ordering mechanism in life time distributions. A random variable X is said to be stochastically smaller than a random variable Y in the

- stochastic order $(X \le_{st} Y)$ if the associated cdfs satisfy: $F_X(x) \ge F_Y(x)$ for all x.
- hazard rate order $(X \le_{hr} Y)$ if the associated hrfs satisfy: $h_X(x) \ge h_Y(x)$ for all x.
- likelihood ratio order $(X \le_{lr} Y)$ if the ratio of the associated pdfs given by $f_X(x)/f_Y(x)$ decreases in x.

When the supports of X and Y have a common finite left end-point, the following chain of implications holds:

$$X \le_{lr} Y \quad \Rightarrow \quad X \le_{hr} Y \quad \Rightarrow \quad X \le_{st} Y.$$

Further details can be found in Shaked and Shanthikumar (1994). Let X be a random variable following a distribution from the T-G family with parameter λ_1 with pdf $f_X(x)$ and Y be a random variable following a distribution from the T-G family with parameter λ_2 with pdf $f_Y(x)$. Let us prove that $X \le_{lr} Y$ under some assumptions on λ_1 and λ_2. We have

$$f_X(x) = \left[1 - \lambda_1 + \frac{2\lambda_1}{(1+G(x))^2}\right] g(x) = \frac{[1+\lambda_1 + (1-\lambda_1)[2+G(x)]G(x)]}{(1+G(x))^2} g(x)$$

and

$$f_Y(x) = \frac{[1+\lambda_2 + (1-\lambda_2)[2+G(x)]G(x)]}{(1+G(x))^2} g(x).$$

Hence

$$\frac{f_X(x)}{f_Y(x)} = \frac{1+\lambda_1 + (1-\lambda_1)z}{1+\lambda_2 + (1-\lambda_2)z},$$

where $z = [2+G(x)]G(x)$. Thus

$$\frac{d}{dx}\frac{f_X(x)}{f_Y(x)} = \frac{2(\lambda_2 - \lambda_1)z'}{(1+\lambda_2+(1-\lambda_2)z)^2},$$

where $z' = 2[1+G(x)]g(x)$. Since $z' \geq 0$ and $(1+\lambda_2+(1-\lambda_2)z)^2 > 0$, if $\lambda_2 < \lambda_1$, we have $d(f_X(x)/f_Y(x))/dx < 0$, implying that $f_X(x)/f_Y(x)$ decreases in x, so $X \leq_{lr} Y$.

2.8 *Stress-strength reliability*

In reliability theory, a common situation is that the life of a component has a random strength subjected to random stress. The random strength can be modeled by a random variable X and the random stress can be modeled by a random variable Y. The probability that the component functions satisfactorily is given by $R = P(Y < X)$, which is a well-known measure of component reliability with many applications. Let X be a random variable following a distribution from the T-G family with parameter λ_1 with pdf $f_X(x)$ and Y be a random variable following a distribution from the T-G family with parameter λ_2 with cdf $F_Y(x)$, independent of X. Owing to the expansions (2.1) and (2.2), we have

$$\begin{aligned} R = P(Y < X) &= \int_{-\infty}^{\infty} P(Y < X \mid \{X = x\}) f_X(x)dx = \int_{-\infty}^{\infty} f_X(x)F_Y(x)dx \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_j^{(\lambda_1)} a_i^{(\lambda_2)} \int_{-\infty}^{\infty} h_{j+1}(x)H_{i+1}(x)dx, \end{aligned}$$

where $a_j^{(\lambda)}$ is defined by $a_0^{(\lambda)} = 1+\lambda$ and $a_j^{(\lambda)} = 2\lambda(-1)^j$ for $j \geq 1$ and $\lambda \in \{\lambda_1, \lambda_2\}$. Now, we can remark that

$$\int_{-\infty}^{\infty} h_{j+1}(x)H_{i+1}(x)dx = (j+1)\int_{-\infty}^{\infty} g(x)G^{j+i+1}(x)dx = \frac{j+1}{j+i+2},$$

implying that

$$R = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_j^{(\lambda_1)} a_i^{(\lambda_2)} \frac{j+1}{j+i+2}.$$

2.9 *Order statistics*

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n of X and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the corresponding order statistics obtained from the sample. The pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(x)(1-F(x))^{n-i} f(x)$$
$$= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j F^{j+i-1}(x) f(x).$$

Using finite and generalized binomial series expansions, we have

$$F^{j+i-1}(x) = \left[1-\lambda+\frac{2\lambda}{1+G(x)}\right]^{j+i-1} G^{j+i-1}(x)$$
$$= G^{j+i-1}(x) \sum_{k=0}^{j+i-1} \binom{j+i-1}{k} (1-\lambda)^{j+i-1-k} (2\lambda)^k [1+G(x)]^{-k}$$
$$= \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} b_{i,j,k,\ell} H_{j+i+\ell-1}(x),$$

where

$$b_{i,j,k,\ell} = \binom{j+i-1}{k} \binom{-k}{\ell} (1-\lambda)^{j+i-1-k} (2\lambda)^k.$$

