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Convex polytopes of constant local multiset dimension with an application

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Abstract

Determining the minimum monitoring points in a road network which remains constant, even as the network expands plays a crucial role in efficient traffic monitoring, congestion detection and adaptive signal control. As traffic is the major issue in many cities, the minimum monitoring points can be located using a variation of metric dimension obtained by incorporating distance based multiset to distinguish adjacent vertices more effectively, called local multiset dimension(lmd). A convex polytope is a bounded geometric structure defined by a collection of convex points in n-dimensional Euclidean space $\mathbb{R}^n$, which can be used to design and analyze networks. The lmd of N-gonal circular ladder is discussed and the result is extended to answer a road traffic problem.

Subject Classification (2020): *05C10, 05C12, 51M20.*

Keywords: *Convex polytope, Local multiset dimension, Metric dimension.*

1. Introduction

Let $G(V,E)$ denote a graph with a vertex set V and edge set E. Convex polytope is one of the geometric shape that is formed by the intersection of finite number of flat surfaces in $2D$ called as polygons and $3D$ called as polyhedra. Convex polytopes play a crucial role in representing real-world situations, especially in areas where optimization, geometry, and combinatorics intersect. Convex polytopes come in various types, each with distinct properties and structures, which can be used in modeling different types of network.

Bengaluru's traffic is a growing challenge due to high vehicle density, bottlenecks at major junctions, and inefficient routing. These issues can be addressed through effective monitoring and management of the road network which remains consistent even when the road network expands. Here, we consider the road network of Bengaluru's ring roads, which comprises of an inner and outer ring road major junctions and path connecting them, consisting of minor junctions forming an N-gonal circular ladder.

The graph of a N-gonal circular ladder, denoted as $N_{n,l}$ is a variation of hendecagonal circular ladder H_n[1]. It has an n sided cycle, with each side

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connected to a $(2l+3)$ sided polygon. It consists of $V(N_{n,l}) = \{u_{i,j}, v_i: 0 \leq i \leq n-1, 0 \leq j \leq l\}$ and $E(N_{n,l}) = \{u_{i,0}u_{i+1,0}, u_{i+1,l}v_i: 0 \leq i \leq n-2\} \cup \{u_{0,0}u_{n-1,0}, u_{0,l}v_{n-1}\} \cup \{u_{i,j}u_{i,j+1}, u_{i,l}v_i, : 0 \leq i \leq n-1, 0 \leq j \leq l-1\}$. There are two types of induced cycle in this graph, called as inner cycle defined by the vertices $\{u_{i,0}: 0 \leq i \leq n-1\}$ and the other is called the outer cycle defined by the vertices $\{u_{i,l}, v_i: 0 \leq i \leq n-1\}$. Between the vertices $u_{i,0}$ and $u_{i,l}$, there lies a path of $l-1$ vertices, that is $\{u_{i,j}: 0 \leq i \leq n-1, 1 \leq j \leq l-1\}$. Thus $|V(N_{n,l})| = (2+l)n$ and $|E(N_{n,l})| = (3+l)n$. Figure 1 shows the graph of polytope $N_{8,6}$.

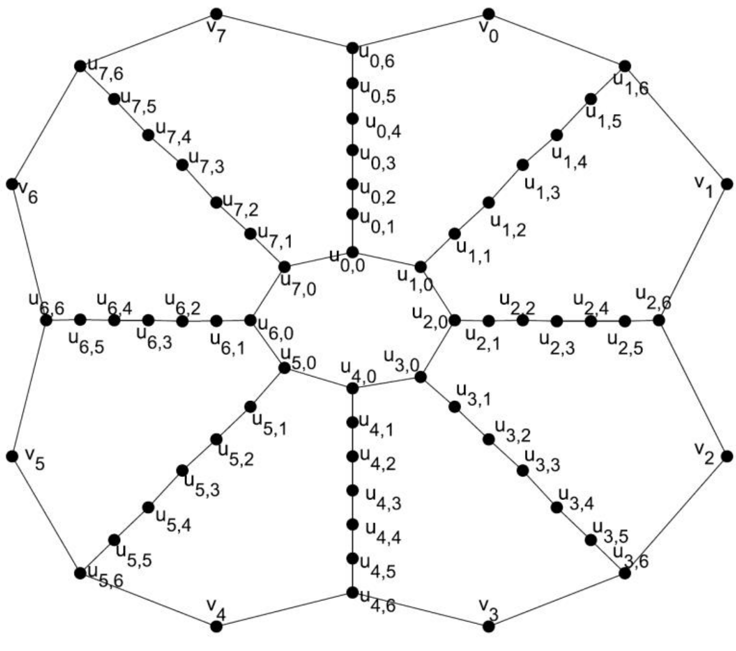

Figure 1
Graph of $N_{8,6}$

The major junctions of the Outer ring road like Hebbal, K R Puram, Marathahalli forms the outer cycle vertices. The major junctions of Inner ring road like Domlur, Ejipura, Sony World Signal forms the inner cycle vertices. Depending on the number of major junctions the value of n can be varied. The minor junctions connecting the major junctions forms a path of vertices. Depending on the number of minor junctions the value of l can be varied. Each major junction of the outer ring road leads to two minor junction.

