

Hybrid block approach based on four-step Adams-Moulton scheme for solving stiff control models in operations research

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Abstract

In this work, a hybrid block method based on the four-step Adams-Moulton scheme specifically tailored for solving stiff control problems has been presented. Tests for convergence and stability are conducted to verify the validity of the method. Comparative numerical examples are used to demonstrate the level of accuracy exhibited by the new method over the analytical process. The findings indicate the method's consistency, zero-stability, and $A(\alpha)$ -stability characteristics, and therefore, suitable for practical applications to stiff dynamical systems in operations research. Numerical results are compared with the exact solutions, and indicate a good level of accuracy.

Subject Classification: 65L04, 65L05, 65L06, 65L20.

Keywords: Adams-Moulton method, Hybrid block method, Stability analysis, Stiff systems, Operations research.

1. Introduction

Lenhart and Workman [1], and Sethi and Thompson [2] showed that stiff ordinary differential equation (ODE) systems arise in control models applied in operations research (OR) problems, especially in dynamic systems that deal with optimal control, production planning, and workforce management. Lambert [3], and Ascher and Petzold [4] noted that such models possess rapidly and slowly varying components, which can lead to stiffness in the resulting mathematical

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formulation. This stiffness creates significant hurdles and makes it extremely difficult for traditional numerical techniques due to stability and efficiency constraints. In such cases, special numerical methods with stability as a primary concern have to be applied to obtain stable and efficient solutions of the stiff ODE defined by:

$$y' = f(x, y(x)), y(x_0) = y_0, x \in [a, b], y \in \mathbb{R} \quad (1)$$

Implicit methods are usually employed for solving stiff ODEs because of their good stability characteristics and high level of order of accuracy. The direct implementation of these methods on large stiff systems can be computationally costly.

To overcome these challenges, recent research has focused on block and hybrid techniques, which enhance the efficiency of traditional methods by calculating multiple time steps simultaneously and blending explicit and implicit strategies within the same framework. Abhulimen and Onumanyi [5] developed block methods for solving stiff ODEs, while Yahaya and Sulaiman [6] developed a class of implicit block methods for stiff ODEs. A second derivative block hybrid method suitable for continuous integration for (1) by Yakubu et al. [7]. The hybrid method was stated to compete favourably with some strong stability stiff integrators. Recently, Hambagda, Atsi, and Aina [8] developed a generalized four-step Adams-Moulton second derivative method for solving equation (1). The stability analysis indicated the suitability of the method for solving stiff systems. Using Chebyshev polynomials of the first kind as a basis function, Rufai, Duromola, and Ganiyu [9], derive a new continuous hybrid block method and the new proposed method for the solution of initial value problems of systems of (1). Atsi et al. [10] developed a second-derivative block method based on a family of modified backward differentiation (BDF) type methods, while Atsi and Kumleng [11] derived a family of modified BDF-type block methods stiff ODEs. Kumleng, Longwap, and Adeo [12] investigated an A-stable linear multistep methods (LMMs) for stiff initial value problems (IVPs). Feichtinger et al. [13] considered stiff numerical treatment of optimal control problems in economics and management science. These studies demonstrate the importance of stable and efficient numerical methods for stiff systems. Furthermore, recent study in the journal of interdisciplinary mathematics has examined high-accuracy Runge-Kutta methods for stiff ODE systems [15]. Interpolation and collocation techniques have also been adopted in the development of block methods for solving stiff systems [4].

This paper proposes a second-derivative hybrid block method using the class of four-step Adam-Moulton schemes adopted by Hambagda, Atsi, and Aina [8] to improve the numerical treatment of stiff systems in OR models. These applications demonstrate how effective the method is in delivering stable, accurate solutions for models that might struggle or even fail with traditional approaches due to stiffness [13], [1].

This paper is further structured in the following parts: Part 2 is the methodology of the new hybrid-block method; Part 3 contains a discussion of the results; and Part 4 presents the conclusion.

2. Methodology

This section presents the development and implementation of the hybrid block method.