Owing to the expansion (2.1), we can write

$$F^{j+i-1}(x) f(x) = \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} a_m b_{i,j,k,\ell} H_{j+i+\ell-1}(x) h_{m+1}(x),$$

therefore

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} c_{j,k,\ell,m} H_{j+i+\ell-1}(x) h_{m+1}(x),$$

where

$$c_{j,k,\ell,m} = \frac{n!}{(i-1)!(n-i)!} \binom{n-i}{j} (-1)^j a_m b_{i,j,k,\ell}.$$

The r^{th} moment of $X_{i:n}$ is given by

$$E(X_{i:n}^r) = \int_{-\infty}^{\infty} x^r f_{i:n}(x) dx = \sum_{j=0}^{n-i} \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} c_{j,k,\ell,m} \int_{-\infty}^{\infty} x^r H_{j+i+\ell-1}(x) h_{m+1}(x) dx.$$

Note that the integral term only depends on the $g(x)$ and $G(x)$ the pdf and cdf of baseline distribution, respectively.

2.10 *Rényi entropy*

The Rényi entropy of X is given by

$$I_\delta = \frac{1}{1-\delta} E[f^{\delta-1}(X)] = \frac{1}{1-\delta}\int_{-\infty}^{\infty} f^\delta(x)dx \tag{2.9}$$

for $\delta > 0$ and $\delta \neq 1$. Using finite and generalized binomial series expansions (noticing that $\left|\lambda\left(\frac{2}{(1+G(x))^2}-1\right)\right| < 1$), we have

$$\begin{aligned} f^\delta(x) &= \left[1+\lambda\left(\frac{2}{(1+G(x))^2}-1\right)\right]^\delta g^\delta(x) = g^\delta(x)\sum_{k=0}^{\infty}\binom{\delta}{k}\lambda^k\left(\frac{2}{(1+G(x))^2}-1\right)^k \\ &= g^\delta(x)\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{\delta}{k}\binom{k}{\ell}\lambda^k(-1)^{k-\ell}2^\ell[1+G(x)]^{-2\ell} \\ &= g^\delta(x)\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=0}^{\infty}\binom{\delta}{k}\binom{k}{\ell}\binom{-2\ell}{m}\lambda^k(-1)^{k-\ell}2^\ell H_m(x). \end{aligned} \tag{2.10}$$

Hence, we have

$$I_\delta = \frac{1}{1-\delta}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=0}^{\infty}\binom{\delta}{k}\binom{k}{\ell}\binom{-2\ell}{m}\lambda^k(-1)^{k-\ell}2^\ell\int_{-\infty}^{\infty}H_m(x)g^\delta(x)dx.$$

It is worth noting that the integral term only depends on the $g(x)$ and $G(x)$ the pdf and cdf of the baseline distribution, respectively. Finally, let us mention that the Shannon entropy of a random variable X is defined by $E(-\log[f(X)])$ is the special case of the Rényi entropy when $\delta \to 1$.

2.11 *Mathai-Haubold entropy*

Classical Shannon entropy has been generalized in many directions. One of them is the δ-generalized entropy introduced by Mathai and Haubold (2013). It is defined by

$$f_{MH}(x) = \frac{1}{\delta-1}\left[\int_{-\infty}^{\infty} f^{2-\delta}(x)dx - 1\right].$$

Similar arguments to (2.10) gives

$$f^{2-\delta}(x) = g^{2-\delta}(x)\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=0}^{\infty}\binom{2-\delta}{k}\binom{k}{\ell}\binom{-2\ell}{m}\lambda^k(-1)^{k-\ell}2^\ell H_m(x).$$

Therefore,

$$f_{MH}(x) = \frac{1}{\delta - 1}\left[\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=0}^{\infty}\binom{2-\delta}{k}\binom{k}{\ell}\binom{-2\ell}{m}\lambda^k(-1)^{k-\ell}2^{\ell}\int_{-\infty}^{\infty}H_m(x)g^{2-\delta}(x)dx - 1\right].$$

The solution of the above integral depends on any arbitrary baseline distribution.

3. Special distributions

Let us now introduce some special distributions of the T-G family based on some well-known distributions.

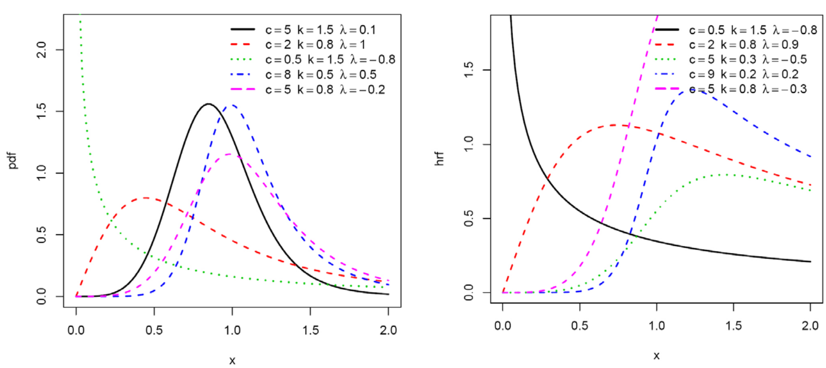
3.1 *Transmuted Burr (TB) distribution*

The Burr XII distribution has the pdf and cdf given by $g(x) = ckx^{c-1}(1+x^c)^{-k-1}$, $k, c, x > 0$, and $G(x) = 1-(1+x^c)^{-k}$, respectively. Then the cdf and pdf of the T-B distribution are, respectively, given by

$$F(x) = 1-(1+x^c)^{-k} + \lambda\frac{[1-(1+x^c)^{-k}][1+x^c]^{-k}}{2-(1+x^c)^{-k}}, \quad x > 0$$

and

$$f(x) = ckx^{c-1}(1+x^c)^{-k-1} + \lambda\frac{ckx^{c-1}(1+x^c)^{-k-1}\{1-[1-(1+x^c)^{-k}][3-[1+x^c]^{-k}]\}}{[2-[1+x^c]^{-k}]^2}, \quad x > 0.$$



(a) (b)

Figure 1

Plots of (a) pdf and (b) hrf for the TB distribution with different parameter values.