A modification of metric dimension, referred as local multiset dimension(lmd) defined by Alfarisi et al. [2] can be used to locate the minimum monitoring points in a network. This will help in identifying congestion points near any junction so that traffic can be diverted by adjusting the traffic signal timings depending on the traffic flow, which results in reduced travel time.

In a graph G, $d(u,v)$ represents the minimum length among all the path connecting the vertices $u,v \in V$. Eccentricity of any vertex v is the maximum of $d(u,v)$ for all $u \in V$. The central vertices of G are those with the least eccentricity. The subset M of V is a resolving set of G if all the vertices can be distinguished uniquely with respect to the ordered set M. A basis is a resolving set with the least cardinality, and its cardinality is known as the metric dimension, this was first revealed by Slater [3] while working on a real world problem on sonar and loran stations also Harry and Melter [4] independently introduced the same. The significance of resolvability lies mainly in solving problems that involve locating and distinguishing vertices in a graph. This aids in analysing and effectively determing the defects in the network, in monitoring and identifying traffic congestion effectively, identifies the location of functional groups which contributes in pharmacological research in characterizing compounds and advancing drug discovery. Extensive research have been carried out on metric dimension, its variations [5], [6] and applications.

Multiset dimension is one of the recently introduced variation by Simanjuntak, Siagian, and Vetrík [7] A subset $\mathcal{M}$ of V is a resolving multiset if all the vertices can be identified uniquely with respect to $\mathcal{M}$ in terms of a multiset and not as an ordered tuple. A multiset basis is a resolving multiset with the least cardinality, and its cardinality is known as the multiset dimension. This concept can be used in the situations where the order of the distance is irrelevant.

The subset $\mathcal{M}$ of V is a local resolving multiset, if all the adjacent vertices can be distinguished uniquely with respect to the set $\mathcal{M}$. A local multiset basis(lmsb) is a local resolving multiset with the least cardinality, and its cardinality is known as the local multiset dimension(lmd) of G. Let $l_m(u|\mathcal{M})$ denotes the multiset of vertex u with respect to $\mathcal{M}$, where $\mathcal{M} = \{m_1, m_2, m_3 \cdots m_k\} \subseteq V$. Then $l_m(u|\mathcal{M}) = \{d(m_1,u), d(m_2,u),\ d(m_3, u), \ldots d(m_k,u)\}$ where each $d(m_i,u)$ represents the shortest distance from the respective m_i to u. For every adjacent vertices $u,v \in V$, $l_m(u|\mathcal{M}) \neq l_m(v|\mathcal{M})$ then $\mathcal{M}$ is known as the lmsb and $\mu_{lm}(G)$ is the lmd of G. The lmd of various graphs like path, cycle, complete graph [8], unicyclic graph [9], graph with homogenous pendant edges [10] has been discussed by Alfarisi et al.

A graph family is said to have constant lmd if its value remains the same independent of the number of vertices and edges. One such family is path and star, whose lmd is one [8]. We refer the following results.

Theorem 1.1: [12] *A graph is bipartite if and only if all its cycles are even.*

Theorem 1.2: [13] *The lmd of G is one if and only if G is a bipartite graph.*

Lemma 1.3: [11] *Consider a graph G with lmsb $\mathcal{M} = \{m_1, m_2\}$, then $d(m_1, m_2) \equiv 0(mod\ 2)$.*

Theorem 1.4: [8] *For a cycle graph* C_n $(n \geq 3)$, $\mu_{lm}(C_n) = \begin{cases} 1, & if\ n\ is\ even. \\ 3, & if\ n\ is\ odd. \end{cases}$

Variety of metric dimension of different types of convex polytopes has been discussed [1], [14]. Section 1 presented the prior results and definitions utilized in our work. Section 2 contains preliminary results that support the main results of the paper. The main results are presented in Sections 3 and 4. Section 5 concludes the paper with a summary of the results and explores the applications of our results.

2. Results on the properties of the lmsb of G

In this section, lemmas on the properties of the lmsb of G are presented, which are useful in determning the lmd of the graphs considered.

Lemma 2.1: *Let* $\mathcal{M} = \{m_1, m_2\}$ *be an lmsb of* G*, Then the following holds.*

(1) *If* G *contains a subgraph* $H = P' \cup (K_4 - uv) \cup P''$*, such that* P' *and* P'' *are the shortest path between* m_1 *and* u*,* v *and* m_2 *respectively in* G*,* $d(m_1, m_2)$ *is even, for all* $x, y \in V(K_4) - \{u, v\}$*, there exist a vertex* $w \in V(P' \cup P'') - \{u, v\}$ *such that either* $w \sim x$ *or* $w \sim y$.

(2) *If* G *contains a connected subgraph* $H = P \cup C_{2k+1}$*, such that* P *is the shortest path between* m_1 *and* m_2*, then* $|V(C_{2k+1}) \cap V(P)| \neq 1$.

Proof:

(1) Let G contains a subgraph $H = P' \cup (K_4 - uv) \cup P''$. Assume that $x, y \in V(K_4) - \{u, v\}$ is not adjacent to any $w \in V(P' \cup P'') - \{u, v\}$, then the vertices x, y receive the same multiset, a contradiction to $\mathcal{M} = \{m_1, m_2\}$ being an lmsb of G.

(2) Let G contains a subgraph $H = P \cup C_{2k+1}$. Assume $V(C_{2k+1}) \cap V(P) = v$, then the two central vertices of the cycle C_{2k+1} receives the same multiset, a contradiction to $\mathcal{M} = \{m_1, m_2\}$ being an lmsb of G.

