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Optimization bisection technique by using quantum calculus approach

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Abstract

This paper presents a numerical simulation approach for solving nonlinear equations using a quantum Bisection Optimization technique. We propose an enhanced version of Bisection technique based on quantum calculus optimization. The results are analyzed for different values of quantum parameter, q, & rate of convergence is determined for every $q \in (0, 1)$. Additionally, it is demonstrated that the modified method is always convergent and for each interval there exists $q \in (0, 1)$ for which the exact solution to the problem and first approximation of the root coincides. This study highlights the potential of the modified Bisection optimization method as a valuable tool for simulating and solving complex nonlinear problems encountered in science and engineering applications.

Subject Classification: *65A05, 65D05, 65D30, 65D32.*

Keywords: *Numerical simulation, Nonlinear optimization, Bisection method, Root-finding, Convergence.*

Introduction:

Nonlinear equations arise in various branches of engineering and sciences, including heat transfer, chemical kinetics, and structural mechanics, fluid dynamics. Solving nonlinear equations is crucial for predicting and understanding the behaviour of complex systems. Numerical simulation techniques play a vital role in approximating the solutions of nonlinear equations when analytical methods are infeasible or impractical.

The Bisection method is a well-known iterative algorithm for finding roots of continuous functions. It is based on the intermediate value theorem and

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This article has been corrected with minor changes. These changes do not impact the academic content of the article.

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guarantees convergence to a root under certain conditions. The method involves repeatedly bisecting an interval and selecting a subinterval containing the root for further processing. Despite its simplicity and reliability, the standard Bisection method has limitations in terms of convergence speed and efficiency, especially for highly nonlinear equations. Wang, Zhu, and Du [12] proposed a unified approach to investigating some different types of quantum integral inequalities such as midpoint-type, Simpson-type, averaged midpoint-trapezoid-type and trapezoid-type inequality. Also, they give an upper bound for quantum integral inequalities via product of two s-preinvex functions.

To find the approximation solution of the complicated problems we focus on the numerical techniques. Numerical analysis is interested in approximating the roots of nonlinear equations. To solve nonlinear equations, several numerical techniques are used, such as the Bisection technique, Regula Falsi technique, Secant technique, Newton- Raphson technique etc. [1]. Jackson published some significant results on q-calculus at the beginning of the 20th century. Because of its many applications in the fields of number theory, relativity, mechanics, combinatorics, and cryptography, this field has recently experienced a significant increase in research [2–6]. The quantum calculus optimization methodology has been widely used by scholars for further research on numerical methods. The literature has quantum analogues for a variety of root obtaining techniques [7–9].

Singh, Mishra, and Pathak [10] studied the q-analogue of iterative techniques, specifically the q-analogue of the proposed Newton-Raphson technique, using q-Taylor formula. They then compared the accuracy of their findings with those of the classical techniques. Using q-differential equations, numerous linear and nonlinear models that arise in scientific and engineering problems can be represented. He developed an iterative approach in [11] to increase the rate of convergence of an old Chinese algorithm. According to results, the revised Chinese algorithms that were built are more efficient and accurate.

The following is Chun-Hui He's iterative technique:

Step 1: Estimating the first approximation using the Chinese algorithm

$$x_2 = x_0 - \frac{f(x_0)}{\frac{f(x_0) - f(x_1)}{x_0 - x_1}} = x_0 - \frac{f(x_0)}{R(x_0,x_1)} \tag{1}$$

Where $R(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$ and $f(x_0)f(x_1) < 0$.

Step 2: Using x_2 as an initial approximation in the Newton technique

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Step 3: Putting $x_2 = x_0$ and $x_3 = x_1$ in equation (1) and repeat the procedure until the desired level of accuracy is achieved.

The current study compares the bisection method's q-analogue approximations to traditional iterative approaches. Although the q-bisection approach always converges, its order of convergence is linear. The definition of the quantum number for $q \in (0,1)$ and n ∈ N is as follows:

$$[n]_q = \frac{1-q^n}{1-q} \text{ and when } q \to 1 \text{ then } [n]_q = n$$

The rest of the paper is designed as follows: section 2 presents proposed methodology, section 3 presents convergence analysis, section 4 presents numerical experiments and comparison of results, section 5 presents result discussion and conclusions.

