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Targeting synchronization of different-dimensional chaotic systems

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Abstract

This work presents targeting chaotic synchronization of different-dimensional dynamic system. Here we use extensive theory of coupling of different-dimensional dynamic chaos systems with mismatch parameters. A framework of chaotic synchronization (generalized) between two oscillators with mismatched or varying parameters was displayed through linear diffusive coupling. Different chaotic systems are considered for effectiveness of our targeting engineering synchronization. To analyze the effectiveness of the proposed scheme, numerical simulation are provided.

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1. Introduction

From 1990, chaos synchronization has remained a concept that has received much attention. Engineers have expressed desired for the use of the synchronization of chaos for communication purposes. In identical sychronization [1], the span between their corresponding states of two dynamical systems goes zero as time goes to infinity. Rulkov et.al. [2] outlined a refinement of this view for unidirectionally coupled dynamics. This type of behaviour known as generalized synchronization (GS). Instead of that, different types of synchronization like complete synchronization (CS) [3,4], anti-synchronization (AS) [5,6], phase synchronization (PS) [7,8], projective synchronization (PrS) [9,10] etc. have been documented in earlier studies. In CS, the state variable of a coupled system comprehensively coincides whereas in AS, the coupled system synchronizes reciprocally. When both CS and AS co-exist in a coupled system, then it is called mixed synchronization [11,12]. Mixed synchronization provides cryptographic modulation for digital signals through parameter alteration. In GS, the general relation of the drive-response system are proposed with their functional relationship. This functional relationship for GS is typically not known under standard diffusive coupling for targeting engineering synchronization states, though different coupling configurations like unidirectional [13], bidirectional [14], repulsive [15] etc. have been investigated. Targeting synchronization, also known as phase synchronization or phase locking, is a fundamental concept in engineering systems that involves aligning the phases of different signals or components [16]. The specific application of targeting synchronization in engineering systems is a communication system [17]. In wireless communication, targeting synchronization ensures that the transmitter and receiver are phase-aligned. This synchronization is crucial for demodulating and decoding the transmitted signal accurately. Targeting synchronization is essential in radar and solar systems for precise target detection and tracking [18]. By synchronizing the phase of the broadcasted signal with the detected signal, these systems can accurately measure the time delay and doppler shift, enabling the determination of target range, velocity, and direction. Overall, targeting synchronization is instrumental in enhancing the performance, reliability, the efficiency of engineering systems around diverse applications, making it a fundamental concept in the design and operation of modern technology.

Of late, the outline of engineering synchronization in nonlinear oscillators has found applicability in practical arena [19]. In this paper, we have proposed the use of linear feedback control to target the generalized engineering synchronization state [20,21]. Engineering linear and nonlinear generalized synchronizations are also found in this paper through suitable choices of target functions [22,23, 26]. Numerical simulation results are proposed for two different dimensional chaotic systems (Lorenz [24] and Lorenz-Stenflo systems [25]) to display the effectiveness of our proposed schemes.

Section-wise, we divide this paper into these components: in section-2, a extensive theory of coupling of the drive and response systems with mismatch parameters is proposed, in section-3, we establish the mixed type synchronization of identical Lorenz systems with mismatch parameters. In section-4, studied the targeting synchronization of different as well as different dimensional chaotic systems. In the first case, we consider the drive system as a Lorenz system (3-D) and the response systems as a Lorenz-Stenflo (4-D) system, but in the second case the drive system is considered as a Lorenz-Stenflo (4-D) system and the response system as a Lorenz system (3-D). In section-5, numerical simulation results are showed as proof of the efficiency of our proposed schemes. Finally, section-6 highlights the conclusions.

2. Extensive Coupling Theory with Mismatch Parameter

We deem the drive system as

$$\dot{x} = f(x,\xi) + \Delta f(x,\xi), \tag{1}$$

here x is an element of $\mathbb{R}^n$, ξ represent the vector of parameters, $\Delta f(x,\xi) = f(x,\xi+\delta\xi) - f(x,\xi)$ and $\delta\xi$ construes the mismatch parameters. If all the parameters are in linear form in $f(.)$, the drive system becomes:

$$\Delta f(x,\xi) = f(x,\delta\xi). \tag{2}$$

For the time being, we deem the response system as follows:

$$\dot{w} = f(w,\xi) + L, \tag{3}$$

We want to drive L (controller) using lyapunov stability theory. The coupled system's error vector can be implied as:

$$\begin{aligned} e &= w - g(x), \\ &= w - \alpha\phi(x), \end{aligned} \tag{4}$$

where $g(x)$ is the target (goal) function and $\phi: \mathbb{R}^n \to \mathbb{R}^m$ is a function (continuously differentiable). From (4), the error dynamics is expressed by:

$$\begin{aligned} \dot{e} &= \dot{w} - Dg(x)\dot{x} \\ &= f(w,\xi) + L - Dg(x)[f(x,\xi) + f(x,\delta\xi)], \end{aligned} \tag{5}$$

where $Dg(x)$ implies the Jacobian matrix of $g(x)$. We select L (controller) in such a manner that the error system approaches to zero as time goes to infinity.