Lemma 2.2: *Let* G *be a graph and* $\mathcal{M} = \{m_1, m_2\} \subseteq V(G)$ *with* $d(m_1, m_2) \equiv 1(mod 2)$*, then* $\mathcal{M}$ *is not an lmsb of* G.

Proof: For an lmsb $\mathcal{M} = \{m_1, m_2\}$ with $d(m_1, m_2) \equiv 1(\text{mod } 2)$, there are two central vertices along the shortest $m_1 - m_2$ path of length l receives the same multiset $\{\frac{l-1}{2}, \frac{l+1}{2}\}$. Hence the proof

Lemma 2.3: *Let* G *be a graph,* $\mathcal{M} = \{m_1, m_2\} \subseteq V(G)$ *and* $d(m_1, m_2)$ *is even. If* G *contains an induced odd cycle* C_m *with a subpath* $x_1 - x_2 - x_3 - \ldots\ldots - x_{k+1}$ *in any shortest path between* m_1 *and* m_2 *such that* $(m-k) + |d(m_1, x_1) - d(x_{k+1}, m_2)| \equiv 1(mod\ 2)$ *, for all* $k \geq 2$*,* $m \geq 2k+1$ *and* $|d(m_1, x_1) - d(x_{k+1}, m_2)| \leq k - 2$*, then* $\mathcal{M}$ *is not an lmsb of* G.

Proof: Given that $\mathcal{M} = \{m_1, m_2\} \subseteq V(G)$ and $d(m_1, m_2)$ is even. Also given that, G contains an induced odd cycle C_m along a shortest $m_1 - m_2$ path P, such that there exist a subpath $P_k\colon x_1 - x_2 - x_3 - \cdots - x_{k+1}$ of length k, where $V(P_k) \subseteq V(C_m) \cap V(P)$. Let $P_q\colon y_1 - y_2 - y_3 - \cdots - y_{q-1}$ be a path, where $V(P_q) \subseteq V(C_m) - V(P)$ such that $x_1 \sim y_1$, $y_i \sim y_{i+1}$ and $y_{q-1} \sim x_{k+1}$ and q be the length of the path along P' : $x_1 - y_1 - y_2 - \cdots - y_{q-1} - x_{k+1}$. Let $d(m_1, x_1) = t$ and $d(x_{k+1}, m_2) = t + s$(no loss in assuming that $d(m_1, x_1) \leq d(x_{k+1}, m_2)$ else interchange m_1 and m_2).

To prove that if $(m - k) + |d(m_1, x_1) - d(x_{k+1}, m_2)| \equiv 1(\text{mod}2)$ and for all $k \geq 2$, $m \geq 2k + 1$, $|d(m_1, x_1) - d(x_{k+1}, m_2)| \leq k - 2$, then $\mathcal{M}$ is not an lmsb of G. Since $d(m_1, m_2) = 2t + k + s$ is even, this implies that $k + s$ is even. Depending on the value of k, two cases are considered.

Case 1: $k \equiv 1(\text{mod } 2)$

This implies that s is also odd. Since C_m is an induced odd cycle, q is even. Therefore $q + s$ is odd.

Case 2: $k \equiv 0(\text{mod}2)$

This implies that s is also even. Since C_m is an induced odd cycle, q is odd. Therefore $q + s$ is odd.

From both the cases, $q + s$ is odd, this implies that $2t + q + s$ is also odd. This forms an odd path $m_1 - \cdots - x_1 - y_1 - \cdots - y_{q-1} - x_{k+1} - \cdots - m_2$ connecting m_1 and m_2, only when $(m - k) + |d(m_1, x_1) - d(x_{k+1}, m_2)| \equiv 1(\text{mod } 2)$.

To prove the conditions for $\mathcal{M} = \{m_1, m_2\}$ is not an lmsb of G.

Now the condition $k \geq 2$, is proved by contradiction.

From Lemma 2.1 $k \neq 0$. Thus assume $k = 1$. Let $v_1, v_2 \ldots\ldots v_m$ be the vertices of an induced odd cycle C_m.

$d(v_1, v_i) = d(v_1, v_j) = i$, for all $2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil$ and $j = m + 2 - i$.

$d(v_m, v_i) = d(v_m, v_j) = i - 1$, for all $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $j = m - i$.

Since P is a path of even length, no two adjacent vertices along the path receives the same multiset.

The multiset representation of $V(C_m)$ is given by.

$$r_m(v_i|\mathcal{M}) = \begin{cases} \{t, t + s + 1,\} & i = 1 \\ \{t + 1, t + s\} & i = m \\ \{t + d(v_1, v_i), t + s + d(v_m, v_i)\}, & 2 \leq i \leq m - 1 \end{cases}$$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset. Therefore $k \geq 2$.

Now the condition $q \geq k + 1$, is proved by contradiction. Depending on the variation of q with respect to k, two cases are considered.

Case 1: $q < k$

This is a contradiction to P being the shortest path between m_1 and m_2.

Case 2: If $q = k$

This is a contradiction to C_m being an induced odd cycle C_m.

Therefore $q \geq k+1$, this implies that $m-k \geq k+1$. Therefore $m \geq 2k+1$.

Now the condition $s \leq k-2$, is proved by contradiction.

