

A critical note on moments of Cauchy distribution based on LOTUS theorem

G. S. David Sam Jayakumar *

Department of Management Studies

Jamal Institute of Management

Jamal Mohamed College (Affiliated to Bharathidasan University)

Trichy 620020

Tamil Nadu

India

W. Samuel [†]

Department of Business Administration

Jamal Mohamed College (Affiliated to Bharathidasan University)

Trichy 620020

Tamil Nadu

India

Bejoy John Thomas [§]

Department of Business Administration

Rajagiri College of Social Sciences

Rajagiri Business School

Cochin 683104

Kerala

India

Abstract

The moments of Cauchy distribution in statistics are still unexplored and the statisticians already concluded that they are undefined when it involves direct integration.

* ORCID-ID: 0000-0002-6630-8783

† ORCID-ID: 0000-0002-9112-9744

§ ORCID-ID: 0000-0002-8395-3063

* E-mail: samjaya77@gmail.com (Corresponding Author)

† E-mail: wsamuel1365@gmail.com;
wsm@jmc.edu

§ E-mail: bejoy@rajagiri.edu

The authors explored the higher order moments of the Cauchy distribution based on LOTUS theorem (Law of the unconscious statistician). At first, the theorem is applied to compute the moments of the normal distribution and the result of the moments are similar when we compared it with the existing results. Hence, this motivates the authors to derive the finite sample moments of Cauchy distribution are numerically displayed. The results derived by the authors are surprising because the real valued random variable of a Cauchy distribution is having finite moments in the complex plane and the moments are expressed in complex numbers. The first ten sample finite moments of the Cauchy distribution are visualized with the help of polar plots to know the magnitude and proximity among the moments.

Subject Classification: 62G30, 62H10.

Keywords: Moments, Lotus theorem, Cauchy distribution, Normal distribution, Complex plane, Complex numbers, Polar plots.

Some preliminaries

Explicit expressions for the Central and Non-Central moments of the Normal and Cauchy distribution involves several special functions (Prudnikov, Brychkov, and Marichev [1] and Gradshteyn and Ryzhik [5]) and they are given as follows:

1. The Generalized hyper-geometric function of order p and q is defined as

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \left(\frac{x^k}{k!} \right)$$

2. The rising factorial or Pochhammer symbol is given as

$$(a)_k = a(a+1)(a+2) \dots (a+k-1)$$

3. The integral representation of the Gamma function is

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

4. The Binomial series expression form of the following series is given as

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

5. If $X \sim ND(\mu, \sigma^2)$, then it's density function with mean μ and Variance σ^2 are given as

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $-\infty < x < +\infty$, $-\infty < \mu < +\infty$, $\sigma > 0$

6. If $X \sim CD(\mu, \sigma)$, then it's density function with Location μ and Scale σ are given as

$$f_x(x) = 1/\pi\sigma \left(1 + \left(\frac{x-\mu}{\sigma} \right)^2 \right)^{-1}$$

where $-\infty < x < +\infty$, $-\infty < \mu < +\infty$, $\sigma > 0$

7. The Lotus theorem of the Continuous random variable 'X' belongs to the function $g_x(x)$, then the expected value of the function is given as

$$E(g_x(x)) = \int g_x(x) f_x(x) dx, \text{ where } g \neq f$$

$$E(f_x(x)) = \int (f_x(x))^2 dx, \text{ where } g = f$$

1. Cauchy distribution, undefined moments and related reviews

The Cauchy distribution, first introduced by Augustin Louis Cauchy in 1824, exhibits unique properties compared to other symmetric unbounded distributions. A key characteristic of this distribution is the absence of raw higher-order moments while possessing lower-order absolute moments, making it widely applicable in various fields such as spectroscopy, exponential growth models, hydrology, electrical permittivity, and computational finance. Several authors, including Stigler [14], Poisson [13], and Eisenhart [3], have extensively studied the Cauchy distribution. Nadarajah [11] pointed out a significant drawback of the Cauchy distribution, ie: the absence of finite moments. This limitation makes it an unrealistic choice as a statistical model, leading to the introduction of a truncated version to address this issue. In this manuscript, the authors note that existing methods for computing higher-order moments of the Cauchy distribution suffer from mathematical

inadequacies, such as the non-convergence of moment integrals. Ogunsanya, Akarawak, and Ekum [12] introduces a new three-parameter Raleigh–Cauchy Distribution (RCD) as an extension of the Cauchy distribution using the T–R{Y} family. It develops the theoretical properties, reliability measures, simulation results, and parameter estimation methods for the distribution, and demonstrates its practical usefulness through a real data application.