2.1 Derivation of the New Method

The new block method is formulated from the foundation of the four-step Adams-Moulton method, which has been tweaked to include higher-order derivative terms and mid-point evaluations. The method can be generally expressed as:

$$\bar{y}(x) = \alpha_3(x)y_{n+3} + h \sum_{j=\frac{1}{2}, 1, 2}^4 \beta_j(x)f_{n+j}(x) + h^2 \sum_{j=\frac{1}{2}, 1, 2}^4 \gamma_j(x)g_{n+j}(x), \quad (2)$$

here,

h is the step size,

$f_{n+j} = f(x_{n+j}, y_{n+j})$ is the first derivative (right-hand side),

$g_{n+j} = f'(x_{n+j}, y_{n+j})$ represents second-derivative terms,

$f_{n+1/2}$ and $g_{n+1/2}$ are off-step point evaluations,

$\alpha_3(x)$, $\beta_j(x)$, and $\gamma_j(x)$ are the continuous coefficients to be determines.

The interpolation and collocation approach ([4]) has been adopted in this work. We obtain a D matrix defined by:

$$D = \begin{bmatrix} d1 \\ d2 \\ d3 \end{bmatrix}$$

where,

$$\begin{aligned} d1 &= [1 \quad x_n + 3h \quad (x_n + 3h)^2 \quad \cdots \quad (x_n + 3h)^{10}] \\ d2 &= \begin{bmatrix} 0 & 1 & 2\left(x_n + \frac{1}{2}h\right) & 3\left(x_n + \frac{1}{2}h\right)^2 & \cdots & 10\left(x_n + \frac{1}{2}h\right)^9 \\ 0 & 1 & 2(x_n + h) & 3(x_n + h)^2 & \cdots & 10(x_n + h)^9 \\ 0 & 1 & 2(x_n + 2h) & 3(x_n + 2h)^2 & \cdots & 10(x_n + 2h)^9 \\ 0 & 1 & 2(x_n + 3h) & 3(x_n + 3h)^2 & \cdots & 10(x_n + 3h)^9 \\ 0 & 1 & 2(x_n + 4h) & 3(x_n + 4h)^2 & \cdots & 10(x_n + 4h)^9 \end{bmatrix} \\ d3 &= \begin{bmatrix} 0 & 0 & 2 & 6\left(x_n + \frac{1}{2}h\right) & 12\left(x_n + \frac{1}{2}h\right)^2 & \cdots & 90\left(x_n + \frac{1}{2}h\right)^8 \\ 0 & 0 & 2 & 6(x_n + h) & 12(x_n + h)^2 & \cdots & 90(x_n + h)^8 \\ 0 & 0 & 2 & 6(x_n + 2h) & 12(x_n + 2h)^2 & \cdots & 90(x_n + 2h)^8 \\ 0 & 0 & 2 & 6(x_n + 3h) & 12(x_n + 3h)^2 & \cdots & 90(x_n + 3h)^8 \\ 0 & 0 & 2 & 6(x_n + 4h) & 12(x_n + 4h)^2 & \cdots & 90(x_n + 4h)^8 \end{bmatrix} \end{aligned}$$

The values of the continuous coefficients are presented from the inverse of the D matrix, and they are given by:

$$\alpha_3 = 1$$

$$h\beta_{\frac{1}{2}} = \frac{128}{364651875h^9} (3h + \tau)^3 (322995h^7 + 8542530\tau^2h^5 + 11618190\tau^3h^4 \\ + 8323695\tau^4h^3 + 3182823\tau^5h^2 + 603722h\tau^6 + 44352\tau^7 \\ + 2996685\tau h^6)$$

$$h\beta_1 = -\frac{1}{272160h^9} (3h + \tau)^3 (34095h^7 + 497610h^5\tau^2 + 597150h^4\tau^3 \\ + 354105h^3\tau^4 + 108927h^2\tau^5 + 16688h\tau^6 + 1008\tau^7 \\ + 180945\tau h^6)$$

$$h\beta_2 = -\frac{1}{272160h^9} (3h + \tau)^3 (17313h^7 + 49887h^6\tau + 191862h^5\tau^2 \\ + 282330h^4\tau^3 + 229905h^3\tau^4 + 88767h^2\tau^5 + 15568h\tau^6 \\ + 1008\tau^7)$$