Using these functions, we immediately obtain the expression of the hrf. Figure 1 shows some plots of pdfs and hrfs for the TB distribution with arbitrary choices for the parameters (c,k,λ).

3.2 *Transmuted Gompertz (TGz) distribution*

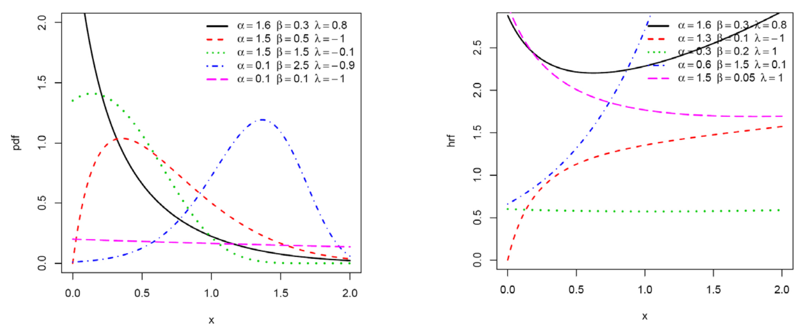
The Gompertz distribution has the pdf and cdf given by $g(x)=a\exp[bx-a(e^{x}-1)/b]$, $a,b,x>0$, and $G(x)=1-\exp[-a(e^{bx}-1)/b]$, respectively. Then the cdf and pdf of the T-Gz distribution are, respectively, given by

$$F(x)=\left[1-\exp\left[-\tfrac{a}{b}(e^{bx}-1)\right]\right]\left[1+\lambda\frac{\exp\left[-\tfrac{a}{b}(e^{bx}-1)\right]}{2-\exp\left[-\tfrac{a}{b}(e^{bx}-1)\right]}\right],\quad x>0$$

and

$$f(x)=a\exp\left[bx-\tfrac{a}{b}(e^{x}-1)\right]\left[1+\lambda\frac{\left\{1-\left[1-\exp\left[-\tfrac{a}{b}(e^{bx}-1)\right]\right]\left[3-\exp\left[-\tfrac{a}{b}(e^{bx}-1)\right]\right]\right\}}{\left[2-\exp\left[-\tfrac{a}{b}(e^{bx}-1)\right]\right]^{2}}\right],\quad x>0.$$

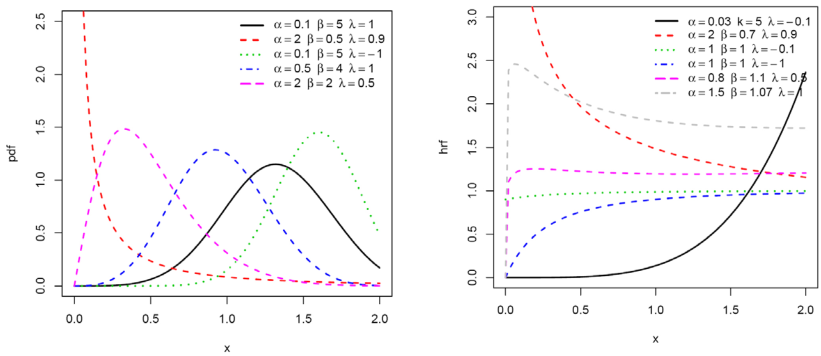
The expression of the hrf follows immediately. Figure 2 shows some plots of pdfs and hrfs for the TGz distribution with arbitrary choices for the parameters (a,b,λ).



(a) (b)

Figure 2

Plots of (a) pdf and (b) hrf for the TGz distribution with different parameter values.



(a) (b)

Figure 3

Plots of (a) pdf and (b) hrf for the TW distribution with different parameter values.

3.3 *Transmuted Weibull (TW) distribution*

The Weibull distribution has the pdf and cdf given by $g(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$, $\alpha, \beta, x > 0$, and $G(x) = 1 - e^{-\alpha x^{\beta}}$, respectively. It has received a wide variety of generalizations and extensions. We may refer to the studies of Dey et al. (2020) and Chakrabarty and Chowdhury (2020), and the references therein. Based on it, the cdf and pdf of the T-W distribution are, respectively, given by

$$F(x) = 1 - e^{-\alpha x^{\beta}} + \frac{\lambda[1 - e^{-\alpha x^{\beta}}]}{2 - e^{-\alpha x^{\beta}}} e^{-\alpha x^{\beta}}, \quad x > 0$$