The term “moment,” originating from physics, has significant applications in statistics. Summarizing a probability distribution or observed data set using moments or cumulants is a standard statistical practice. When fitting a parametric distribution to data, it is common to estimate parameters by equating sample moments with those of the theoretical distribution. However, despite their long-standing use in statistics, moment-based methods are not always reliable. Higher-order moments do not always provide clear insights into a distribution’s shape, sample moments may differ significantly from their population counterparts (especially in small samples), and parameter estimates obtained via the method of moments are often less accurate than those derived from other techniques, such as maximum likelihood estimation (Hosking [7]). Comprehensive discussions on moments can be found in works by Ewis [8], Galkin and Ufimtsev [15], Hosking [7], Charalambides and Papadatos [2], Broca [4], and Breton and Prévault [6]. Without altering the classical form of the Cauchy density and its bounds, the authors apply the Lotus theorem (Ross, [10]) to explore the higher-order non-central moments of the Cauchy distribution, which is examined in detail in the following section.

2. Moments of Normal distribution

Theorem 3.1: *The Non-Central moment of r^{th} order of the Normal distribution is given as*

$$E_x(x^r) = \mu^r {}_2F_0\left(-r/2, (1-r)/2; 2(\sigma/\mu)^2\right) \quad (2.1)$$

Where ${}_2F_0(\)$ is the generalized hyper-geometric function.

Proof: We Know from Preliminaries 5, based on the density function of Normal distribution solve it for the random variable ‘X’, we get

$$x = \mu + \sigma \sqrt{2} \left(\log \left(\frac{1}{\sigma \sqrt{2\pi} f_x(x)} \right) \right)^{1/2} \quad (2.2)$$

Now from (2.2) take mathematical expectation for the r^{th} order, it follows on writing

$$E_x(x^r) = \mu^r E_x \left[1 + \left(\frac{\sigma}{\mu} \right) \sqrt{2} \left(\log \left[\frac{1}{\sigma \sqrt{2\pi} f_x(x)} \right] \right)^{1/2} \right]^r \quad (2.3)$$

From Binomial expansion, Preliminaries 7 and Substitution of the density function of Normal distribution in (2.3), further it can be simplified as

$$E_x(x^r) = \mu^r \sum_{k=0}^r \binom{r}{k} \left(\frac{1}{\mu} \right)^k E_x(x - \mu)^k \quad (2.4)$$

We know the odd power of the central moments of Normal distribution are vanished in (2.4) and rewrite in terms of the even order moments as

$$E_x(x^r) = \mu^r \sum_{k=0}^r \binom{r}{2k} \left(\frac{1}{\mu} \right)^{2k} E_x(x - \mu)^{2k} \quad (2.5)$$

By using the result $E_x(x - \mu)^{2k} = \int_{-\infty}^{+\infty} (x - \mu)^{2k} f_x(x) dx = \frac{2^k \sigma^{2k}}{\sqrt{\pi}} \Gamma(k + 1/2)$ in (2.5) and expand the binomial series, then the final form of non-central moment of r^{th} is

$$E_x(x^r) = \mu^r {}_2F_0 \left(-r/2, (1-r)/2; 2(\sigma/\mu)^2 \right) \quad (2.6)$$

From (2.6), if $r = 1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10, then the following order of non-central moments are derived as follows:

$$E_x(x) = \mu$$

$$E_x(x^2) = \mu^2 + \sigma^2$$

$$E_x(x^3) = \mu^3 + 3\mu\sigma^2$$

$$E_x(x^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$$

$$E_X(x^5) = \mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4$$

$$E_X(x^6) = \mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$$

$$E_X(x^7) = \mu^7 + 21\mu^5\sigma^2 + 105\mu^3\sigma^4 + 105\mu\sigma^6$$

$$E_X(x^8) = \mu^8 + 28\mu^6\sigma^2 + 210\mu^4\sigma^4 + 420\mu^2\sigma^6 + 105\mu^8$$

$$E_X(x^9) = \mu^9 + 36\mu^7\sigma^2 + 378\mu^5\sigma^4 + 1260\mu^3\sigma^6 + 945\mu\sigma^8$$

$$E_X(x^{10}) = \mu^{10} + 45\mu^8\sigma^2 + 630\mu^6\sigma^4 + 3150\mu^4\sigma^6 + 4725\mu^2\sigma^8 + 945\sigma^{10}$$