3. Engineering Synchronization for Mismatch Parameters

Engineering synchronization for mismatch parameters involves aligning or compensating for discrepancies or differences between various parameters in an engineering system. These parameters could include physical properties, characteristics, or specifications that may vary between different components or sub-systems within the system. For engineering synchronization with mismatch parameters, deem the Lorenz system as:

$$\begin{aligned} \dot{x}_{11} &= \sigma(x_{22} - x_{11}) \\ \dot{x}_{22} &= r x_{11} - x_{22} - x_{11}x_{33} \\ \dot{x}_{33} &= x_{11}x_{22} - \beta x_{33}, \end{aligned} \tag{6}$$

where x_{11}, x_{22}, x_{33} are the variables (state) and σ, r, β are the parameters of the system mentioned earlier. 2D phase diagram of Lorenz System (6) in x–z plane is shown in Figure 1(a). With mismatch parameters of σ, r and β, we deem the equation (6) as the drive system in following way:

$$\begin{aligned}\dot{x}_{11} &= (\sigma + \delta\sigma)(x_{22} - x_{11})\\ \dot{x}_{22} &= (r + \delta r)x_{11} - x_{22} - x_{11}x_{33}\\ \dot{x}_{33} &= x_{11}x_{22} - (\beta + \delta\beta)x_{33},\end{aligned} \tag{7}$$

here $\delta\sigma$, δr and $\delta\beta$ are the mismatch parameters corresponding to the parametrs σ, r and β. Bifurcation plot and variation of Lyapunov exponent for the mismatch parameter $\delta\sigma(\sigma = 10, r = 28, \beta = 2.66, \delta r = 0.5, \delta\beta = 2.5)$, for the mismatch parameter $\delta r(\sigma = 10, r = 28, \beta = 2.66, \delta\sigma = 0.5, \delta\beta = 2.5)$, for the mismatch parameter $\delta\beta(\sigma = 10, r = 28, \beta = 2.66, \delta\sigma = 0.5, \delta r = 2.5)$, are shown in Figure 2, Figure 3 and Figure 4 respectively.We consider Lorenz-Stenflo system as the response system with the following system of non-linear equations:

$$\begin{aligned}\dot{w}_{11} &= \sigma(w_{22} - w_{11}) + sw_{44} + l_1\\ \dot{w}_{22} &= rw_{11} - w_{22} - w_{11}w_{33} + l_2\\ \dot{w}_{33} &= w_{11}w_{22} - \beta w_{33} + l_3\\ \dot{w}_{44} &= -w_{11} - \sigma w_{44} + l_4,\end{aligned} \tag{8}$$

Here, our target is to regulate the controllers (l_1, l_2, l_3, l_4) for the objective of synchronization of two different-dimensional chaotic systems with mismatch parameters. 2D phase diagram of above Lorenz-Stenflo system (8) without controllers l_1, l_2, l_3, and l_4 are shown in Figure 1(b). Let the target state be defined as

$g(z) = (\alpha_{11}x_{11}, \alpha_{22}x_{22}, \alpha_{33}x_{33}, \alpha_{44}(x_{11} + x_{22}))^T$, where $\alpha = diag(\alpha_{11}, \alpha_{22}, \alpha_{33}, \alpha_{44})$ is the scaling matrix. Then the error stands as:

$$\begin{aligned}e_{11} &= w_{11} - \alpha_{11}x_{11}\\ e_{22} &= w_{22} - \alpha_{22}x_{22}\\ e_{33} &= w_{33} - \alpha_{33}x_{33}\\ e_{44} &= w_{44} - \alpha_{44}(x_{11} + x_{22})\end{aligned} \tag{9}$$

where α_{11}, α_{22}, α_{33}, α_{44} are the scaling factors. Then the error dynamics between the systems (7) and (8) can be expressed as:

$$\begin{aligned}\dot{e}_{11} &= \dot{w}_{11} - \alpha_{11}\dot{x}_{11}\\ &= \sigma(w_{22} - w_{11}) + sw_{44} - \alpha_{11}[(\sigma + \delta\sigma)(x_{22} - x_{11})] + l_1\\ &= \sigma e_{22} - \sigma e_{11} - \alpha_{11}\delta\sigma(x_{22} - x_{11}) + sw_{44} + l_1\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{e}_{22} &= \dot{w}_{22} - \alpha_{22}\dot{x}_{22}\\ &= rw_{11} - w_{22} - w_{11}w_{33} - \alpha_{22}[(r + \delta r)x_{11} - x_{22} - x_{11}x_{33}] + l_2\\ &= re_{11} - e_{22} - w_{11}w_{33} - \alpha_{22}\delta r x_{11} + \alpha_{22}x_{11}x_{33} + l_2\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{e}_{33} &= \dot{w}_{33} - \alpha_{33}\dot{x}_{33}\\ &= w_{11}w_{22} - \beta w_{33} - \alpha_{33}(x_{11}x_{22} - (\beta + \delta\beta)x_{33}) + l_3\\ &= -\beta e_{33} + w_{11}w_{22} - \alpha_{33}x_{11}x_{22} + \alpha_{33}\delta\beta x_{33} + l_3\end{aligned} \tag{12}$$

$$\begin{aligned}
\dot{e}_{44} &= \dot{w}_{44} - \alpha_{44}(\dot{x}_{11} + \dot{x}_{22}) \\
&= -w_{11} - \sigma w_{44} - \alpha_{44}[(\sigma + \delta\sigma)(x_{22} - x_{11}) + (r + \delta r)x_{11} \\
&\quad -x_{22} - x_{11}x_{33}] + l_4 \\
&= -e_{11} - \alpha_{44}x_{11} - \sigma(e_{44} + \alpha_{44}(x_{11} + x_{22})) - \alpha_{44}\sigma x_{22} + \alpha_{44}\sigma x_{11} \\
&\quad -\alpha_{44}\delta\sigma(x_{22} - x_{11}) - \alpha_{44}rx_{11} \\
&\quad -\alpha_{44}\delta rx_{11} + \alpha_{44}x_{22} + \alpha_{44}x_{11}x_{33} + l_4 \\
&= -e_{11} - \sigma e_{44} - \alpha_{11}x_{11} - 2\alpha_{44}\sigma x_{22} - \alpha_{44}\delta\sigma(x_{22} - x_{11}) \\
&\quad -\alpha_{44}rx_{11} - \alpha_{44}\delta rx_{11} + \alpha_{44}x_{22} + \alpha_{44}x_{11}x_{33} + l_4.
\end{aligned} \tag{1}$$

Now we choose the controllers l_1, l_2, l_3 and l_4 in the following manner.

$$\begin{aligned}
l_1 &= \alpha_{11}\delta\sigma(x_{22} - x_{11}) - sy_4 + m_1 \\
l_2 &= w_{11}w_{33} + \alpha_{22}\delta rx_{11} - \alpha_{22}x_{11}x_{33} + m_2 \\
l_3 &= -w_{11}w_{22} + \alpha_{33}x_{11}x_{22} - \alpha_{33}\delta\beta x_{33} + m_3 \\
l_4 &= \alpha_{11}x_{11} + 2\alpha_{44}\sigma x_{22} + \alpha_{44}\delta\sigma(x_{22} - x_{11}) \\
&\qquad +\alpha_{44}rx_{11} + \alpha_{44}\delta rx_{11} - \alpha_{44}x_{22} - \alpha_{44}x_{11}x_{33} + m_4,
\end{aligned} \tag{14}$$

where m_1, m_2, m_3, m_4 are the linear feedback controllers. Therefore, the error system stands as:

$$\begin{aligned}
\dot{e}_{11} &= \sigma e_{22} - \sigma e_{11} + m_1 \\
\dot{e}_{22} &= re_{11} - e_{22} + m_2 \\
\dot{e}_{33} &= -e_{11} + \beta e_{33} + m_3 \\
\dot{e}_{44} &= -e_{11} - \sigma e_{44} + m_4.
\end{aligned} \tag{15}$$

By choosing of the linear feedback control $m_1 = 0$, $m_2 = -re_{11} + re_{44}$, $m_3 = 0$, $m_4 = 0$, the error system stands as:

$$\begin{aligned}
\dot{e}_{11} &= \sigma(e_{22} - e_{11}) \\
\dot{e}_{22} &= -e_{22} + re_{44} \\
\dot{e}_{33} &= -e_{11} + \beta e_{33} \\
\dot{e}_{44} &= -e_{11} - \sigma e_{44}.
\end{aligned} \tag{16}$$

The error dynamics shown here (16) can be defined as $\dot{e} = Ae$, where,

$$A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ 0 & -1 & 0 & r \\ -1 & 0 & \beta & 0 \\ -1 & 0 & 0 & -\sigma \end{bmatrix} \tag{17}$$

Using the choices of the values of the parameters $\sigma = 1, r = 1.5$ and $\beta = 2.66,$ and applying a MATLAB software package, we observe that the eigen values of A are $-2.66, -2.1447$ and $-0.4276 \pm i0.9914$. Since the eigen values of A have no positive real parts, therefore synchronization of two chaotic systems (7) and (8) has been achieved with mismatch parameters. The Figure 5 represents engineering synchronization error state paths of (7) and (8), which confirms that (7) and (8) are in synchronized steady state.