$$h\beta_3 = -\frac{(3h+\tau)}{1260000h^9} (384255h^9 + 1592235h^8\tau + 6162375h^7\tau^2 \\ + 12237075h^6\tau^3 + 14405175h^5\tau^4 + 10332387h^4\tau^5 \\ + 4515291h^3\tau^6 + 1160103h^2\tau^7 + 159824h\tau^8 + 9072\tau^9)$$

$$h\beta_4 = -\frac{1}{93350880h^9} (3h + \tau)^3 (160089h^7 + 1106631h^6\tau + 3845286h^5\tau^2 \\ + 6689610h^4\tau^3 + 6486345h^3\tau^4 + 3434823h^2\tau^5 \\ + 910672h\tau^6 + 89712\tau^7)$$

$$h^2\gamma_{\frac{1}{2}} = \frac{128}{3472875h^8} (3h + \tau)^3 (870h^7 + 5850h^6\tau + 14580h^5\tau^2 \\ + 18360h^4\tau^3 + 12555h^3\tau^4 + 4662h^2\tau^5 + 868h\tau^6 + 63\tau^7)$$

$$h^2\gamma_1 = \frac{1}{90720h^8} (3h + \tau)^3 (5763h^7 + 47997h^6\tau + 142722h^5\tau^2 \\ + 213030h^4\tau^3 + 165645h^3\tau^4 + 67347h^2\tau^5 + 13328h\tau^6 \\ + 1008\tau^7)$$

$$h^2\gamma_2 = \frac{1}{90720h^8} (3h + \tau)^3 (2805h^7 + 24075h^6\tau + 78030h^5\tau^2 \\ + 129690h^4\tau^3 + 115965h^3\tau^4 + 54747h^2\tau^5 + 12208h\tau^6 \\ + 1008\tau^7)$$

$$h^2\gamma_3 = \frac{1}{28000h^8} (905h^8 + 5370h^7\tau + 19715h^6\tau^2 + 36660h^5\tau^3 \\ + 39325h^4\tau^4 + 24534h^3\tau^5 + 8659h^2\tau^6 + 1568h\tau^7 \\ + 112\tau^8)(3h + \tau)^2$$

$$h^2\gamma_4 = \frac{1}{4445280h^8} (3h + \tau)^3 (1677h^7 + 11763h^6\tau + 40878h^5\tau^2 \\ + 71370h^4\tau^3 + 69525h^3\tau^4 + 37107h^2\tau^5 + 9968h\tau^6 + 1008\tau^7)$$

(3)

Substituting the values of the continuous coefficients into the continuous form (2), we have the following:

$$\begin{aligned}
\bar{y}(x) = & y_{n+3} + \left(\frac{128}{364651875h^9} (3h + \tau)^3 (322995h^7 + 8542530\tau^2h^5 + \right. \\
& 11618190\tau^3h^4 + 8323695\tau^4h^3 + 3182823\tau^5h^2 + 603722h\tau^6 + \\
& \left. 44352\tau^7 + 2996685\tau h^6) \right) f_{\frac{1}{2}} \\
& + \left(-\frac{1}{272160h^9} (3h + \tau)^3 (34095h^7 + 497610h^5\tau^2 + 597150h^4\tau^3 + \right. \\
& 354105h^3\tau^4 + 108927h^2\tau^5 + 16688h\tau^6 + 1008\tau^7 + \\
& \left. 180945\tau h^6) \right) f_{n+1} \\
& + \left(-\frac{1}{272160h^9} (3h + \tau)^3 (17313h^7 + 49887h^6\tau + 191862h^5\tau^2 + \right. \\
& 282330h^4\tau^3 + 229905h^3\tau^4 + 88767h^2\tau^5 + 15568h\tau^6 + \\
& \left. 1008\tau^7) \right) f_{n+2} \\
& + \left(-\frac{(3h+\tau)}{1260000h^9} (384255h^9 + 1592235h^8\tau + 6162375h^7\tau^2 + \right. \\
& 12237075h^6\tau^3 + 14405175h^5\tau^4 + 10332387h^4\tau^5 + 4515291h^3\tau^6 + \\
& \left. 1160103h^2\tau^7 + 159824h\tau^8 + 9072\tau^9) \right) f_{n+3} \\
& + \left(-\frac{1}{93350880h^9} (3h + \tau)^3 (160089h^7 + 1106631h^6\tau + 3845286h^5\tau^2 + \right. \\
& 6689610h^4\tau^3 + 6486345h^3\tau^4 + 3434823h^2\tau^5 + 910672h\tau^6 + \\
& \left. 89712\tau^7) \right) f_{n+4} \\
& + \left(\frac{128}{3472875h^8} (3h + \tau)^3 (870h^7 + 5850h^6\tau + 14580h^5\tau^2 + \right. \\
& 18360h^4\tau^3 + 12555h^3\tau^4 + 4662h^2\tau^5 + 868h\tau^6 + 63\tau^7) \right) g_{\frac{1}{2}} \\
& + \left(\frac{1}{90720h^8} (3h + \tau)^3 (5763h^7 + 47997h^6\tau + 142722h^5\tau^2 + \right. \\
& 213030h^4\tau^3 + 165645h^3\tau^4 + 67347h^2\tau^5 + 13328h\tau^6 + \\
& \left. 1008\tau^7) \right) g_{n+1} \\
& + \left(\frac{1}{90720h^8} (3h + \tau)^3 (2805h^7 + 24075h^6\tau + 78030h^5\tau^2 + \right. \\
& 129690h^4\tau^3 + 115965h^3\tau^4 + 54747h^2\tau^5 + 12208h\tau^6 + \\
& \left. 1008\tau^7) \right) g_{n+2} \\
& + \left(\frac{1}{28000h^8} (905h^8 + 5370h^7\tau + 19715h^6\tau^2 + 36660h^5\tau^3 + \right. \\
& 39325h^4\tau^4 + 24534h^3\tau^5 + 8659h^2\tau^6 + 1568h\tau^7 + 112\tau^8) (3h + \\
& \left. \tau)^2 \right) g_{n+3}
\end{aligned}$$