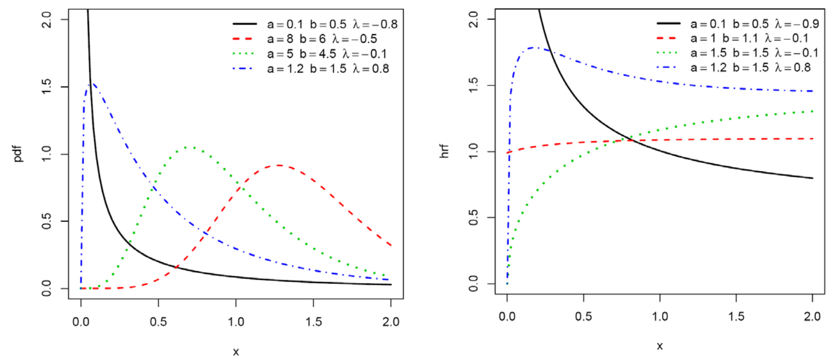
and

$$f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}} + \frac{\lambda\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}}{[2 - e^{-\alpha x^{\beta}}]^2}\{1 - [1 - e^{-\alpha x^{\beta}}][3 - e^{-\alpha x^{\beta}}]\}, \quad x > 0.$$

The expression of the hrf follows immediately. Figure 3 presents some plots of pdfs and hrfs for the TW distribution with arbitrary choices for the parameters (α, β, λ).

3.4 *Transmuted gamma (TGa) distribution*

The gamma distribution has the pdf and cdf given by $g(x) = b^a x^{a-1}e^{-bx} / \Gamma(a)$, $a, b, x > 0$, and $G(x) = \gamma(a, b, x) / \Gamma(a)$, where



(a) (b)

Figure 4

Plots of (a) pdf and (b) hrf for the TGa distribution with different parameter values.

$\gamma(a,b,x) = b^a \int_0^x t^{a-1} e^{-bt} dt$, respectively. Then, the cdf and pdf of the T-Ga distribution are, respectively, given by

$$F(x) = \frac{\gamma(a,b,x)}{\Gamma(a)}\left[1+\lambda\frac{1-\frac{\gamma(a,b,x)}{\Gamma(a)}}{1+\frac{\gamma(a,b,x)}{\Gamma(a)}}\right], \quad x>0$$

and

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}\left[1+\lambda\frac{\left\{1-\left[\frac{\gamma(a,b,x)}{\Gamma(a)}\left[2-\frac{\gamma(a,b,x)}{\Gamma(a)}\right]\right]\right\}}{\left[1+\frac{\gamma(a,b,x)}{\Gamma(a)}\right]^2}\right], \quad x>0.$$

We can easily obtain the hrf expression. Figure 4 presents some plots of pdfs and hrfs for the TGa distribution with arbitrary choices for the parameters (a,b,λ).

Figures 1, 2, 3 and 4 show a wide variety of shapes, illustrating the richness of the T-G family.

4. Bivariate extension

We now propose a bivariate version of the T-G family of distributions. Let (X,Y) be a bivariate random variable with the joint cdf given by

$$F_{X,Y}(x,y)=\left[1-\lambda+\frac{2\lambda}{1+G(x,y)}\right]G(x,y), \qquad \lambda\in[-1,1],(x,y)\in\mathbb{R}^2,$$

where $G(x,y)$ denotes a bivariate cdf. Let $G_1(x)$ and $G_2(y)$ be the corresponding marginal cdfs, $g_1(x)$ and $g_2(y)$ be the corresponding marginal pdfs and $g(x,y)$ be the corresponding pdf. Then, the marginal cdfs of (X,Y) are given by

$$F_X(x)=\left[1-\lambda+\frac{2\lambda}{1+G_1(x)}\right]G_1(x)$$

and

$$F_Y(y)=\left[1-\lambda+\frac{2\lambda}{1+G_2(y)}\right]G_2(y).$$

The marginal pdfs of (X,Y) are given by

$$f_X(x)=\left[1-\lambda+\frac{2\lambda}{(1+G_1(x))^2}\right]g_1(x)$$

and

$$f_Y(y)=\left[1-\lambda+\frac{2\lambda}{(1+G_2(y))^2}\right]g_2(y).$$

The pdf of (X,Y) is given by

$$f(x,y)=\left[1-\lambda+\frac{2\lambda}{(1+G(x,y))^2}\right]\theta(x,y),$$

where

$$\theta(x,y)=g(x,y)-\frac{4\lambda}{(1+G(x,y))[(1-\lambda)(1+G(x,y))^2+2\lambda]}\frac{\partial G(x,y)}{\partial x}\frac{\partial G(x,y)}{\partial y},$$

provided that it is positive, depending on the values of λ and the property of $g(x,y)$.