Theorem 3.2: The Central moment of r^{th} order of the Normal distribution is given as

$$E_X(x - \mu)^r = \mu^r \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} {}_2F_0\left(-k/2, (1-k)/2; 2(\sigma/\mu)^2\right) \quad (2.7)$$

where ${}_2F_0(\)$ is the generalized hyper-geometric function

Proof: The central moments of the normal distribution are if $r=1, 2, 3, 4, 3, 5, 6, 7, 8, 9$ and 10 are derived as follows:

$$E_X(x - \mu) = 0, E_X(x - \mu)^2 = \sigma^2$$

$$E_X(x - \mu)^3 = 0, E_X(x - \mu)^4 = 3\sigma^4$$

$$E_X(x - \mu)^5 = 0, E_X(x - \mu)^6 = 15\sigma^6$$

$$E_X(x - \mu)^7 = 0, E_X(x - \mu)^8 = 105\sigma^8$$

$$E_X(x - \mu)^9 = 0, E_X(x - \mu)^{10} = 945\sigma^{10}$$

3. Moments of cauchy distribution

Theorem 3.1: The non-central moment of r^{th} order of the Cauchy distribution is given as

$$S_r = (\mu + \sigma i)^r \quad (3.1)$$

where $\mu + \sigma i$ is the Complex number, μ and σ are the real and imaginary part respectively.

Proof: We Know from Preliminaries 6, if $X \sim CD(\mu, \sigma)$, then the inverse of the density function of Cauchy distribution can be written in terms of the random variable X is

$$x = \mu + \sigma \sqrt{\frac{1}{\pi\sigma} \left(\frac{1}{f_x(x)} \right) - 1} \quad (3.2)$$

Now from (3.2) take mathematical expectation for the r^{th} order, it follows on writing

$$E_x(x^r) = \mu^r E_x \left(1 + \frac{\sigma}{\mu} \sqrt{\frac{1}{\pi\sigma} \left(\frac{1}{f_x(x)} \right) - 1} \right)^r \quad (3.3)$$

By using binomial expansion in (3.3), further it can be simplified as

$$E_x(x^r) = \mu^r \sum_{k=0}^r (-1)^{k/2} \binom{r}{k} \left(\frac{\sigma}{\mu} \right)^k E_x \left(1 - \frac{1}{\pi\sigma} \left(\frac{1}{f_x(x)} \right) \right)^{k/2} \quad (3.4)$$

From (3.4), the index ' k ' is fractional and again using binomial expansion for the expectation term and rewrite as

$$E_x \left(1 - \frac{1}{\pi\sigma} \left(\frac{1}{f_x(x)} \right) \right)^{k/2} = \sum_{l=0}^{\infty} (-1)^l \binom{k/2}{l} \left(\frac{1}{\pi\sigma} \right)^l E_x \left(\frac{1}{f_x(x)} \right)^l \quad (3.5)$$

Now from (3.5) and Preliminaries.7, the result of the expectation term is derived as

$$E_x \left(\frac{1}{f_x(x)} \right)^l = \frac{1}{\pi^{1-l} \sigma^{1-l}} \int_{-\infty}^{+\infty} \frac{1}{\left(1 + \left(\frac{x-\mu}{\sigma} \right)^2 \right)^{1-l}} dx = \frac{\sigma^l}{\pi^{1-l}} B(1/2, (1/2)-l) \quad (3.6)$$

Substitute (3.6) in (3.5), we get the result of $E_x \left(1 - \frac{1}{\pi\sigma} \left(\frac{1}{f_x(x)} \right) \right)^{k/2} = 1$, then the final result of (3.6) found as

$$E_X(x^r) = (\mu + \sigma i)^r \quad (3.7)$$

From (3.7), if $r=1, 2, 3, 4, 3, 5, 6, 7, 8, 9$ and 10, then the following order of non-central moments of Cauchy distribution are given as follows:

$$E_X(x) = \mu + \sigma i$$

$$E_X(x^2) = (\mu^2 - \sigma^2) + (2\mu\sigma)i$$

$$S_3 = (\mu^3 - 3\mu\sigma^2) + (3\mu^2\sigma - \sigma^3)i$$

$$E_X(x^4) = (\mu^4 - 6\mu^2\sigma^2 + \sigma^4) + (4\mu^3\sigma - 4\mu\sigma^3)i$$