Since P is a path of even length, no two adjacent vertices along the path receives the same multiset. If $s \geq k-1$, then the two central vertices of the odd path $m_1 - \cdots - x_1 - y_1 - \cdots - y_{q-1} - x_{k+1} - \cdots - m_2$ lies along the path $x_{k+1} - m_2 \subseteq V(P)$ and the adjacent vertices do not receive the same multiset, a contradiction to $s \geq k-1$. Hence $s \leq k-2$.

Therefore if $(m-k) + |d(m_1, x_1) - d(x_{k+1}, m_2)| \equiv 1(\text{mod}2)$ and for all $k \geq 2$, $m \geq 2k+1$, $s \leq k-2$, the two central vertices along the path $m_1 - \cdots - x_1 - y_1 - \cdots - y_{q-1} - x_{k+1} - \cdots - m_2$ receives the same multiset $\{\frac{2t+s+q+1}{2}, \frac{2t+s+q-1}{2}\}$.

Therefore $\mathcal{M} = \{m_1, m_2\}$ is not an lmsb of G.

3. Local multiset dimension of a N-gonal circular ladder $N_{n,l}$

Theorem 3.1: *For any even integer* $n \geq 4$ *and* $l > 0$, $\mu_{lm}(N_{n,l}) = 2$.

Proof: $N_{n,l}$ contains an odd cycle C_{2l+3}, hence from Theorem 1.2 and Theorem 1.1 $\mu_{lm}(N_{n,l}) \geq 2$. However for the upper bound, lmsb $\mathcal{M}$ is considered based on the value of n and justify that there exist an unique multiset for every adjacent vertices.

Case 1: $\frac{n}{2}$ is odd.

Let $\mathcal{M} = \{u_{0,0}, u_{\frac{n}{2},1}\} \subseteq V(N_{n,l})$, for all $u_{i,0}$, $l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i\}$, where ξ_i and η_i is the multiset generated due to the basis vertices $u_{0,0}$ and $u_{\frac{n}{2},1}$ respectively and are given by,

$$l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i\} = \begin{cases} \{i,\ (\frac{n}{2} - i + 1)\} & \text{for } 0 \leq i \leq \frac{n}{2} - 1 \\ \{(n-i), (i - \frac{n}{2} + 1)\}, & \text{for } \frac{n}{2} \leq i \leq n-1 \end{cases}$$

For all $u_{i,j}$,

$$l_m(u_{i,j}|\mathcal{M}) = \begin{cases} \{\xi_i + j, \eta_i + j - 2\}, & \text{for } i = \frac{n}{2} \\ \{\xi_i + j, \eta_i + j\}, & \text{otherwise} \end{cases}$$

$$l_m(u_{i,l}|\mathcal{M}) = \begin{cases} \{\xi_i + l, \eta_i + l - 1\}, & \text{for } i = \frac{n}{2} - 1, \frac{n}{2} + 1 \\ \{\xi_i + l, \eta_i + l - 2\}, & \text{for } i = \frac{n}{2} \\ \{\xi_i + l, \eta_i + l\}, & \text{otherwise} \end{cases}$$

For all v_i,

$$l_m(v_i|\mathcal{M}) = \begin{cases} \{\xi_i + l + 1, \eta_i + l\}, & \text{for } 0 \le i \le \frac{n}{2} - 3 \\ \{\xi_i + l + 1, \eta_i + l - 1\}, & \text{for } i = \frac{n}{2} - 2 \\ \{\xi_i + l + 1, \eta_i + l - 2\}, & \text{for } i = \frac{n}{2} - 1 \\ \{\xi_i + l, \eta_i + l - 1\}, & \text{for } i = \frac{n}{2} \\ \{\xi_i + l, \eta_i + l\}, & \text{for } i = \frac{n}{2} + 1 \\ \{\xi_i + l, \eta_i + l + 1\}, & \text{for } \frac{n}{2} + 2 \le i \le n - 1 \end{cases}$$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset. Therefore $\mu_{lm}(N_{n,l}) \le 2$.

Case 2: $\frac{n}{2}$ is even.

Let $\mathcal{M} = \{u_{0,0}, u_{\frac{n}{2},0}\} \subseteq V(N_{n,l})$, for all $u_{i,0}$, $l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i\}$, where ξ_i and η_i is the multiset generated due to the basis vertices $u_{0,0}$ and $u_{\frac{n}{2},0}$ respectively and are given by,

$$l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i\} = \begin{cases} \left\{i, \left(\frac{n}{2} - i\right)\right\} & \text{for } 0 \le i \le \frac{n}{2} - 1 \\ \left\{(n - i), \left(i - \frac{n}{2}\right)\right\}, & \text{for } \frac{n}{2} \le i \le n - 1 \end{cases}$$

For all $u_{i,j}$,

$l_m(u_{(i,j)}|\mathcal{M}) = \{\xi_i + j,\ \eta_i + j\}$, for $0 \le i \le n - 1, 1 \le j \le l$.

For all v_i,

$$l_m(v_i|\mathcal{M}) = \begin{cases} \{\xi_i + l + 1, \eta_i + l\}, & \text{for } 0 \le i \le \frac{n}{2} - 1 \\ \{\xi_i + l, \eta_i + l + 1\}, & \text{for } \frac{n}{2} \le i \le n - 1 \end{cases}$$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset. Therefore $\mu_{lm}(N_{n,l}) \le 2$.