The conditional cdfs are given by

$$F_{X|Y}(x\mid y)=\frac{(1-\lambda)(1+G(x,y))+2\lambda}{(1-\lambda)(1+G_2(y))+2\lambda}\times\frac{G(x,y)(1+G_2(y))}{(1+G(x,y))G_2(y)}$$

and

$$F_{Y|X}(y \mid x) = \frac{(1-\lambda)(1+G(x,y))+2\lambda}{(1-\lambda)(1+G_1(x))+2\lambda} \times \frac{G(x,y)(1+G_1(x))}{(1+G(x,y))G_1(x)}.$$

The conditional pdfs are given by

$$f_{X|Y}(x \mid y) = \frac{(1-\lambda)(1+G(x,y))^2+2\lambda}{(1-\lambda)(1+G_2(y))^2+2\lambda} \times \frac{\theta(x,y)(1+G_2(y))^2}{(1+G(x,y))^2 g_2(y)}$$

and

$$f_{Y|X}(y \mid x) = \frac{(1-\lambda)(1+G(x,y))^2+2\lambda}{(1-\lambda)(1+G_1(x))^2+2\lambda} \times \frac{\theta(x,y)(1+G_1(x))^2}{(1+G(x,y))^2 g_1(x)}.$$

5. Different estimation methods

In this section, we will give the estimates of the T-G family by five different methods of estimation, i.e., the maximum likelihood method, the least square method, the weighted least square method, the maximum product spacing method and the Cramér-von Mises method.

5.1 *Maximum likelihood method (MLE)*

Here, we determine the maximum likelihood estimates (MLEs) of the model parameters of T-G from complete samples only. Let $x_1, x_2, \ldots x_n$ be n observed values from the T-G distribution with parameters λ and ξ. Let $\Theta = (\lambda, \xi)^T$ be the parameter vector. The total log-likelihood function for Θ is given by

$$\ell(\Theta) = \sum_{i=1}^{n} \log[g(x_i)] - 2\sum_{i=1}^{n} \log[1-G(x_i)] + \sum_{i=1}^{n} \log[(1+\lambda)+(1-\lambda)G(x_i)].$$

Maximizing $\ell(\Theta)$ with respect to λ and ξ, we have following system of nonlinear equations:

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \sum_{i=1}^{n} \left[\frac{1-G(x_i)\left\{2+G(x_i)\right\}}{(1+\lambda)+(1-\lambda)G(x_i)\left\{2+G(x_i)\right\}} \right] = 0,$$

$$\frac{\partial \ell(\Theta)}{\partial \xi} = \sum_{i=1}^{n} \left[\frac{g'(x_i)_\xi}{g(x_i)} \right] - 2\sum_{i=1}^{n} \left[\frac{G'(x_i)_\xi}{1+G(x_i)} \right] + \sum_{i=1}^{n} \left[\frac{2(1-\lambda)G'(x_i)_\xi \{1+G(x_i)\}}{(1+\lambda)+(1-\lambda)G(x_i)\{2+G(x_i)\}} \right] = 0,$$

where $g'(x_i)_\xi = \frac{\partial}{\partial \xi} g(x_i)$ and $G'(x_i)_\xi = \frac{\partial}{\partial \xi} G(x_i)$. This system of nonlinear equations can be solved numerically by any software to obtain the estimates $\hat{\lambda}_{MLE}$ and $\hat{\xi}_{MLE}$.

5.2 *Least square method (LSE)*

Let $x_1, x_2, \ldots, x_n$ be n observed values from a distribution of the T-G family with parameters λ and ξ, in increasing order. By considering the associated order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, note that $E[F(X_{i:n})] = i/(n+1)$. Least square estimates can be obtained by minimizing the following expression:

$$S(\Theta) = \sum_{i=1}^{n} [F(x_i) - E[F(X_{i:n})]]^2 = \sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1} \right]^2 .$$

Minimizing $S(\Theta)$ with respect to λ and ξ, we have following system of non linear equations:

$$\frac{\partial S(\Theta)}{\partial \lambda} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i)_\lambda = 0,$$

$$\frac{\partial S(\Theta)}{\partial \xi} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i)_\xi = 0,$$

where $F'(x_i)_\lambda = \frac{\partial}{\partial \lambda} F(x_i)$ and $F'(x_i)_\xi = \frac{\partial}{\partial \xi} F(x_i)$. This system of nonlinear equations can be solved numerically by any software to obtain the estimates $\hat{\lambda}_{LSE}$ and $\hat{\xi}_{LSE}$.

5.3 *Minimum spacing method (MPS)*

This method is based on the idea that the spacings between consecutive points should be identically distributed. Let $x_1, x_2, \ldots, x_n$ be n observed values from a distribution of the T-G family with parameters λ and ξ, in increasing order. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i},$$

where $D_i = F(x_i) - F(x_{i-1})$ with $F(x_0) = 0$ and $F(x_{n+1}) = 1$. We have

$$\log(GM) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log[F(x_i) - F(x_{i-1})].$$

Using the cdf in (1.2), we have

$$\log(GM) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[\left[1-\lambda+\frac{2\lambda}{1+G(x_i)}\right]G(x_i) - \left[1-\lambda+\frac{2\lambda}{1+G(x_{i-1})}\right]G(x_{i-1})\right].$$