$$E_X(x^5) = (\mu^5 - 10\mu^3\sigma^2 + 15\mu\sigma^4) + (15\mu^4\sigma - 10\mu^2\sigma^3 + \sigma^5)i$$

$$E_X(x^6) = (\mu^6 - 15\mu^4\sigma^2 + 15\mu^2\sigma^4 - \sigma^6) + (6\mu^5\sigma - 20\mu^3\sigma^3 + 6\mu\sigma^5)i$$

$$E_X(x^7) = (\mu^7 - 21\mu^5\sigma^2 + 35\mu^3\sigma^4 - 7\mu\sigma^6) + (7\mu^6\sigma - 35\mu^4\sigma^3 + 21\mu^2\sigma^5 - \sigma^7)i$$

$$E_X(x^8) = (\mu^8 - 28\mu^6\sigma^2 + 70\mu^4\sigma^4 - 28\mu^2\sigma^6 + \sigma^8) + (56\sigma^5\mu^3 - 56\sigma^3\mu^5 - 8\mu\sigma^7 + 8\mu^7\sigma)i$$

$$E_X(x^9) = (\mu^9 - 36\mu^7\sigma^2 + 126\mu^5\sigma^4 - 84\mu^3\sigma^6 + 9\mu\sigma^8) + (\sigma^9 - 84\sigma^3\mu^6 + 126\mu^4\sigma^5 + 9\mu^8\sigma - 36\mu^2\sigma^7)i$$

$$E_X(x^{10}) = (\mu^{10} - 45\mu^8\sigma^2 + 210\mu^6\sigma^4 - 210\mu^4\sigma^6 + 45\mu^2\sigma^8 - \sigma^{10}) + (-120\mu^7\sigma^3 + 252\mu^5\sigma^5 + 10\mu^9\sigma - 120\mu^3\sigma^7 + 10\mu\sigma^9)i$$

Theorem 4.2: The Central r^{th} order moments of the Cauchy distribution is given as

$$E_X(x - E_X(x))^r = 0 \quad (3.8)$$

Proof: The Central r^{th} order moments of the distribution can be represented in the binomial series form as

$$\begin{aligned} E_X\left((x - E_X(x))^r\right) &= E_X\left(\sum_{k=0}^r \binom{r}{k} x^k (-E_X(x))^{r-k}\right) \\ &= \sum_{k=0}^r \binom{r}{k} E_X(x^k) (-E_X(x))^{r-k} \end{aligned} \quad (3.9)$$

Then substitute the central moments from (3.7) in (3.9) we will get

$$\begin{aligned}
 &= \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} (\mu + i\sigma)^k (\mu + i\sigma)^{r-k} \\
 &= (\mu + i\sigma)^r \sum_{k=0}^r (-1)^{r-k} \binom{r}{k}
 \end{aligned} \tag{3.10}$$

Then, simplify (3.10), we came to know the central moments of Cauchy distribution of all order is zero and it is proved in (3.8).

Result 4.3: The Skewness and Kurtosis of the Cauchy distribution based on the Central moments from theorem 4.2 are indeterminant form

4. Numerical illustration

Result 4.1: In this section, the authors demonstrate the application and computation of sample central and non-central moments for the Normal and Cauchy distributions using three different datasets. First, the authors utilize the experimental Iris plant dataset (1936), which contains four variables—sepal length, sepal width, petal length, and petal width—for three species of Iris (Setosa, Versicolor, and Virginica) measured in centimeters, with a total of 150 instances. Next, the authors analyze a time series dataset consisting of the daily stock prices of Netflix from May 23, 2002, to January 31, 2025. This dataset, sourced from Yahoo Finance, includes 5,710 observations, from which stock returns are computed and utilized. Finally, a simulated dataset of 500 observations is used, denoted as variable 'X,' which follows a standard normal distribution with a mean of 0 and a variance of 1. Moreover, Math wave Easy fit professional software 5.5 utilized for performing maximum likelihood estimation of parameters and fitting of normal and Cauchy distribution. Similarly, Maple version 18 also used for the computation of the sample moments of the distributions and the moments of Cauchy distribution are also transformed into Polar Co-ordinates to find the radius and angle of each higher order moment and its magnitude. The following table-1 visualizes the results of maximum likelihood estimation of parameters and table 2-7 shows the results of fitted Sample central and non-central moments of both the distributions along with the Polar Plots of each moment of Cauchy distribution.