From both the cases, we have $\mu_{lm}(N_{n,l}) = 2$.

Corollary 3.2: *For any even integer* $n \ge 4$, $\mu_{lm}(H_n) = 2$.

Lemma 3.3: *For any odd integer* $n \ge 3$ *and* $l > 0$, $\mathcal{M} = \{m_1, m_2\} \subseteq V(N_{n,l})$ *is not an lmsb of* $N_{n,l}$ *when* $d(m_1, m_2)$ *is even.*

Proof: Assume to the contradiction that $\mathcal{M} = \{m_1, m_2\} \subseteq V(N_{n,l})$ is an lmsb of $N_{n,l}$. Depending on the positions of m_1 and m_2 the values of t, s, m and k as defined in Lemma 2.3 vary as shown in the following cases. Consider $d(m_1, m_2)$ is even in all the below considered cases.

Case 1: m_1 and m_2 lies on the inner cycle.
Here $t = 0,\ s = 0,\ m = n,\ k = d(m_1, m_2),\ q + s$ is odd.

Case 2: m_1 and m_2 lies on the same C_{2l+3} cycle.
Here $t = 0,\ s = 0,\ m = 2l + 3,\ k = d(m_1, m_2),\ q + s$ is odd.

Case 3: m_1 lies on inner cycle and m_2 lies on any C_{2l+3} cycle. Depending on the position of the vertex m_2, two subcases are considered.

Subcase 1: $m_1 = u_{0,0}$ and $m_2 = u_{i,j} (j \geq 1)$.

$d(m_1, m_2)$	Inference
$\leq 2j,\ 2 \leq j \leq l$	$t = 0,\ s = d(m_1, m_2) - k,\ m = 2l + 3,\ k = j + 1,\ q + s$ is odd.
$\geq 2j + 2,\ 1 \leq j \leq l$	$t = 0,\ s = j,\ m = n,\ k = d(m_1, m_2) - j,\ q + s$ is odd.

Subcase 2: $m_1 = u_{0,0}$ and $m_2 = v_i$.

$d(m_1, m_2)$	Inference
$\leq 2l$	$t = 0,\ s = d(m_1, m_2) - k,\ m = 2l + 3,\ k = l + 1,\ q + s$ is odd.
$\geq 2l + 2$	$t = 1,\ s = l - 1,\ m = 2l + 3(d(m_1, m_2)) - 27,\ k = d(m_1, m_2) - l - 1,\ q + s$ is odd.

Case 4: m_1 and m_2 lies on different C_{2l+3} cycle. Depending on the position of the vertices m_1 and m_2, three subcases are considered.

Subcase 1: $m_1 = u_{0,j}$ and $m_2 = u_{(i,j)}$.
here $t = j,\ s = 0,\ m = n,\ k = d(m_1, m_2) - 2j,\ q + s$ is odd.

Subcase 2: $m_1 = u_{0,j_1}$ and $m_2 = u_{(i,j_2)}$ (assume $j_1 < j_2$).

$d(m_1, m_2)$	Inference
$\leq 2j_2,\ 1 \leq j_1 < j_2 \leq l$	$t = 0,\ s = d(m_1, m_2) - k,\ m = 2l + 3,\ k = j_2 + 1,\ q + s$ is odd.
$\geq 2j_2 + 2,\ 1 \leq j_1 < j_2 \leq l$	$t = j_1,\ s = j_2 - j_1,\ m = n,\ k = d(m_1, m_2) - j_1 - j_2,\ q + s$ is odd.

Subcase 3: $m_1 = u_{0,j}$ and $m_2 = v_i$.

$d(m_1, m_2)$	Inference
$\leq 2l$	$t = 0,\ s = d(m_1, m_2) - k,\ m = 2l + 3,\ k = l + 1,\ q + s$ is odd.
$\geq 2l + 2$	$t = 1,\ s = l - 1,\ m = 2l + 3(d(m_1, m_2)) - 27,\ k = d(m_1, m_2) - l - 1,\ q + s$ is odd.

In all the above cases, $k \geq 2$, $m \geq 2k+1$, $s \leq k-2$ and $q+s$ is odd. Therefore from Lemma 2.3, $\mathcal{M} = \{m_1, m_2\}$ is not an lmsb of $N_{n,l}$.

Theorem 3.4: *For any odd integer* $n \geq 3$ *and* $l > 0$, $\mu_{lm}(N_{n,l}) = 3$.

Proof: $N_{n,l}$ contains an odd cycle C_{2l+3}, hence from the Theorem 1.2 and Theorem 1.1, $\mu_{lm}(N_{n,l}) \geq 2$. Suppose that $\mu_{lm}(N_{n,l}) = 2$, from Lemma 2.2, $d(m_1, m_2)$ can not be odd and from Lemma 3.3 $d(m_1, m_2)$ can not be even, producing a contradiction for all possible pairs of vertices $\mathcal{M} = \{m_1, m_2\} \subseteq V(N_{n,l})$ as lmsb. Hence $\mu_{lm}(N_{n,l}) \geq 3$.

However for the upper bound, lmsb $\mathcal{M}$ is considered based on the value of n and justify that there exist an unique multiset for every adjacent vertices.

Case 1: $n = 3$

Based on l values, there are two cases.