4. Targeting Synchronization of the Different Dimensional System

It is derived through this section the targeting synchronization of two different dimensional chaotic systems. In Case 1, we consider the drive system to be a Lorenz (three-dimensional) system and the response system a Lorenz-Stenflo (four-dimensional) system but in Case 2, we consider the drive system to be four-dimensional whereas the response system is three-dimensional for the Lorenz-Stenflo and Lorenz systems respectively.

4.1 *Case 1*

Here a Lorenz (three-dimensional) system is considered to be the drive system and a Lorenz-Stenflo (four-dimensional) system is considered to be the response system. We deem the Lorenz system as the drive system as follows:

$$\begin{aligned} \dot{x}_{11} &= \sigma(x_{22} - x_{11}) \\ \dot{x}_{22} &= r x_{11} - x_{22} - x_{11} x_{33} \\ \dot{x}_{33} &= x_{11} x_{22} - \beta x_{33} \end{aligned} \tag{18}$$

where x_{11}, x_{22}, x_{33} are the variables (state) and σ, β, r, are the parameters of aforementioned system and Lorenz-Stenflo system with controllers n_1, n_2, n_3 and n_4 is considered to be response system as follows:

$$\begin{aligned} \dot{w}_{11} &= \sigma(w_{22} - w_{11}) + s w_{44} + n_1 \\ \dot{w}_{22} &= r w_{11} - w_{11} w_{33} - w_{22} + n_2 \\ \dot{w}_{33} &= w_{11} w_{22} - \beta w_{33} + n_3 \\ \dot{w}_{44} &= -w_{11} - \sigma w_{44} + n_4. \end{aligned} \tag{19}$$

Here our target is to regulate the controllers for the sake of generalized linear synchronization of two chaos systems with different-dimension. Let the error vectors be defined as:

$$e = w - \alpha x \tag{20}$$

where, $w = \begin{bmatrix} w_{11} \\ w_{22} \\ w_{33} \\ w_{44} \end{bmatrix}$, $\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \end{bmatrix}$ and $x = \begin{bmatrix} x_{11} \\ x_{22} \\ x_{33} \end{bmatrix}$, then the error dynamics become:

$$\begin{aligned} \dot{e}_{11} &= \dot{w}_{11} - \alpha_{11}\dot{x}_{11} - \alpha_{12}\dot{x}_{22} - \alpha_{13}\dot{x}_{33} \\ \dot{e}_{22} &= \dot{w}_{22} - \alpha_{21}\dot{x}_{11} - \alpha_{22}\dot{x}_{22} - \alpha_{23}\dot{x}_{33} \\ \dot{e}_{33} &= \dot{w}_{33} - \alpha_{31}\dot{x}_{11} - \alpha_{32}\dot{x}_{22} - \alpha_{33}\dot{x}_{33} \\ \dot{e}_{44} &= \dot{w}_{44} - \alpha_{41}\dot{x}_{11} - \alpha_{42}\dot{x}_{22} - \alpha_{43}\dot{x}_{33}. \end{aligned} \tag{21}$$

The error dynamics can be highlighted as,

$$\dot{e}_{11} = \sigma e_{22} - \sigma e_{11} + s e_{44} + m_1, \tag{22}$$

where $m_1 = \sigma(\alpha_{21}x_{11} + \alpha_{22}x_{22} + \alpha_{23}x_{33}) - \sigma(\alpha_{11}x_{11} + \alpha_{12}x_{22} + \alpha_{13}x_{33}) + s(r x_{11} - x_{22} - x_{11}x_{33}) - \alpha_{13}(x_{11}x_{22} - \beta x_{33} + n_1) - r w_{33} + r\alpha_{31}x_{11} + r\alpha_{32}x_{22} + r\alpha_{33}x_{33}$,