Table 1
Maximum Likelihood estimation of Parameters

Variables	Normal distribution		Maximised Log-likelihood	Cauchy distribution		Maximised Log-likelihood
	Parameters		Max ($\log L(\hat{\mu}, \hat{\sigma})$)	Parameters		Max ($\log L(\hat{\mu}, \hat{\sigma})$)
	$\hat{\mu}$	$\hat{\sigma}$		$\hat{\mu}$	$\hat{\sigma}$	
Sepal length	5.8433	0.82807	-184.0414	5.7815	0.55914	-219.3640
Sepal Width	3.0573	0.43587	-87.7778	3.0183	0.23694	-108.4249
Petal length	3.758	1.7653	-297.5887	4.4893	1.041	-335.1405
Petal Width	1.1993	0.76224	-171.6163	1.3729	0.51676	-213.6932
Netflix Stock Returns	0.05431	3.62900	-15458.8865	0.02779	1.41223	-14553.4755
X	0	1	-677.4035	0.040465	0.594227	-778.20800

Table 2
First ten Sample Non-central and Central moments of Sepal Length

Order	Normal distribution		Cauchy distribution	
r	Moments			
	Non-central	Central	Non-central	Central
1	5.8433	0	5.7815+0.55914 <i>i</i>	0
2	34.8298	0.68569	33.1131+6.46533 <i>i</i>	0
3	211.5347	0	187.8283+55.8942 <i>i</i>	0
4	1307.7097	1.41055	1054.677+428.174 <i>i</i>	0
5	8221.5378	0	5858.206+3065.204 <i>i</i>	0
6	52524.3944	4.83655	32155.341+20997.035 <i>i</i>	0
7	340740.8413	0	174170+139370 <i>i</i>	0
8	2243162.77	23.24000	929010.30+903172.099 <i>i</i>	0
9	14976640.77	0	4866073.412+5741136.31 <i>i</i>	0

10	101356233.90	136.99294	24923104.48+35913195.87 <i>i</i>	0
----	--------------	-----------	----------------------------------	---

Table 3**First ten Sample Non-central and Central moments of Sepal Width**

Order	Normal distribution		Cauchy distribution	
r	Moments			
	Non-central	Central	Non-central	Central
1	3.0573	0	3.0183+0.23694 <i>i</i>	0
2	9.5370	0.18998	9.05399+1.43031 <i>i</i>	0
3	30.3193	0	26.98877+6.46236 <i>i</i>	0
4	98.1309	0.10828	79.92902+25.90007 <i>i</i>	0
5	323.0563	0	235.11300+97.11257 <i>i</i>	0
6	1080.896	0.10285	686.63171+348.82255 <i>i</i>	0
7	3672.874	0	1989.81048+1215.54164 <i>i</i>	0
8	12666.538	0.13702	5717.83456+4140.33503 <i>i</i>	0
9	44307.667	0	16277.12908+13851.55696 <i>i</i>	0
10	157119.635	0.244080	45847.27078+45664.85733 <i>i</i>	0

Table 4**First ten Sample Non-central and Central moments of Petal Length**

Order	Normal distribution		Cauchy distribution	
r	Moments			
	Non-central	Central	Non-central	Central
1	3.758	0	4.4893+1.041 <i>i</i>	0
2	17.23884	3.11628	19.07013+9.34672 <i>i</i>	0
3	88.20558	0	75.88161+61.81225 <i>i</i>	0
4	492.64002	29.13368	276.30876+356.48649 <i>i</i>	0
5	2950.83581	0	869.33051+1888.01225 <i>i</i>	0
6	18765.27231	453.94417	1937.26470+9380.82646 <i>i</i>	0
7	125693.7496	0	-1068.4778+44130.03679 <i>i</i>	0
8	881702.5479	9902.331	-50736.086+197000.6887 <i>i</i>	0
9	6447017.631	0	-432847.228+831578.925 <i>i</i>	0
10	48956612.85	27726.00	-2808854.724+3282613.30 <i>i</i>	0