$$+ \left(\frac{1}{4445280h^8} (3h + \tau)^3 (1677h^7 + 11763h^6\tau + 40878h^5\tau^2 + 71370h^4\tau^3 + 69525h^3\tau^4 + 37107h^2\tau^5 + 9968h\tau^6 + 1008\tau^7) \right) g_{n+4} \quad (4)$$

Five discrete methods are obtained from the continuous scheme (4) by interpolation for $\tau = 0, \frac{1}{2}, h, 2h$, and $4h$. These can be presented as:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{1}{2}} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} \\ & + h \left(\begin{bmatrix} \frac{804821}{272160} & -\frac{1166557312}{364651875} & \frac{206203}{272160} & \frac{560669}{1260000} & \frac{733895}{18670176} \\ \frac{37243553}{28628} & -\frac{248107894}{1136218112} & \frac{52170581}{2099} & \frac{28477193}{18692} & \frac{932613293}{120133} \\ \frac{13934592}{8505} & -\frac{72930375}{364651875} & \frac{69672960}{1701} & \frac{64512000}{39375} & \frac{23897825280}{2917215} \\ \frac{28628}{2273} & -\frac{1136218112}{2756224} & \frac{2099}{5771} & \frac{18692}{25617} & \frac{120133}{53363} \\ \frac{672}{28864} & -\frac{900375}{1025376256} & \frac{3360}{15488} & \frac{28000}{47296} & \frac{1152480}{1155068} \\ \frac{8505}{364651875} & -\frac{1025376256}{364651875} & \frac{8505}{8505} & \frac{39375}{39375} & \frac{2917215}{2917215} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{1}{2}} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} + \right) \\ & + h^2 \left(\begin{bmatrix} -\frac{170993}{90720} & -\frac{3070208}{3472875} & -\frac{79831}{90720} & -\frac{5809}{28000} & -\frac{38879}{4445280} \\ -\frac{42774481}{23224320} & -\frac{4980127}{5556600} & -\frac{4050757}{4644864} & -\frac{295149}{1433600} & -\frac{9882049}{1137991680} \\ -\frac{1000}{567} & -\frac{3033088}{3472875} & -\frac{2833}{2835} & -\frac{192}{875} & -\frac{1271}{138915} \\ -\frac{1921}{1120} & -\frac{7424}{8575} & -\frac{187}{224} & -\frac{1629}{5600} & -\frac{559}{54880} \\ -\frac{4288}{2835} & -\frac{2883584}{3472875} & -\frac{142}{42835} & -\frac{64}{875} & -\frac{1256}{27783} \end{bmatrix} \begin{bmatrix} g_{n+1} \\ g_{n+\frac{1}{2}} \\ g_{n+2} \\ g_{n+3} \\ g_{n+4} \end{bmatrix} \right) \end{aligned} \quad (5)$$