$$\dot{e}_{22} = r e_{11} - e_{22} + r e_{33} + m_2, \tag{23}$$

where $m_2 = r\alpha_{11}x_{11} + r\alpha_{12}x_{22} + r\alpha_{13}x_{33} - w_{11}w_{22} - \alpha_{21}x_{11} - \alpha_{22}x_{22} - \alpha_{23}x_{33} - \sigma\alpha_{21}x_{22} + \sigma\alpha_{21}x_{11} - r\alpha_{22}x_{11} + \alpha_{22}x_{22} + \alpha_{22}x_{11}x_{33} - \alpha_{23}x_{11}x_{22} + \beta\alpha_{23}x_{33} + n_2 - w_{11} + \alpha_{11}x_{11} + \alpha_{12}x_{22} + \alpha_{13}x_{33}$,

$$\dot{e}_{33} = e_{11} - \beta e_{33} + m_3, \tag{24}$$

where $m_3 = w_{11}w_{22} - \sigma\alpha_{31}x_{22} + \sigma\alpha_{31}x_{11} - \beta\alpha_{31}x_{11} - \beta\alpha_{32}x_{22} - r\alpha_{32}x_{11} + \alpha_{32}x_{22} + \alpha_{32}x_{11}x_{33} - \alpha_{33}x_{11}x_{22} + n_3$,

$$\dot{e}_{44} = -e_{11} - \sigma e_{44} + m_4, \tag{25}$$

where $m_4 = -\alpha_{11}x_{11} - \alpha_{12}x_{22} - \alpha_{13}x_{33} - \sigma\alpha_{42}x_{22} - \sigma\alpha_{43}x_{33} - \sigma\alpha_{41}x_{22} - r\alpha_{42}x_{11} + \alpha_{42}x_{22} + \alpha_{42}x_{11}x_{33} - \alpha_{43}x_{11}x_{22} + \beta\alpha_{43}w_{33} + n_4$. By the choice of control parameters $m_1 = 0, m_2 = -re_{33}, m_3 = -e_{11}$ and $m_4 = 0$, the error system stands as:

$$\begin{aligned} \dot{e}_{11} &= \sigma(e_{22} - e_{11}) + se_{44} \\ \dot{e}_{22} &= re_{11} - e_{22} \\ \dot{e}_{33} &= -\beta e_{33} \\ \dot{e}_{44} &= -e_{11} - \sigma e_{44}. \end{aligned} \tag{26}$$

The error dynamics shown here can be viewed as $\dot{e} = Be$, where

$$B = \begin{bmatrix} -\sigma & \sigma & 0 & s \\ r & -1 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ -1 & 0 & 0 & -\sigma \end{bmatrix}. \tag{27}$$

For the same choices of the parameter values $\sigma = 1, r = 1.5, \beta = 2.66$ and $s = 1.5$, when we apply MATLAB software package we observe that the characteristic values of the matrix B are $-1, -1, -1$ and -2.66. Since all characteristic values of the matrix B are negative, therefore the error system (26) is stable (asymptotically) with the above choices m_i, s for $i = 1,2,3,4$. This signifies that the synchronization of two different chaos systems has been accomplished. The Figure 6 represents generalized synchronization error state paths of (18) and (19).

4.2 *Case 2*

Case 2 highlights a four-dimensional Lorenz-Stenflo system which is considered to be drive system and a three-dimensional Lorenz system consider as response system. Drive Lorenz system, shown as follows:

$$\begin{aligned} \dot{x}_{11} &= \sigma(x_{22} - x_{11}) + sx_{44} \\ \dot{x}_{22} &= rx_{11} - x_{22} - x_{11}x_{33} - x_{22} \\ \dot{x}_{33} &= x_{11}x_{22} - \beta x_{33} \\ \dot{x}_{44} &= x_{11} - \sigma x_{44}, \end{aligned} \tag{28}$$

where $x_{11}, x_{22}, x_{33}, x_{44}$ are the variables (state) and σ, r, β and s are this parameters of this system and the response system with controllers n_1, n_2 and n_3 are mentioned below:

$$\begin{aligned} \dot{w}_{11} &= \sigma(w_{22} - w_{11}) + n_1 \\ \dot{w}_{22} &= rw_{11} - w_{22} - w_{11}w_{33} + n_2 \\ \dot{w}_{33} &= w_{11}w_{22} - \beta w_{33} + n_3. \end{aligned} \tag{29}$$