Minimizing $\log(GM)$ with respect to λ and ξ, we have following system of nonlinear equations:

$$\frac{\partial\log(GM)}{\partial\lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{\left[-1+\frac{2}{1+G(x_i)}\right]G(x_i) - \left[-1+\frac{2}{1+G(x_{i-1})}\right]G(x_{i-1})}{\left[\left[1-\lambda+\frac{2\lambda}{1+G(x_i)}\right]G(x_i) - \left[1-\lambda+\frac{2\lambda}{1+G(x_{i-1})}\right]G(x_{i-1})\right]}\right] = 0,$$

$$\frac{\partial\log(GM)}{\partial\xi} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{\left\{\left[1-\lambda+\frac{2\lambda}{1+G(x_i)}\right]G'(x_i)_\xi + \left[\frac{2\lambda}{[1+G(x_i)]^2}\right]G(x_i)\right\}}{\left[\left[1-\lambda+\frac{2\lambda}{1+G(x_i)}\right]G(x_i) - \left[1-\lambda+\frac{2\lambda}{1+G(x_{i-1})}\right]G(x_{i-1})\right]}\right.$$

$$\left. - \frac{\left\{\left[1-\lambda+\frac{2\lambda}{1+G(x_{i-1})}\right]G'(x_{i-1})_\xi + \left[\frac{2\lambda}{[1+G(x_{i-1})]^2}\right]G(x_{i-1})\right\}}{\left[\left[1-\lambda+\frac{2\lambda}{1+G(x_i)}\right]G(x_i) - \left[1-\lambda+\frac{2\lambda}{1+G(x_{i-1})}\right]G(x_{i-1})\right]}\right] = 0.$$

This system of nonlinear equations can be solved numerically by any software to obtain the estimates $\hat{\lambda}_{MPE}$ and $\hat{\xi}_{MPE}$.

5.4 *Weighted least square (WLS)*

Let $x_1, x_2, \ldots, x_n$ be *n* observed values from a distribution of the T-G family with parameters λ and ξ, in increasing order. The likelihood function for weighted least square estimates is given by

$$W(\Theta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i) - \frac{i}{n+1}\right]^2.$$

Minimizing $W(\Theta)$ with respect to λ and ξ, we have following system of nonlinear equations:

$$\frac{\partial W(\Theta)}{\partial \lambda} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i)_\lambda = 0,$$

$$\frac{\partial W(\Theta)}{\partial \xi} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i)_\xi = 0.$$

These equations can be solved numerically to obtain the estimates $\hat{\lambda}_{wls}$ and $\hat{\xi}_{wls}$.

5.5 *Cramér-von Mises (CVM)*

Cramér-von Mises estimators are type of minimum distance estimators. Let $x_1, x_2, \ldots, x_n$ be n observed values from a distribution of the T-G family with parameters λ and ξ, in increasing order. The likelihood function for Cramér-von Mises estimates is given by

$$C(\Theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right]^2.$$

Minimizing $C(\Theta)$ with respect to λ and ξ, we have following system of nonlinear equations:

$$\frac{\partial C(\Theta)}{\partial \lambda} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\lambda = 0,$$

$$\frac{\partial C(\Theta)}{\partial \xi} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\xi = 0.$$

This system of nonlinear equations can be solved numerically to obtain the estimates $\hat{\lambda}_{CME}$ and $\hat{\xi}_{CME}$.

6. Simulation results and applications to real-life data

In this section, we present some experimental results to compare the performance of the different estimators proposed in the previous section. Then, we use two real-life data sets set to compare the TB, TW, TGa and TGz models to the TGE (transmuted generalized exponential), TLE (transmuted linear exponential), GT-W (generalized transmuted Weibull), Burr XII, Weibull, gamma and Gompertz models.

6.1 *Simulation results*

We perform extensive Monte Carlo simulations to compare the performance of the different estimators (CVM, LSE, WLS, MPS and MLE), mainly with respect to their biases and mean-squared errors (MSEs) for different sample sizes and for different parameter values. For obtaining different estimators, the number of replications is 10000 in all the simulations. We have considered the following different sample sizes: $n = 10, 50, 100$ and 250. For obtaining frequentist estimators, we consider the TGz model with $a = 10,$ $b = 2.5$ $\lambda = 0.5,$ because it was the superior among the considered sub-models of the T-G family of models as it will be shown in the next subsection. The simulation results are provided in Table 1.

Table 1

Simulation results for the TGz model.

n	parameters	CVM	LSE	WLS	MPS	MLE
20	Bias ($\hat{a}$)	1.334	4.044	3.495	0.385	0.275
	MSE ($\hat{a}$)	1.643	5.158	4.291	0.949	1.191
	Bias ($\hat{b}$)	0.300	0.240	0.264	0.017	0.111
	MSE ($\hat{b}$)	0.300	0.336	0.400	0.388	0.480
	Bias ($\hat{\lambda}$)	0.077	0.092	0.085	0.041	0.050
	MSE ($\hat{\lambda}$)	0.125	0.203	0.185	0.070	0.089
50	Bias ($\hat{a}$)	1.121	3.777	3.870	0.580	0.033
	MSE ($\hat{a}$)	1.233	4.153	4.096	1.296	0.799
	Bias ($\hat{b}$)	0.300	0.268	0.368	0.201	0.034
	MSE ($\hat{b}$)	0.300	0.294	0.438	0.347	0.339
	Bias ($\hat{\lambda}$)	0.071	0.091	0.092	0.032	0.034
	MSE ($\hat{\lambda}$)	0.114	0.201	0.205	0.048	0.058
100	Bias ($\hat{a}$)	1.166	3.459	4.179	0.535	0.010
	MSE ($\hat{a}$)	1.869	3.685	4.313	1.265	0.589
	Bias ($\hat{b}$)	0.307	0.249	0.427	0.179	0.035
	MSE ($\hat{b}$)	0.465	0.285	0.472	0.326	0.269
	Bias ($\hat{\lambda}$)	0.021	0.088	0.099	0.021	0.023
	MSE ($\hat{\lambda}$)	0.033	0.191	0.219	0.032	0.040
250	Bias ($\hat{a}$)	0.747	3.452	4.263	0.290	0.028
	MSE ($\hat{a}$)	1.510	3.560	4.337	1.049	0.438
	Bias ($\hat{b}$)	0.198	0.261	0.440	0.102	0.024
	MSE ($\hat{b}$)	0.346	0.281	0.462	0.240	0.221
	Bias ($\hat{\lambda}$)	0.014	0.089	0.100	0.012	0.017
	MSE ($\hat{\lambda}$)	0.022	0.191	0.223	0.020	0.028