2.2 Block Implementation

The method is implemented in block form, thereby aiding the simultaneous computation of multiple solution values within a single step. This helps in improving the computational efficiency of the new hybrid-block method in (5). The following matrix finite difference equation represents the method in (5):

$$P^{(0)}Y_m = P^{(1)}y_m + hQ^{(0)}F_m + h^2R^{(0)}G_m \quad (6)$$

here,

$$\begin{aligned} Y_m &= [y_{n+1} \quad y_{n+\frac{1}{2}} \quad y_{n+2} \quad y_{n+3} \quad y_{n+4}]^T, \\ y_m &= [y_{n-4} \quad y_{n-3} \quad y_{n-2} \quad y_{n-1} \quad y_n]^T, \\ F_m &= [f_{n+1} \quad f_{n+\frac{1}{2}} \quad f_{n+2} \quad f_{n+3} \quad f_{n+4}]^T, \text{ and} \\ G_m &= [g_{n+1} \quad g_{n+\frac{1}{2}} \quad g_{n+2} \quad g_{n+3} \quad g_{n+4}]^T \end{aligned}$$

and the coefficient matrices $P^{(0)}$, $P^{(1)}$, $Q^{(0)}$, and $R^{(0)}$ are 5 by 5 matrices, derived from the scalar α_3 , β_j , and γ_j .

2.3 Stability Analysis of the Hybrid Block Methods

2.3.1 Definition (Zero-Stability)

A block method is zero stable if all the roots r_i of the first characteristic polynomial, $\rho(r)$, satisfy $|r_i| \leq 1$, and any root with $|r_i| = 1$ is simple (Chollom, Ndam, and Kumleng [14]).

The new hybrid-block method is zero-stable since $|r| = 0, 0, 0$, or 1.

2.3.2 Definition (Order and Error Constants)

A block method is said to be of order p if $c_0 = c_1 = c_2 = \dots = c_p = 0$, but $c_{p+1} \neq 0$, where c_{p+1} is defined as the error constant (Kumleng, Longwap, and Adee [12]).

This implies that the new hybrid block method is of order ten with

$$p = [10 \quad 10 \quad 10 \quad 10 \quad 10]^T,$$

and the error constant of:

$$C_{p+1} = \left[\frac{417451}{100590336000} \quad \frac{26557079}{6437781504000} \quad \frac{13429}{3143448000} \quad \frac{5531}{1241856000} \quad \frac{113}{19646550} \right]^T$$

2.3.3 Consistency

The order $p \geq 1$ implies that the new block method is consistent. (Chollom, Ndam, and Kumleng [14])

2.3.4 Definition (Absolute stability)

A block method is said to be absolutely stable if for $\bar{h}$, all the roots r_s of the stability polynomial:

$$\pi(r; \bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0$$

Satisfy $|r_s| < 1$, $s = 1, 2, \dots, k$. (Kumleng, Longwap, and Adee [12])

The method (5) was reformulated as a general linear method and is plotted using the Matlab program, as revealed in Figure 1, to be $A(\alpha)$ -stable.

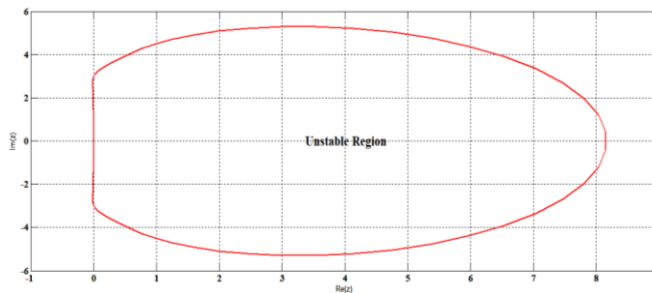


Figure 1
Region of Absolute stability of the new hybrid block method

2.3.5 Definition (Convergence)

A block method is said to be convergent if the computed solution approaches the exact solution as the step size, $h \rightarrow 0$ (Rufai, Duromola, and Ganiyu [9]).