Here our target is to regulate the controllers for the purpose of GLS of two different-dimensional systems. Let the error vectors be defined as:

$$e = w - \alpha x, \tag{30}$$

where, $y = \begin{bmatrix} w_{11} \\ w_{22} \\ w_{33} \end{bmatrix}$, $\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \end{bmatrix}$ and $z = \begin{bmatrix} x_{11} \\ x_{22} \\ x_{33} \\ x_{44} \end{bmatrix}$. Then the error dynamics becomes:

$$\begin{aligned} \dot{e}_{11} &= \dot{w}_{11} - \alpha_{11}\dot{x}_{11} - \alpha_{12}\dot{x}_{22} - \alpha_{13}\dot{x}_{33} - \alpha_{14}\dot{x}_{44} \\ \dot{e}_{22} &= \dot{w}_{22} - \alpha_{21}\dot{x}_{11} - \alpha_{22}\dot{x}_{22} - \alpha_{23}\dot{x}_{33} - \alpha_{24}\dot{x}_{44} \\ \dot{e}_{33} &= \dot{w}_{33} - \alpha_{31}\dot{x}_{11} - \alpha_{32}\dot{x}_{22} - \alpha_{33}\dot{x}_{33} - \alpha_{34}\dot{x}_{44} \end{aligned} \tag{31}$$

With the help of (28) and (29), the error dynamics (31) can be put forward as:

$$\dot{e}_{11} = \sigma e_{22} - \sigma e_{11} + m_1, \tag{32}$$

where $m_1 = \sigma(\alpha_{22}x_{11} + \alpha_{22}x_{22} + \alpha_{23}x_{33} + \alpha_{24}x_{44} - \alpha_{12}x_{22} - \alpha_{13}x_{33} - \alpha_{11}x_{22})$ $-s\alpha_{11}x_{44} - r\alpha_{12}x_{11} + \alpha_{12}x_{11}x_{33} + \alpha_{12}x_{22} - \alpha_{13}x_{11}x_{22} + \beta\alpha_{13}x_{33} + \alpha_{14}x_{11} + n_1)$,

$$\dot{e}_{22} = re_{11} - e_{22} + m_2, \tag{33}$$

where $m_2 = r\alpha_{11}x_{11} + r\alpha_{12}x_{22} + r\alpha_{13}x_{33} + r\alpha_{14}x_{44} - \alpha_{21}x_{11} - \alpha_{23}x_{33} - \alpha_{24}x_{44}$ $-w_{11}w_{33} - \sigma\alpha_{21}x_{22} + \sigma\alpha_{21}x_{11} - r\alpha_{22}x_{11} + \alpha_{22}x_{11}x_{33} + \alpha_{23}x_{11}x_{22} + \beta\alpha_{23}x_{33}$ $+\alpha_{24}x_{11} + \sigma\alpha_{24}x_{44} - s\alpha_{21}x_{44} + n_2$, and

$$\dot{e}_{33} = -\beta e_{33} + m_3, \tag{34}$$

where $m_3 = -\beta\alpha_{31}x_{11} - \beta\alpha_{32}x_{22} - \beta\alpha_{33}x_{33} - \beta\alpha_{34}x_{44} - \sigma\alpha_{31}x_{22} + \sigma\alpha_{31}x_{11}$ $-s\alpha_{31}x_{44} - r\alpha_{32}x_{11} + \alpha_{32}x_{22} + \alpha_{32}x_{11}x_{33} - \alpha_{33}x_{11}x_{22} + \beta\alpha_{33}x_{33} + \alpha_{34}x_{11}$ $+\sigma\alpha_{34}x_{44}$. By choices of $m_1 = 0, m_2 = -2re_{11}$ and $m_3 = 0$, the error dynamics of (32), (33) and (34) becomes:

$$\begin{aligned} \dot{e}_{11} &= \sigma(e_{22} - e_{11}) \\ \dot{e}_{22} &= -re_{11} - e_{22} \\ \dot{e}_{33} &= -\beta e_{33}. \end{aligned} \tag{35}$$

The error system derived here can be expressed as $\dot{e} = Ce$, where

$$C = \begin{bmatrix} -\sigma & \sigma & 0 \\ -r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}. \tag{36}$$

Here we choose the parameter values as $\sigma = 10, r = 28$ and $\beta = 2.66$, so the eigen values of C are $-5.5 \pm i16.1168$ and -2.66. Since the eigenvalues of C have no positive real parts, hence the error system (35) is stable (asymptotically). This construes that the synchronization of two different chaos systems (28) and (29) has been achieved. Time evolution of error system (35) represent in Figure 7.

Numerical Simulation

From this section, we derived that computer simulations are perforced to expound the synchronization of two different chaotic systems. The 4th order RK method is applied with fixed step size 0.01. The parameter values are selected as $\sigma = 10, r = 28, \beta = 2.66, s = 1.5, \delta\sigma = 0.5, \delta r = 0.5,$ and $\delta\beta = 2.5$. For all simulations we consider the initial values of the state variables as $x_{11}(0) =$

$0.01, x_{22}(0) = 0.01, x_{33}(0) = 0.01, x_{44}(0) = 0.01$ and $w_{11}(0) = -0.01, w_{22}(0) = -0.01, w_{33}(0) = -0.01, w_{44}(0) = -0.01$.