The results indicate that the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.

6.2 *Empirical reliability data examples*

In this subsection, we evaluate the performance of the T-G family of distributions by fitting four sub-models of this family, namely the TB, TW, TGa and TGz models, to two reliability data sets. The data sets are described as follows.

The first data set (data set 1) refers to the 50 observations with hole and sheet thickness are 12 mm and 3.15 mm reported by Dasgupta (2011). The data are: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

The second data set (data set 2) relates to the strengths of 1.5 cm glass fibers given by Oguntunde et al. (2017). This set was obtained by workers at the UK National Physical Laboratory. The observations are as follows: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24.

The descriptive statistics of the two data sets are given in Table 2. From this table, both data sets are under-dispersed. The data set 1 is approximately symmetric and the data set is moderately skewed. Moreover, the first data set is platykurtic while the second one is leptokurtic. Moreover, some measures of the TGz model for both data sets are obtained by using its theoretical properties and are shown in Table 3. Making use of Tables 2 and 3, we can note the closeness between measures of the TGz model and the corresponding descriptive statistics of both data sets.

For all the compared distributions, the maximum likelihood method is used to estimate the parameters and also their standard errors are obtained. The model adequacy measures: Anderson-Darling (A*), Cramér-von Mises (W*) and Kolmogorov-Smirnov (K-S) statistics with its p-value are used to compare these distributions, where the smaller values of these statistics and larger p-value give the best fit to the data. The obtained results are given in Tables 4 and 5 for data sets 1 and 2, respectively, and we conclude that the considered sub-models of the family are good competitors to the compared distributions. It is also discovered that the TGz model has the smallest values of A*, W*, K-S, and largest values of p-value, indicating that it outperforms the other distributions. This conclusion is asserted again by Figures 5 and 6 for data sets 1 and 2, respectively.

Also, the variance-covariance matrix of the MLEs of the TGz model for data set 1 is given as

$$\begin{pmatrix} 5.779 & -11.185 & -2.841 \\ -11.185 & 22.275 & 5.421 \\ -2.841 & 5.421 & 1.509 \end{pmatrix}.$$

The variance-covariance matrix of the MLEs of the TGz model for data set 2 is given as

$$\begin{pmatrix} 0.001 & -0.007 & -0.006 \\ -0.007 & 0.226 & 0.163 \\ -0.006 & 0.163 & 0.202 \end{pmatrix}.$$

Note that the diagonal entries of those matrices are the variances of the MLEs of the TGz model parameters for each data set, while the other entries can lead to positive and negative correlations for pairs of the estimates. Also, 95% and 99% confidence intervals of the TGz model parameters are given in Tables 6 and 7 for both data sets, respectively.

Table 2

Descriptive statistics for the two data sets.

Statistics	Mean	SD	Median	MD Mean	MD Median	Skewness	Kurtosis	Entropy
Data set 1	0.16	0.08	0.16	0.065	0.07	0.07	2.21	2.47
Data set 2	1.51	0.32	1.59	0.24	0.23	-0.88	3.92	3.80

Table 3

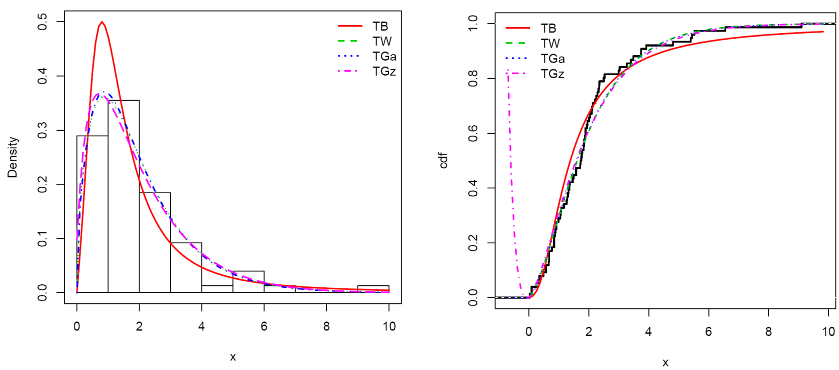
Some moments of the TGz model for the two data sets.

Statistics	Mean	S.D	Median	Variance	Skewness	Kurtosis	Entropy	MD Mean	MD Median
Data set 1	0.163	0.079	0.177	0.006	0.093	2.606	2.68	0.065	0.069
Data set 2	1.52	0.32	1.60	0.11	-0.90	4.20	3.86	0.25	0.23

Table 4

MLEs, their standard errors (in parentheses), A*, W*, KS, and P-value for data set 1.