2.3.6 Numerical Examples

Some linear Stiff Control models in OR of the form (1) are considered. We present the calculated absolute errors in Tables 1 and 2, and the solution curves are shown in Figures 1 and 2, for Examples 1 and 2, respectively.

Example 1: Workforce adjustment model (WAM)

In this example, we present the dynamic adjustment model of a two-tier workforce system using a stiff system of first-order ODEs. The model represents how temporary and permanent staff levels evolve in response to internal adjustment policies and external demand.

3. Model description

Let:

$y_1(x)$ represents the number of temporary workers (fast-adjusting), and $y_2(x)$ represents the number of permanent workers (slow-adjusting).

The dynamics of the system are governed by the following coupled system of ODEs:

$$\begin{aligned}y_1' &= -100y_1 + y_2 \\ y_2' &= -y_1 - y_2\end{aligned}$$

With initial conditions:

$$y_1(0) = -\frac{196}{99}, y_2(0) = 1$$

The system is solved over the interval $t \in [0, 1]$ with the new hybrid-block scheme and compares the result with the exact solution:

$$\begin{aligned}y_1(t) &= \frac{e^{-t}}{99} - \frac{197}{99}e^{-100t} \\ y_2(t) &= e^{-t}\end{aligned}$$

on the range, $0 \leq x \leq 1$ and a step size, $h = 0.1$:

Example 2: Stiff system (Inventory-Production-Backlog Model)

This is a real-world problem of Production, Inventory, and Demand (PID), and it is a stiff system because production changes slowly to a change in demand while inventory changes quickly.

The model system is represented as follows:

$$\begin{aligned}y_1' &= -k_1y_1 + k_2y_2 + D_0 \\ y_2' &= -k_3(y_2 - y_3) \\ y_3' &= -k_4y_3 + P_0\end{aligned}$$

where:

$y_1(t)$ are fast-changing dynamics (Backlog)

$y_2(t)$ are medium dynamics (Inventory)

$y_3(t)$ are slow-changing dynamics (production)

$D(t)$ is input (Demand)

$P(t)$ is Production input

$k_1 \gg k_3, k_4$ are stiffness determinant

Let

$$k_1 = 100, k_2 = 20, k_3 = 10, k_4 = 1, D_0 = 10, P_0 = 5$$

Initial conditions:

$$y_1(0) = 0, y_2(0) = 0, y_3(0) = 0$$

The exact solution for $t \in [0,1]$:

$$y_1(t) = \frac{11}{10} - \frac{1000}{891}e^{-t} + \frac{10}{81}e^{-10t} - \frac{901}{8910}e^{-100t}$$

$$y_2(t) = 5 - \frac{50}{9}e^{-t} + \frac{5}{9}e^{-10t}$$

$$y_3(t) = 5(1 - e^{-t})$$

4. Results

Table 1

Error comparison between the new block method & the exact solution in Example 1

t	Error of y_1	Error of y_2
0	0	0
0.1000000000000000	0.000356650743287	0.017252682269680
0.2000000000000000	0.000149902305328	0.014774123067522
0.3000000000000000	0.000128116824162	0.012606683657438
0.4000000000000000	0.000108971355711	0.010718663378744
0.5000000000000000	0.000092315839596	0.009076444657313
0.6000000000000000	0.000077849194255	0.007650282607516
0.7000000000000000	0.000065305101767	0.006413873389819
0.8000000000000000	0.000054448161974	0.005343968103130
0.9000000000000000	0.000045070414727	0.004420029054397
1.0000000000000000	0.000036988244521	0.003623923831923