Subcase 1: $l = 1,2$

Let $\mathcal{M} = \{u_{0,1}, u_{1,1}, v_1\} \subseteq V(N_{3,l})$, multiset for all the vertices are mentioned in Tables 1-2.

Table 1

Multiset of $V(N_{3,1})$

i	0	1	2
$l_m(u_{i,0}\|\mathcal{M})$	{1,2,3}	{2,1,2}	{2,2,2}
$l_m(u_{i,1}\|\mathcal{M})$	{0,2,3}	{2,0,1}	{2,2,1}
$l_m(v_i\|\mathcal{M})$	{1,1,2}	{3,1,0}	{1,3,2}

Table 2

Multiset of $V(N_{3,2})$

i	0	1	2
$l_m(u_{i,0}\|\mathcal{M})$	{1,2,4}	{2,1,3}	{2,2,3}
$l_m(u_{i,1}\|\mathcal{M})$	{0,3,4}	{3,0,2}	{3,3,2}
$l_m(u_{i,2}\|\mathcal{M})$	{1,3,3}	{3,1,1}	{3,3,1}
$l_m(v_i\|\mathcal{M})$	{2,2,2}	{4,2,0}	{2,4,2}

Subcase 2: $l \geq 3$

Let $\mathcal{M} = \{u_{0,1}, u_{1,1}, u_{1,2}\} \subseteq V(N_{3,l})$, for all $u_{i,0}$ is given by $l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i, \zeta_i\}$, where ξ_i, η_i and ζ_i is the multiset generated due to the basis vertices $u_{0,1}$, $u_{1,1}$ and $u_{1,2}$ respectively. The multiset for all $u_{i,j}$ and v_i are mentioned in the Tables 3-6.

Table 3

Multiset for all $u_{i,0}$

i	$l_m(u_{i,0}\|\mathcal{M})$
0	{1,2,3}
1	{2,1,2}
2	{2,2,3}

Table 4

Multiset for all $u_{i,j}(j \geq 1)$

$l_m(u_{0,j}\|\mathcal{M})$	$\{\xi_0 + j - 2, \eta_0 + j, \zeta_0 + j\}$	$1 \leq j \leq l - 2$
$l_m(u_{0,l-1}\|\mathcal{M})$	$\{\xi_0 + l - 3, \eta_0 + l - 1, \zeta_0 + l - 2\}$	
$l_m(u_{0,l}\|\mathcal{M})$	$\{\xi_0 + l - 2, \eta_0 + l - 1, \zeta_0 + l - 3\}$	
$l_m(u_{1,1}\|\mathcal{M})$	$\{\xi_1 + 1, \eta_1 - 1, \zeta_1 - 1\}$	
$l_m(u_{1,j}\|\mathcal{M})$	$\{\xi_1 + j, \eta_1 + j - 2, \zeta_1 + j - 4\}$	$2 \leq j \leq l - 1$
$l_m(u_{1,l}\|\mathcal{M})$	$\{\xi_1 + l - 1, \eta_1 + l - 2, \zeta_1 + l - 4\}$	
$l_m(u_{2,j}\|\mathcal{M})$	$\{\xi_2 + j, \eta_2 + j, \zeta_2 + j\}$	$1 \leq j \leq l - 2$
$l_m(u_{2,l-1}\|\mathcal{M})$	$\{\xi_2 + l - 1, \eta_2 + l - 1, \zeta_2 + l - 2\}$	
$l_m(u_{2,l}\|\mathcal{M})$	$\{\xi_2 + l - 1, \eta_2 + l - 1, \zeta_2 + l - 3\}$	

Table 5

Multiset for all $v_i(l = 3)$

i	$l_m(v_i\|\mathcal{M})$
0	{3,3,2}
1	{5,3,2}
2	{3,5,4}

Table 6

Multiset for all $v_i(l \geq 4)$

$l_m(v_0\|\mathcal{M})$	$\{\xi_i + l - 1, \eta_i + l - 2, \zeta_i + l - 4\}$
$l_m(v_1\|\mathcal{M})$	$\{\xi_i + l, \eta_i + l - 1, \zeta_i + l - 3\}$
$l_m(v_2\|\mathcal{M})$	$\{\xi_i + l - 2, \eta_i + l, \zeta_i + l - 2\}$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset. Therefore $\mu_{lm}(N_{3,l}) \leq 3$.

Case 2: $n \geq 5$

Let $\mathcal{M} = \{u_{0,1}, u_{\frac{n-1}{2},0}, u_{0,0}\} \subseteq V(N_{n,l})$, for all $u_{i,0}$ is given by $l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i, \zeta_i\}$, where ξ_i, η_i and ζ_i is the multiset generated due to the basis vertices $u_{0,1}$, $u_{\frac{n-1}{2},0}$ and $u_{0,0}$ respectively.

$$l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i, \zeta_i\} = \begin{cases} \left\{(i+1), \left(\frac{n-1}{2} - i\right), i\right\}, & \text{for } 0 \le i \le \frac{n-1}{2} \\ \left\{(n-i+1), \left(i - \frac{n-1}{2}\right), (n-i)\right\}, & \text{otherwise} \end{cases}$$