Distribution		MLE's		A*	W*	KS	P-value
TB (c,k,λ)	2.148 (0.321)	38.854 (16.111)	-0.0097 (0.555)	0.660	0.107	0.109	0.581
TW (α,β,λ)	35.904 (15.729)	2.108 (0.334)	-0.025 (0.562)	0.642	0.104	0.109	0.587
TGa (a,b,λ)	2.818 (0.678)	18.218 (3.924)	-0.293 (0.442)	1.045	0.173	0.142	0.236
TGz (a,b,λ)	2.285 (1.404)	8.765 (4.719)	-0.421 (0.228)	0.445	0.072	0.097	0.736
TGE (α,θ,λ)	2.693 (0.839)	12.493 (1.666)	-0.547 (0.318)	1.076	0.178	0.147	0.234
TLE (β,θ,λ)	0.612 (1.895)	65.579 (14.184)	-0.378 (0.441)	0.570	0.096	0.107	0.630
GT-W (η,σ,λ)	0.170 (0.023)	1.991 (0.330)	-0.271 (0.430)	0.607	0.098	0.117	0.612
Burr(c, k)	2.154 (0.240)	39.124 (15.013)	- -	0.675	0.108	0.120	0.571
Weibull (α,β)	36.145 (14.393)	2.117 (0.245)	- -	0.649	0.108	0.110	0.569
Gamma(a, b)	3.029 (0.575)	18.564 (3.836)	- -	1.120	0.192	0.145	0.170
Gompertz(a, b)	1.605 (0.512)	10.221 (1.753)	- -	0.475	0.0875	0.135	0.568



(a) (b)

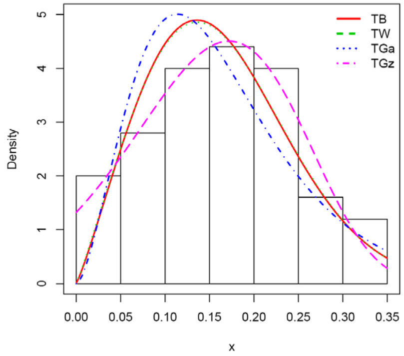
Figure 5

Plots of the estimated (a) pdfs and (b) cdfs for data set 1.

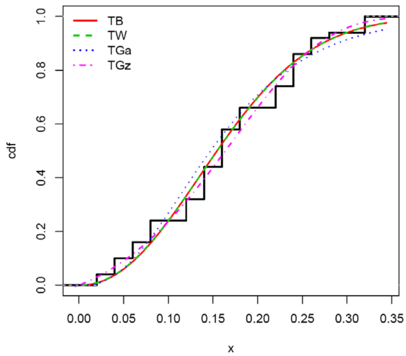
Table 5

MLEs, their standard errors (in parentheses), A*, W*, KS, and P-value for data set 2.

Distribution		MLE's		A*	W*	KS	P-value
TB (c,k,λ)	5.889	0.492	-0.909	5.751	1.055	0.293	0.003
	(1.208)	(0.115)	(0.173)				
TW (α,β,λ)	0.077	5.467	-0.314	1.242	0.226	0.144	0.442
	(0.034)	(0.680)	(0.366)				
TGa (a,b,λ)	16.354	11.241	-0.465	2.921	0.532	0.201	0.223
	(3.177)	(2.105)	(0.266)				
TGz (a,b,λ)	0.020	3.177	-0.629	0.808	0.140	0.124	0.685
	(0.016)	(0.477)	(0.450)				
Burr (c,k)	7.482	0.320	-	6.128	1.132	0.330	0.002
	(1.280)	(0.065)	-				
Weibull (α,β)	0.059	5.77	-	1.303	0.237	0.152	0.307
	(0.020)	(0.575)	-				
Gamma (a,b)	17.439	11.573	-	3.117	0.568	0.216	0.105
	(3.078)	(2.072)	-				
Gompertz (a,b)	0.009	3.618	-	0.842	0.144	0.130	0.437
	(0.004)	(0.294)	-				



(a)



(b)

Figure 6

Plots of the estimated (a) pdf and (b) cdf for data set 2.

Table 6

Confidence intervals of the TGz model for data set 1.

CI	a	b	λ
95%	[0,5.036]	[0,18.014]	[0,0.025]
99%	[0,5.907]	[0,20.940]	[0,0.167]

Table 7

Confidence intervals of the TGz model for data set 2.

CI	a	b	λ
95%	[0, 0.051]	[2.242, 4.111]	[0, 0.253]
99%	[0, 0.061]	[1.946, 4.407]	[0, 0.532]

7. Conclusion

A new transmuted family of distributions has been presented. The investigations concern theory and practice. The theory includes the main properties of the family, such as moments, incomplete moments, the moment generating function, measures of entropy, and order statistics. The practice focuses on the inferential properties of the family, with an emphasis on the adjustment of data sets of various natures by taking into account special models. The few members extracted show skills in this regard, surpassing standard models in the field. We believe other members of the new family may also be the subject of an investigation. Their applications for modern dataset analysis are also options.

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