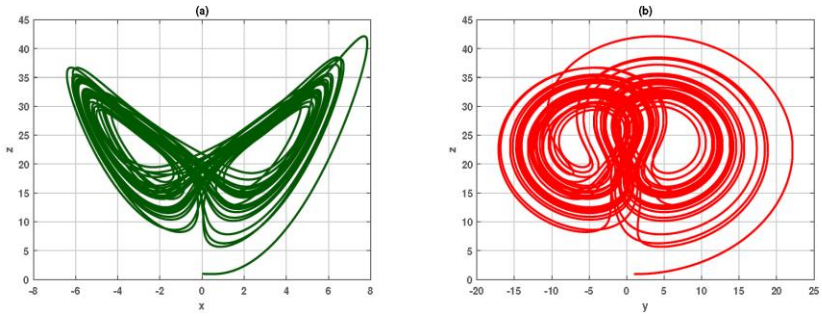

Figure 1
2D phase diagrams of system (a) $x - z$ phase plot of Lorenz system (6) and (b) $y - z$ phase plot of Lorenz-Stenflo system.

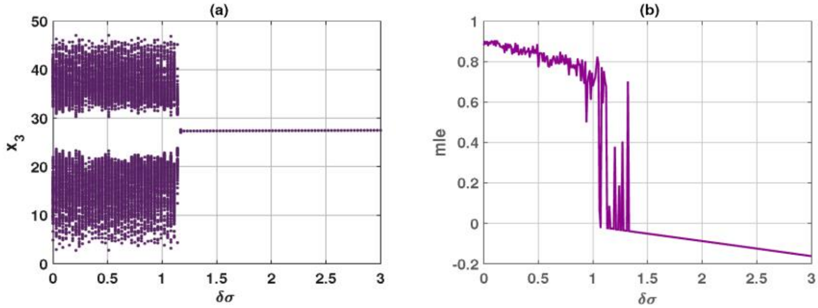

Figure 2
Bifurcation plot and variation of lyapunov exponent for the mismatch parameter $\delta\sigma$ of the Lorenz system (7) for $\sigma = 10, r = 28,\ \beta = 2.66,\ \delta r = 0.5$ and $\delta\beta = 2.5$

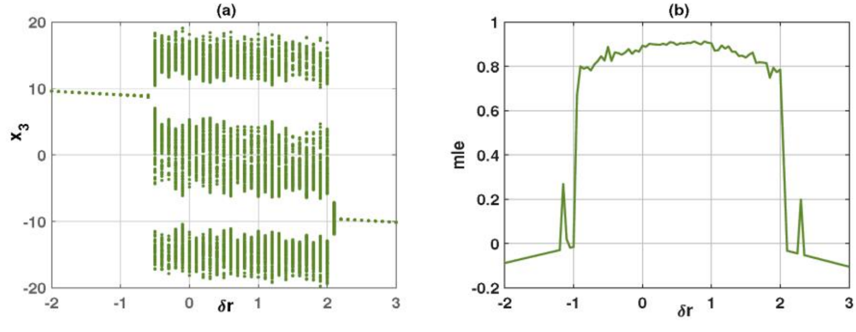

Figure 3
Bifurcation plot and variation of lyapunov exponent for the mismatch parameter δr of the Lorenz system (7) for $\sigma = 10, r = 28,\ \beta = 2.66,\ \delta\sigma = 0.5$ and $\delta\beta = 2.5$

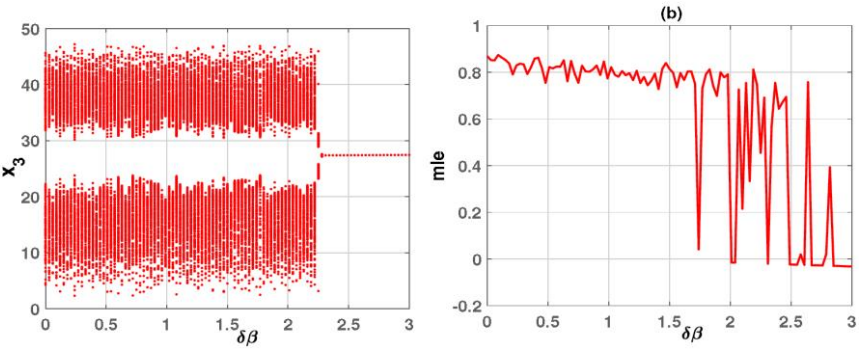

Figure 4

Bifurcation plot and variation of lyapunov exponent for the mismatch parameter $\delta\beta$ of the Lorenz system (7) for $\sigma = 10, r = 28,\ \beta = 2.66,\ \delta\sigma = 0.5$ and $\delta r = 2.5$