Table 5
First ten Sample Non-central and Central moments of Petal Width

Order	Normal distribution		Cauchy distribution	
r	Moments			
	Non-central	Central	Non-central	Central
1	1.19930	0	1.3729+.51676 <i>i</i>	0
2	2.01933	0.58101	1.61781+1.41891 <i>i</i>	0
3	3.81539	0	1.48785+2.78405 <i>i</i>	0
4	8.09555	1.01272	0.60398+4.59109 <i>i</i>	0
5	18.57612	0	-1.54327+6.61523 <i>i</i>	0
6	45.79632	2.94199	-5.53725+8.28454 <i>i</i>	0
7	119.68097	0	-11.88322+8.51242 <i>i</i>	0
8	329.79017	11.96529	-20.71335+5.54593 <i>i</i>	0
9	951.80392	0	-31.30327-3.08982 <i>i</i>	0
10	2866.00038	62.56755	-41.37956-20.41830 <i>i</i>	0

Table 6
First ten Sample Non-central and Central moments of Netflix stock Returns

Order	Normal distribution		Cauchy distribution	
r	Moments			
	Non-central	Central	Non-central	Central
1	0.05431	0	$0.02779 + 1.41223i$	0
2	13.17259	13.16964	$-1.99362+ .078491i$	0
3	2.145889	0	$-0.16625- 2.813270i$	0
4	520.55141	520.31833	$3.9683- 0.31296i$	0
5	141.31354	0	$0.55226+ 5.59554i$	0
6	34285.0506	34262.02821	$-7.886+ 0.93541i$	0
7	13028.3126	0	$-1.54020- 11.112053i$	0
8	3161360.234	3158530.28	$15.6499 - 2.48392i$	0
9	1544319.079	0	$3.94278+ 22.03233i$	0
10	374789686.10	374370388.8	$-31.005+ 6.18039i$	0

Table 7
First ten Sample Non-central and Central moments of Simulated 'X'

Order	Normal distribution		Cauchy distribution	
r	Moments			
	Non-central	Central	Non-central	Central
1	0	0	$0.040465 + 0.594227i$	0
2	0	1	$-0.351468+ 0.048090i$	0
3	0	0	$-0.0427- 0.20690i$	0
4	0	3	$0.1212- 0.0338047i$	0
5	0	0	$0.02499+ 0.0706626i$	0
6	0	15	$-0.04097+ 0.017710i$	0
7	0	0	$-0.012182- 0.0236337i$	0
8	0	105	$0.013550- 0.0081954i$	0
9	0	0	$0.0054182+ 0.0077206i$	0
10	0	945	$-0.0043685+ 0.0035321i$	0

Result 5.2: The Sample Non-central moments of Cauchy distribution are derived in terms of Complex numbers and to find the Proximity of each higher order moments from the origin by using Polar Co-ordinates, the position of a point in a plane is represented by the ordered pair (R, θ) (where R is the radial distance from the origin and θ is the vectorial angle) rather than $\text{Re}(S_r)$, $\text{Im}(S_r)$) (Where $\text{Re}(S_r)$ and $\text{Im}(S_r)$ are the real and Imaginary part of the r^{th} Non-central moment of the Cauchy distribution). They are the natural system for describing positions on a circle. The equations to transform Complex moments of Cauchy distribution into polar coordinates are given below:

$$\text{Re}(S_r) = R_r \cos(\theta_r) \quad (4.1)$$

$$\text{Im}(S_r) = R_r \sin(\theta_r) \quad (4.2)$$

$$R_r = \sqrt{(\text{Re}(S_r))^2 + (\text{Im}(S_r))^2} \quad (4.3)$$

$$\theta_r = \arctan(\text{Im}(S_r) / \text{Re}(S_r)) \quad (4.4)$$

Using (4.3) and (4.4), the Polar plots with spokes of first ten non-central moments for the variables Sepal length, Sepal Width, Petal length, Petal Width, Netflix stock Returns and Simulated

'X' variable shows the proximity among each complex moments of Cauchy distribution are visualized below:

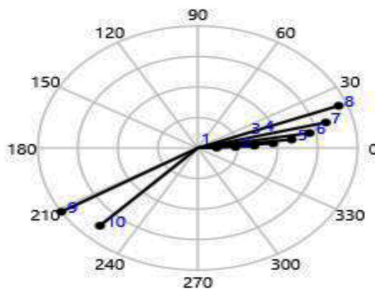


Figure 1
Sepal Length

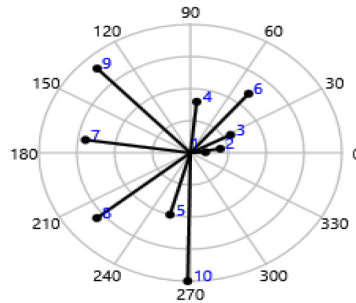


Figure 2
Sepal width

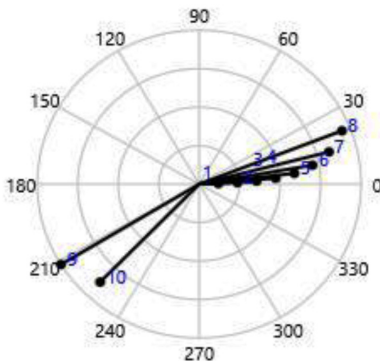


Figure 3
Petal Length

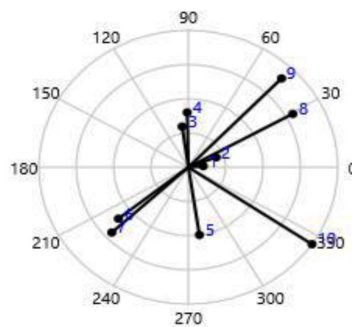


Figure 4
Petal width

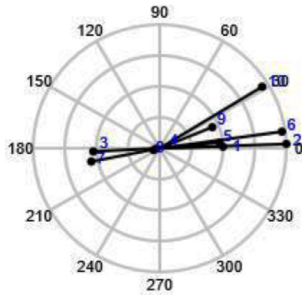


Figure 5
Netflix stock Return

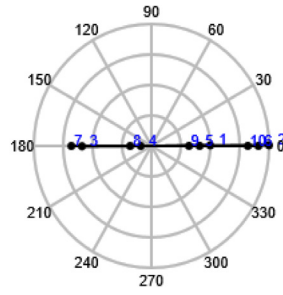


Figure 6
Simulated 'X'

5. Discussion

From the previous sections, it has been established that the finite moments of the Cauchy distribution exist and can be derived using the Lotus theorem rather than direct integration. Initially, the authors applied the Lotus theorem to derive the moments of the normal distribution, and the approach proved successful. Building on this, they extended the method to the Cauchy distribution, ultimately concluding that its moments do exist. This exploration led to the resolution of three previously unsolved problems in statistical literature. First, it debunks the long-standing belief that the moments of the Cauchy distribution do not exist. Additionally, as demonstrated in Section 5, numerical illustrations using three different datasets further support these findings. Table 1 presents the maximum likelihood estimates of the parameters for the Normal and Cauchy distributions, fitted to six variables.

Using the estimated parameters, Tables 2 to 7 present the first ten central and non-central sample moments of the Normal and Cauchy distributions. Secondly, the sample non-central moments of the Cauchy distribution are derived in terms of complex numbers, revealing that all non-central moments are zero. This finding provides clarification on a question raised by Peter McCullagh [9] in his paper *“Conditional Inference and Cauchy Models.”* McCullagh questioned the introduction of complex numbers when dealing with real-valued random variables, stating, *“While there is no obvious reason not to use the complex plane to represent a parameter space, this choice is decidedly unusual in statistics. Why introduce complex numbers when only real-valued random variables are involved? To this question, I can give no better answer.”* The authors were surprised to discover that the

Cauchy distribution, despite being a real-valued random variable, possesses complex moments. These moments can be represented in both the real and complex planes, which may explain why McCullagh [9], either knowingly or unknowingly, considered the parameter space in the entire complex plane when reparametrizing the Cauchy distribution. Furthermore, the authors transformed the non-central sample moments of the Cauchy distribution into polar coordinates using radius and angle, providing insight into the magnitude of these moments relative to the origin. This transformation is visually represented in Figures 1 to 6 through polar plots. Thirdly, the mean of the Cauchy distribution is also complex, while its variance is zero. The complex mean allows for an alternative reparametrization of the Cauchy distribution. Additionally, the literature suggests that the univariate Cauchy distribution can be extended to the bivariate or multivariate case in two forms: correlated and uncorrelated. However, the authors strongly oppose the correlated form of this extension. The results confirm that the location parameter of the Cauchy distribution is complex, and its variance is zero, making it impossible to generalize the distribution to a multivariate form with a correlation structure. Instead, they propose that a multivariate complex Cauchy distribution could be a possible extension. On the other hand, the authors support the uncorrelated form of the multivariate Cauchy distribution as a more natural generalization and strongly recommend it. Similarly, some researchers argue that when the degrees of freedom in a multivariate Student's t distribution equal one, it reduces to a multivariate Cauchy distribution. However, based on the findings discussed above, this claim is incorrect. When the degrees of freedom are one, the covariance structure of the multivariate t distribution becomes negative, making this assumption a statistical fallacy. Therefore, the authors conclude that the uncorrelated form is the most statistically sound extension of the multivariate Cauchy distribution, a conclusion supported by the results derived from the Lotus theorem.