Proposed Methodology: Quantum Bisection for Optimization

Assume that $f(x) = 0$, is a nonlinear equation where $f(x)$ is real continuous function. The following q-bisection optimization iterative formulas are used to approximate root:

$$c = \frac{a+qb}{[2]_q} \tag{2}$$

or

$$c = \frac{a+qb}{[2]_q} \tag{3}$$

where c represents an approximated root value. For the root approximation, one can use either of the iterative formulas. In this study, the standard QBM (2) is used.

Proposition:

Suppose that c is the root of $f(x)$, then

$$q = \frac{c-a}{b-c} \tag{4}$$

gives the value of q for which algorithm (2) rapidly converges to root. Expression equivalent to (4) can also be obtained for (3).

Convergence Analysis:

We rewrite the q-bisection technique as follows:

$$x_{n+1} = \frac{x_{n-1}+qx_n}{[2]_q} \tag{5}$$

If x is root of f, then ϵ_n is difference between x and nth approximation of the root x_n.

$$x = x_n + \epsilon_n$$
$$x = x_{n-1} + \epsilon_{n-1}$$
$$x = x_{n+1} + \epsilon_{n+1}$$

From equation (5), we get

$$x - \epsilon_{n+1} = \frac{x-\epsilon_{n+1} + qx - q\epsilon_n}{[2]_q}$$

$$x - \epsilon_{n+1} = \frac{[2]_q x-\epsilon_{n-1} - q\epsilon_n}{[2]_q}$$

$$\epsilon_{n+1} = \frac{\epsilon_{n-1} + q\epsilon_n}{[2]_q}$$

$$\epsilon_{n+1} = \frac{q\epsilon_n}{[2]_q}\left[1 + \frac{\epsilon_{n-1}}{q\epsilon_n}\right]$$

Neglecting the term $\frac{\epsilon_{n-1}}{q\epsilon_n}$, we obtained

$$\epsilon_{n+1} \approx \frac{q\epsilon_n}{[2]_q}$$

Hence q-bisection optimization technique has a linear order of convergence.

Numerical Experiments and Comparison of Results:

To evaluate performance of proposed Bisection optimization method, numerical simulations were conducted on several benchmark problems involving nonlinear equations. The proposed method was compared with others existing techniques.

The stopping criterion are used in the proposed algorithm is

$$\left|\frac{b-a}{2}\right| < \epsilon \text{ and } |f(c)| < \epsilon = 10^{-16}$$

The efficiency of the proposed method is determined by analyzing solution of some of nonlinear equations. The value of $f(x_n)$, difference between successive approximation δ and number of iterations (IT) all are shown below in table 1-5. In all the problems we take $q = 0.5$, $q = 0.75$ are taken for QBM_a QBM_b and for QBM_c q value is calculated from the equation (4).

Benchmark problems:

Problem 1: $e^x 2^x - 1 + 2\cos x\, 2^x - 6\, 2^x$
Problem 2: $2x\cos x - x^2 + 4x - 4$
Problem 3: $10x - 8 - 2\sin x$
Problem 4: $e^x \sin x - 1$
Problem 5: $\ln(\tan x) - e^{-2x}$

Table 1

$\boldsymbol{f_1(x)}$: $\boldsymbol{e^x 2^x - 1 + 2\cos x\, 2^x - 6\, 2^x}$

Methods	Number of Iterations	x_n	$f(x_n)$	δ
Bisection	43	$-8325792882709777 \times 10^{-16}$	4463097×10^{-20}	2842171×10^{-20}
Regula Falsi	11	$-8325792882709121 \times 10^{-16}$	2513545×10^{-19}	-172617×10^{-15}
Newton-Raphson	5	$-832579288270992 \times 10^{-16}$	0	110223×10^{-22}
QBM_a	53	$-8325792882709923 \times 10^{-16}$	-110223×10^{-21}	666138×10^{-22}

QBM_b	51	$-8325792882709914 \times 10^{-16}$	1776357×10^{-21}	1332268×10^{-21}
QBM_c	1	$-8325792882709914 \times 10^{-16}$	0	1554312×10^{-21}