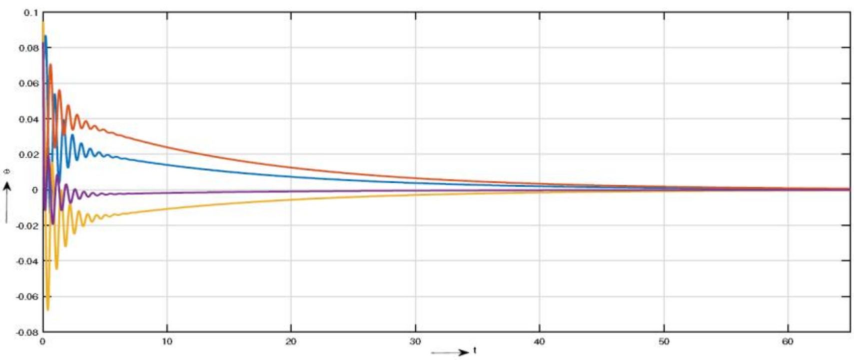

Figure 5

Engineering synchronization error state path of (7) and (8)

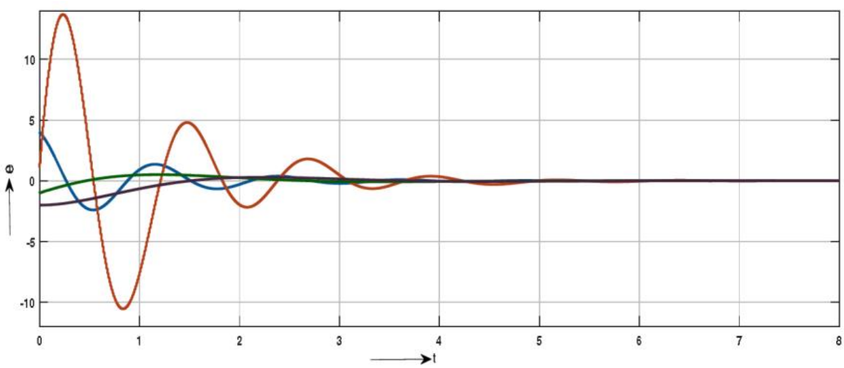

Figure 6

Generalized synchronization error state path of (18) and (19) in case- 1.

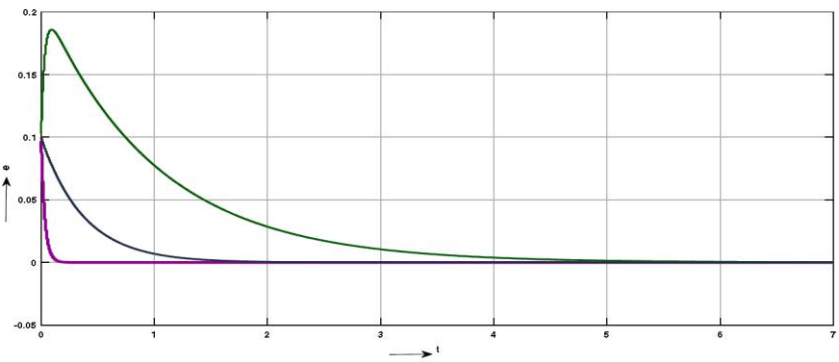

Figure 7
Generalized synchronization error state path of (28) and (29) in case- 2.

6. Conclusion

Through this paper, we proposed targeting the engineering synchronization of different chaotic oscillators with mismatch parameters. The general scheme for the coupling design for a unidirectional couple oscillator has been discussed. Targeting engineering linear and non-linear generalized synchronizations using linear feedback coupling has been demonstrated here. Targeting synchronization with mismatch parameters plays a crucial role in mitigating the effects of parameter discrepancies, improving system robustness, ensuring accurate and reliable operation across diverse engineering systems and also demonstrating chaotic synchronization between systems of different dimensions in various fields ranging from communication and cryptography to control and signal processing. The physical realization of engineering synchronization using electronic circuits is also a challenging problem which will be accomplished in future work.

7. Author Contribution

CR: Methodology development, Writing-original draft, Validation, Simulation and numerical studies and **MAK:**Funding acquisition, Methodology development, Paper writing, Validation, Simulation and numerical studies. Author **CR** is responsible for overall management.

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9. Declarations

9.1. *Conflicts of Interest*

The authors affirm that there are no conflict of interest.

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