6. Conclusion

In this paper, the authors demonstrated that the raw higher-order moments of the Cauchy distribution can be defined and proved that these moments are finite in the complex plane using the Lotus theorem. The maximum likelihood estimation of the parameters for the Normal and Cauchy distributions was examined in detail through three numerical examples based on different datasets. The long-standing belief in the non-

existence of Cauchy distribution moments was revisited, and these moments were successfully derived in terms of complex numbers. Consequently, the previously unexplored and undefined moments of the Cauchy distribution have been thoroughly investigated and resolved in this study. Furthermore, the authors recommend adopting the uncorrelated form of the multivariate generalization of the Cauchy distribution for computing higher-order product moments, utilizing the Lotus theorem as a foundation. Finally, the proposed method can be applied to derive moments for most continuous probability distributions, with some exceptions. For instance, in the case of the Beta and Gamma distributions, deriving moments is challenging due to the non-linearity of parameters in the density function and the absence of a one-to-one mapping. However, this difficulty can be addressed using Lambert's W function. A distribution whose probability density or mass function can be explicitly solved for a random variable is more easily suited for deriving moments.

References

- [1] A. P. Prudnikov, I. A. Brychkov, and O. I. Marichev, *Integrals and Series: Direct Laplace Transforms*, vol. 4. Boca Raton, FL, USA: CRC Press (1986).
- [2] C. A. Charalambides and N. Papadatos, "The q -factorial moments of discrete q -distributions and a characterization of the Euler distribution," in *Advances on Models, Characterizations and Applications*, pp. 57–71 (2005).
- [3] C. Eisenhart, "The development of the concept of the best mean of a set of measurements from antiquity to the present day," presented at the 131st Annual Meeting of the American Statistical Association, Fort Collins, CO, USA (1971).
- [4] D. S. Broca, "Mean and variance through factorial moments," *Teaching Statistics*, vol. 27, no. 2 (2005).
- [5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. New York, NY, USA: Academic Press (2000).
- [6] J. C. Breton and N. Privault, "Factorial moments of point processes," *Stochastic Processes and their Applications*, vol. 124, no. 10, pp. 3412–3428 (2014).
- [7] J. R. Hosking, "L-moments: Analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal*

- Statistical Society: Series B (Statistical Methodology)*, vol. 52, no. 1, pp. 105–124 (1990).
- [8] K. M. Ewis, “Central moments via factorial moments for some discrete distributions,” *American Academic & Scholarly Research Journal*, vol. 4, no. 1, pp. 64–67 (2012).
 - [9] P. McCullagh, “Conditional inference and Cauchy models,” *Biometrika*, vol. 79, no. 2, pp. 247–259 (1992).
 - [10] S. M. Ross, *Introduction to Probability Models*. Amsterdam, Netherlands: Academic Press (2014).
 - [11] S. Nadarajah, “Making the Cauchy work,” *Brazilian Journal of Probability and Statistics*, vol. 25, no. 1, pp. 99–120 (2011).
 - [12] S. Ogunsanya, E. E. Akarawak, and M. I. Ekum, “On some properties of Rayleigh–Cauchy distribution,” *Journal of Statistics and Management Systems*, vol. 24, no. 6, pp. 1213–1231 (2021), doi: 10.1080/09720510.2020.1822499.
 - [13] S. D. Poisson, *Recherches sur la probabilité des jugements en matière criminelle et en matière civile: précédées des règles générales du calcul des probabilités*. Paris, France: Bachelier (1837).
 - [14] S. M. Stigler, “Linear functions of order statistics with smooth weight functions,” *The Annals of Statistics*, vol. 2, no. 4, pp. 676–693 (1974).
 - [15] V. Y. Galkin and M. V. Ufimtsev, “On the moment characteristics of distributions,” *Computational Mathematics and Modeling*, vol. 9, no. 3, pp. 196–202 (1998).

Received August, 2025