For all $u_{i,j}$,

$$l_m(u_{i,j}|\mathcal{M}) = \begin{cases} \{\xi_0 + j - 2, \eta_0 + j, \zeta_0 + j\}, & \text{for } i = 0 \\ \{\xi_i + j, \eta_i + j, \zeta_i + j\}, & \text{for } 1 \le i \le n-1 \end{cases}$$

$$l_m(u_{i,l}|\mathcal{M}) = \begin{cases} \{\xi_i + l - 2, \eta_i + l, \zeta_i + l\}, & \text{for } i = 0 \\ \{\xi_i + l - 1, \eta_i + l, \zeta_i + l\}, & \text{for } i = 1, n-1 \\ \{\xi_i + l, \eta_i + l, \zeta_i + l\}, & \text{otherwise} \end{cases}$$

For all v_i,

$$l_m(v_i|\mathcal{M}) = \begin{cases} \{\xi_i + l - 1, \eta_i + l, \zeta_i + l + 1\}, & \text{for } i = 0 \\ \{\xi_i + l, \eta_i + l, \zeta_i + l + 1\}, & \text{for } i = 1 \\ \{\xi_i + l + 1, \eta_i + l, \zeta_i + l + 1\}, & \text{for } 2 \le i \le \frac{n-1}{2} - 1 \text{ and } n \ne 5 \\ \{\xi_i + l + 1, \eta_i + l + 1, \zeta_i + l + 1\}, & \text{for } i = \frac{n-1}{2} \\ \{\xi_i + l, \eta_i + l + 1, \zeta_i + l\}, & \text{for } \frac{n+1}{2} \le i \le n-3 \text{ and } n \ne 5 \\ \{\xi_i + l - 1, \eta_i + l + 1, \zeta_i + l\}, & \text{for } i = n-2 \\ \{\xi_i + l - 2, \eta_i + l + 1, \zeta_i + l\}, & \text{for } i = n-1 \end{cases}$$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset. Therefore $\mu_{lm}(N_{n,l}) \le 3$.

From both the cases, we have $\mu_{lm}(N_{n,l}) = 3$.

Corollary 3.5: *For any odd integer $n \ge 3$, $\mu_{lm}(H_n) = 3$.*

4. Local multiset dimension of a modified N-gonal circular ladder $\mathbf{N'_{n,l}}$

The modified N-gonal circular ladder is obtained by adding extra l edges to $N_{n,l}$. $N'_{n,l}$ is obtained by adding l edges between the vertices $u_{i,j}$, $u_{i+1,j}$ and $u_{0,j}$, $u_{n-1,j}$ in $N_{n,l}$. The vertex set remains the same but the edge set is increased by l, therefore $|V(N'_{n,l})| = (l+2)n$ and $|E(N'_{n,l})| = (2l+3)n$. It consists of $n\ C_3$ and $ln\ C_4$ connected with each other. As an extension of the result on the local multiset dimension of the N-gonal circular ladder, the lmd of the modified N-gonal circular ladder is determined in this section. It is also shown that the same lmsb as $N_{n,l}$ can locally resolve all the vertices of $N'_{n,l}$.

Corollary 4.1: *For any even integer $n \ge 4$ and $l > 0$, $\mu_{lm}(N'_{n,l}) = 2$.*

Proof: $N'_{n,l}$ is a modified $N_{n,l}$, hence $\mu_{lm}(N'_{n,l}) \ge 2$. However for the upper bound, lmsb $\mathcal{M}$ is considered based on the value of n and justify that there exist an unique multiset for every adjacent vertices.

Case 1: $\frac{n}{2}$ is odd.

Let $\mathcal{M} = \{u_{0,0}, u_{\frac{n}{2},1}\} \subseteq V(N'_{n,l})$, for all $u_{i,0}$ is given by $l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i\}$, where ξ_i and η_i is the multiset generated due to the basis vertices $u_{0,0}$ and $u_{\frac{n}{2},1}$ respectively. The multiset for all $u_{i,0}$ is same as $N_{n,l}$.

For all $u_{i,j}$,

$$l_m(u_{i,j}|\mathcal{M}) = \{\xi_i + j,\ \eta_i + j - 2\},\ \text{for } 0 \leq i \leq n-1.$$

For all v_i,

$$l_m(v_i|\mathcal{M}) = \begin{cases} \{\xi_i + l + 1, \eta_i + l - 2\}, & \text{for } 0 \leq i \leq \frac{n}{2} - 1 \\ \{\xi_i + l, \eta_i + l - 1\}, & \text{otherwise} \end{cases}$$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset. Therefore $\mu_{lm}(N_{n,l}{}') = 2$.

Case 2: $\frac{n}{2}$ is even.

Let $\mathcal{M} = \{u_{0,0}, u_{\frac{n}{2},0}\} \subseteq V(N'_{n,l})$. Since the multiset remains the same as the multiset of $N_{n,l}$ and also $l_m(u_{i,j}|\mathcal{M}) \neq l_m(u_{i+1,j}|\mathcal{M})$, $l_m(u_{0,j}|\mathcal{M}) \neq l_m(u_{n-1,j}|\mathcal{M})$. So no two adjacent vertices receives the same multiset. Therefore $\mu_{lm}(N_{n,l}{}') = 2$.

From both the cases, we have $\mu_{lm}(N'_{n,l}) = 2$.

Corollary 4.2: *For any even integer* $n \geq 4$, $\mu_{lm}(H'_n) = 2$.

Corollary 4.3: *For any odd integer* $n \geq 3$ *and* $l > 0$, $\mu_{lm}(N'_{n,l}) = 3$.