Table 2

$\boldsymbol{f_2(x)\colon 2x\cos x - x^2 + 4x - 4}$

Methods	**Number of Iterations**	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$	$\boldsymbol{\delta}$
Bisection	47	$9484340699196361 \times 10^{-16}$	6661338×10^{-22}	8881784×10^{-22}
Regula Falsi	61	$9484340699196361 \times 10^{-16}$	8881784×10^{-22}	-444089×10^{-21}
Newton-Raphson	7	$948434069919636 \times 10^{-15}$	6661338×10^{-22}	6661338×10^{-22}
QBM_a	57	$9484340699196361 \times 10^{-16}$	6661338×10^{-22}	1554312×10^{-21}
QBM_b	53	$9484340699196359 \times 10^{-16}$	2220446×10^{-22}	4996004×10^{-22}
QBM_c	1	$948434069919636 \times 10^{-15}$	0	0

Table 3

$\boldsymbol{f_3(x)\colon 10x - 8 - 2\sin x}$

Methods	**Number of Iterations**	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$	$\boldsymbol{\delta}$
Bisection	51	$9643338876952226 \times 10^{-16}$	-1387779×10^{-22}	4440892×10^{-22}
Regula Falsi	7	$9643338876952218 \times 10^{-16}$	-8326672×10^{-22}	2985389×10^{-22}
Newton-Raphson	5	$964333887695223 \times 10^{-15}$	24980018×10^{-23}	-131509×10^{-14}
QBM_a	57	$9643338876952231 \times 10^{-16}$	3330669×10^{-22}	8326673×10^{-22}
QBM_b	53	$9643338876952229 \times 10^{-16}$	1387779×10^{-22}	4440892×10^{-22}
QBM_c	1	$964333887695223 \times 10^{-15}$	$-555111512 \times 10^{-25}$	0

Table 4

$f_4(x)$: $e^x \sin x - 1$

Methods	Number of Iterations	x_n	$f(x_n)$	δ
Bisection	51	$5885327439818613 \times 10^{-16}$	2220446×10^{-22}	4440892×10^{-21}
Regula Falsi	23	$588532743981861 \times 10^{-15}$	11102230×10^{-23}	-33306×10^{-20}
Newton-Raphson	5	$588532743981861 \times 10^{-15}$	1110223×10^{-22}	1110223×10^{-22}
QBM_a	58	$5885327439818610 \times 10^{-16}$	-1110223×10^{-22}	2775558×10^{-22}
QBM_b	47	$5885327439818592 \times 10^{-16}$	-2553513×10^{-21}	4440892×10^{-21}
QBM_c	1	$588532743981861 \times 10^{-15}$	0	0

Table 5

$f_5(x)$: $\ln(\tan x) - e^{-2x}$

Methods	Number of Iterations	x_n	$f(x_n)$	δ
Bisection	51	$872312388801615 \times 10^{-15}$	-9436896×10^{-22}	6661338×10^{-22}
Regula Falsi	7	$8723123888016150 \times 10^{-16}$	-8326673×10^{-23}	-111022×10^{-21}
Newton-Raphson	5	$872312388801615 \times 10^{-15}$	-8326673×10^{-23}	0
QBM_a	57	$8723123888016149 \times 10^{-16}$	-4718448×10^{-22}	2220446×10^{-21}
QBM_b	51	$8723123888016111 \times 10^{-16}$	-9436896×10^{-22}	7882583×10^{-21}
QBM_c	1	$872312388801615 \times 10^{-15}$	$-83266726846 \times 10^{-27}$	0

Result Discussion and Conclusions:

The article's main goal is to create an optimized iterative approach for solving nonlinear equations using quantum calculus. The proposed method generalizes the classical bisection method, as it is seen that the quantum bisection optimization methodology converges at varied speeds for different values of quantum parameter q ∈ (0, 1). Despite linear order of convergence of QBM, there exist q for which method converges quickly to root. The algorithm comparison shows that the generalized quantum bisection approach outperforms the classical techniques and is more dependable for specific values of q. Newton's method relies heavily on the initial guess, even if it has a higher order of convergence; if the original guess is wrong, the algorithm may not work.

Similar to this, there are situations in which the Regula Falsi and the conventional bisection method are unable to yield the necessary degree of accuracy in the conclusion. Nonetheless, QBM offers a more precise approximation of the roots for a subset of q values. To improve the technique's efficiency, q-analogues of a few well-known numerical methods might be developed in the future.

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