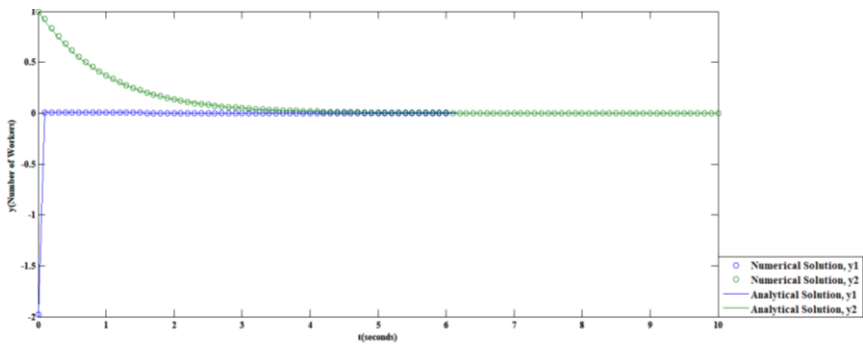


Figure 2
Solution curves for Example 1

Table 2
Error comparison between the new block method and the exact solution in Example 2

t	Error of y_1	Error of y_2	Error of y_3
0	0	0	0
0.1000	0.000026993335459	0.000000125173845	0.0000000000000001
0.2000	0.000000010792271	0.000000092097796	0.0000000000000000
0.3000	0.000000011290552	0.000000050821343	0.0000000000000001
0.4000	0.000000005539594	0.000000024928178	0.0000000000000000
0.5000	0.000000002547380	0.000000011463208	0.0000000000000001
0.6000	0.000000001124555	0.000000005060495	0.0000000000000001
0.7000	0.000000000482650	0.000000002171927	0.0000000000000001
0.8000	0.000000000202923	0.000000000913151	0.0000000000000001
0.9000	0.000000000083982	0.000000000377920	0.0000000000000000
1	0.000000000034328	0.000000000154477	0.0000000000000000

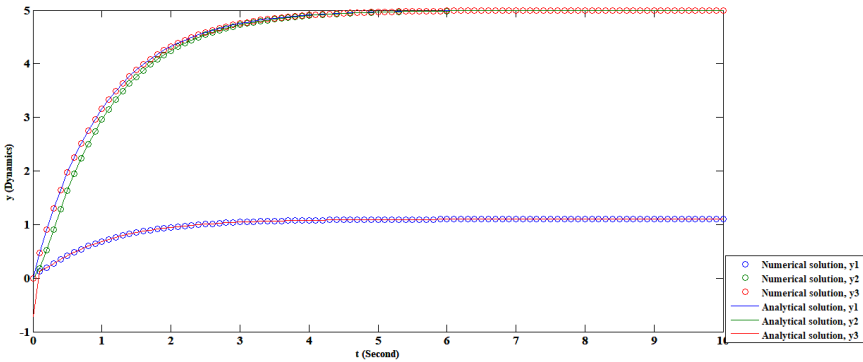


Figure 3
Solution curves for Example 2

5. Discussion of Results

Figure 1 illustrates the region of absolute stability of the new method. It can be observed that the stability region contains a larger portion of the left half of the complex plane. Therefore, the method is $A(\alpha)$ -stable.

The solutions of two stiff problems in OR indicate the suitability of the new block scheme derived from the four-step Adams-Moulton method. The WAM has been presented in Example 1. The new method is efficient in solving the problem over the interval $[0,1]$ with a step size of 0.1. Taking a look at Table 1, the error obtained is minimal, thereby demonstrating the level of consistency and convergence of the new hybrid-block method. The level of accuracy is demonstrated by the solution curve in Figure 2. For example, 2 (PIB model), has to do with a higher level of stiff system, and also demonstrates the performance of the hybrid-block method. The errors for the three variables are favorably low, as shown in Table 2, and solutions curves are presented in Figure 3.

These results further justify the established theoretical properties of the new method.

This level of accuracy, even in a stiff system, really highlights the hybrid block method's superior stability and error-controlling abilities. The results also reinforce the theoretical properties we established earlier.

6. Conclusion

A new hybrid-block method has been successfully derived in this work from a generalized four-step Adams-Moulton method. The multistep collocation technique has been adopted for the construction of the method. The block implementation of the method enables its computational efficiency. To ascertain the suitability of the method, stability analysis has been investigated to ensure consistency, zero-stability, and therefore convergence. Further analysis showed that the method is $A(\alpha)$ -stable.

The method has been tested on two real-world stiff problems: in WAM and PIB systems, reliable and stable results. In the future, this method will be extended to a system of nonlinear ODEs.

7. Declarations

Availability of Data and Material: Data generated or analyzed during this work are all included in this article and their supplementary references.

Conflicts of Interest/Competing Interests: All the authors declare that they have no conflicts of interest.

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Code availability: 'Not applicable.'

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