Proof: $N'_{n,l}$ is a modified $N_{n,l}$, hence $\mu_{lm}(N'_{n,l}) \geq 3$. However for the upper bound, lmsb $\mathcal{M}$ is considered based on the value of n and justify that there exist an unique multiset for every adjacent vertices.

Case 1: $n = 3$

Based on l values, there are two cases.

Subcase 1: $l = 1,2$

Let $\mathcal{M} = \{u_{0,1}, u_{1,1}, v_1\} \subseteq V(N'_{3,l})$, for all $u_{i,0}$ is same as $N_{3,l}$. The multiset of all the other vertices are mentioned in Tables 7-8

Table 7

Multiset of $V(N_{3,1})$

i	0	1	2
$l_m(u_{i,1}\|\mathcal{M})$	{0,1,2}	{1,0,1}	{1,1,1}
$l_m(v_i\|\mathcal{M})$	{1,1,2}	{2,1,0}	{1,2,2}

Table 8

Multiset of $V(N_{3,2})$

i	0	1	2
$l_m(u_{i,1}\|\mathcal{M})$	{0,1,3}	{1,0,2}	{1,1,2}
$l_m(u_{i,2}\|\mathcal{M})$	{1,2,2}	{2,1,1}	{2,2,1}
$l_m(v_i\|\mathcal{M})$	{2,2,2}	{3,2,0}	{2,3,2}

Subcase 2: $l \geq 3$

Let $\mathcal{M} = \{u_{0,1}, u_{1,1}, u_{1,2}\} \subseteq V(N'_{3,l})$, for all $u_{i,0}$ is given by $l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i, \zeta_i\}$, where ξ_i, η_i and ζ_i is the multiset generated due to the basis vertices $u_{0,1}$, $u_{1,1}$ and $u_{1,2}$ respectively. The multiset for all $u_{i,0}$ is same as $N_{n,l}$. The multiset for all $u_{i,j}(j \geq 1)$ and v_i are mentioned in the Tables 9-10.

Table 9

Multiset for all $u_{i,j}(j \geq 1)$

$l_m(u_{i,1}\|\mathcal{M})$	$\{\xi_i - 1, \eta_i - 1, \zeta_i - 1\}$	for $0 \leq i \leq n-1$
$l_m(u_{i,j}\|\mathcal{M})$	$\{\xi_i + j - 2, \eta_i + j - 2, \zeta_i + j - 4\}$	for $2 \leq j \leq l, 0 \leq i \leq n-1$

Table 10

Multiset for all v_i of $V(N_{3,l})$

$l_m(v_0\|\mathcal{M})$	$\{\xi_i + l - 1, \eta_i + l - 2, \zeta_i + l - 4\}$
$l_m(v_1\|\mathcal{M})$	$\{\xi_i + l - 1, \eta_i + l - 1, \zeta_i + l - 3\}$
$l_m(v_2\|\mathcal{M})$	$\{\xi_i + l - 2, \eta_i + l - 1, \zeta_i + l - 3\}$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset.

Case 2: $n \geq 5$

Let $\mathcal{M} = \{u_{0,1}, u_{\frac{n-1}{2},0}, u_{0,0}\} \subseteq V(N'_{n,l})$, for all $u_{i,0}$ is given by $l_m(u_{i,0}|\mathcal{M}) = \{\xi_i, \eta_i, \zeta_i\}$, where ξ_i, η_i and ζ_i is the multiset generated due to the basis vertices $u_{0,1}$, $u_{\frac{n-1}{2},0}$ and $u_{0,0}$ respectively. The multiset for all $u_{i,0}$ is same as $N_{n,l}$.

For all $u_{i,j}$,

$$l_m(u_{i,j}|\mathcal{M}) = \{\{\xi_i + j - 2,\ \eta_i + j,\ \zeta_i + j\},\ \text{for } 0 \leq i \leq n-1, 1 \leq j \leq l.$$

For all v_i,

$$l_m(v_i|\mathcal{M}) = \begin{cases} \{\xi_i + l - 1, \eta_i + l, \zeta_i + l + 1\}, & \text{for } 0 \leq i \leq \frac{n-1}{2} - 1 \\ \{\xi_i + l - 1, \eta_i + l + 1, \zeta_i + l + 1\}, & \text{for } i = \frac{n-1}{2} \\ \{\xi_i + l - 2, \eta_i + l + 1, \zeta_i + l\}, & \text{otherwise} \end{cases}$$

From the above multisets, it is observed that no two adjacent vertices receives the same multiset. Therefore $\mu_{lm}(N'_{n,l}) = 3$.

From both the cases, we have $\mu_{lm}(N'_{n,l}) = 3$.

Corollary 4.4: *For any odd integer* $n \geq 3$, $\mu_{(lm)}(H'_n) = 3$.

5. Conclusion

In this paper, family of convex polytopes having constant local multiset dimension is computed. It is observed that two or three vertices are sufficient to distinguish all the adjacent vertices of this family of convex polytopes. It is noted that the same lmsb locally resolves the graph and the modified graph with extra edges. Introducing new connections, such as road networks or public transport, while using the same lmsb for monitoring transport systems, results in improved connectivity and reduced congestion. The lmsb conidered, is scalable for future expansion. This concept can be extended to a broad class of other networks.

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