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Revisiting BIBDs : 3-frameproof and 2-traceable properties for large block sizes

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Abstract

Stinson and Wei [6] have extensively studied the construction of c-frame proof codes using Balanced Incomplete Block Designs (BIBDs), offering a powerful Combinatorial Approach to Digital Fingerprinting and Traitor Tracing. Their work identifies specific combinatorial properties of BIBDs that ensure frameproof properties of BIBDs, thereby preventing coalitions of users from framing innocent parties. In this study, we revisit these constructions and reinterpret the conditions outlined by Stinson and Wei within the framework of traceable codes-specifically, 2-traceable codes and 3-frameproof codes for the higher values of k i.e. block size in case of (v, b, r, k, λ)-BIBD.

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1. Introduction

In the realm of Combinatorics and Coding Theory, *Design Theory* and *Frameproof Codes* are two significant areas that intersect in applications related to Data Security, Information Distribution, and Digital Fingerprinting. Design Theory is a branch of Combinatorics that focuses on the construction and analysis of combinatorial structures known as *designs*. These structures aim to ensure that subsets of elements (blocks) follow specific balance and symmetry properties. Classic examples include Balanced incomplete block designs (BIBDs),t-designs, and Steiner Systems. The main objective of design theory is often to distribute elements in a way that ensures even coverage and prevents over representation, which is vital in experiment planning, Error-Correcting Codes, and Cryptographic systems. On the other hand, Frameproof Codes are specialized codes used primarily in digital fingerprinting and traitor tracing.

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These codes are designed to prevent unauthorized users (or *pirates*) from framing an innocent user by colluding and combining their codewords. A code is *frameproof* if no coalition of users can generate a new codeword that implicates another user who is not part of the coalition. Frameproof codes thus play a crucial role in protecting copyrighted content and securing digital transmissions against collusion attacks. The intersection of design theory and frameproof codes is a novel area for research. The structured and balanced properties of combinatorial designs often lend themselves well to the construction of Frameproof Codes. By leveraging the principles of design theory, Researchers can construct Frameproof Codes with desirable properties such as minimal code length, optimal size, and high resistance to collusion. This paper explores the foundational aspects of both fields, delves into their interplay, and presents constructions and bounds that enhance our understanding of secure code Design. The synergy between the Mathematical elegance of design theory and the practical importance of frameproof codes highlights the relevance of Combinatorial Techniques in Modern Information Protection.

Our next section explores key findings in Field Theory and Design Theory, which will be crucial for our results. Combinatorial Design Theory, has practical applications, notably in designing experiments.

1.1 *Literature Review*

Boneh and Shaw [12] pioneered the study of frameproof codes in 1994. An extensive study of frameproof codes is discussed in [11]. Application of BIBDs as in terms of frameproof codes were defined by Stinson and Wei in [6]. A new class of 2-frameproof codes has been suggested in [2] by Kathuria, Batra and Arora. The concept of traceable codes was established in [8] and new improved conditions of traceability in terms of equidistant codes were studied in [1]. Different types of Combinatorial Structures/Designs like BIBDs can also be utilised as 2-TA Codes, have been discussed in [5]. These codes provide high security against collusion attacks making them suitable for protecting copyrighted content. Optimized Traceable Codes using Lexicographic Codes and Projective Planes have been studied in [3, 4], while an extensive exploration of Traceable Codes is presented in [10]." A review of all types of Frameproof Codes and traceable codes have been discussed in [14].

2. Basic Definitions of Design Theory;

2.1. *Design*

A pair (T, A) consisting of two parameters i.e.;

1. T: A set of elements is called points.
2. A: A collection of non-empty subsets of T is called blocks.

Each and every block in T contains a specific subset of points from A. The study of Balanced Incomplete Block Designs was begun in 1930 by Fisher and Yates [13]. These structures have many practical Applications in Statistics, Coding Theory, Cryptography and many more..

2.1. Balanced Incomplete Block Design [7]: A Balanced Incomplete Design (BIBD) is a design (T, A) with parameters (t, b, k, λ) satisfying:

(i) $|T| = t$, the set T has t points.

(ii) $|B| = k$;Each block B in A contains exactly $k$points an

(iii) Every pair of distinct points is contained in exactly λ blocks.

A BIBD is called an Incomplete as not all points appear in every block.

Example: An example of (9, 3, 1)-BIBD is;

T = {1, 2, 3, 4, 5, 6, 7, 8, 9}, and

A= {123, 456, 789, 147, 258, 169, 159, 267, 348, 168, 249, 357} ……(1)

2.1.1. Theorem [7]: In a (t, b, k, λ)-BIBD, every point occurs in exactly m blocks, where:

$$m = \frac{\lambda(t-1)}{(k-1)}$$

2.1.2. Theorem [7]: *A (t, b, k)-BIBD has exactly*

$$l = \frac{t.m}{b} = k\left(\frac{t^2-t}{b^2-b}\right) \quad \text{blocks.}$$

2.1.3. Corollary [7]: *In a (t, b, k, λ)-BIBD there exists the following relationships*

$$k(t-1) \equiv 0(mod\ b-1)$$

and $kt(t-1) \equiv 0(modb(b-1))$ holds.

Solution: As each point appears in r blocks. Therefore, it implies that $kt(t-1)$ must $be\ divisible\ by\ b(b-1)$. For example, a particular type of (4, 6, 2, 1)-BIBD does not exist because $kt(t-1)$=24 is not congruent to $4(mod\ 30)$, which is not a multiple of 30.Therefore it completes the proof .

In the next steps we dive into the basic definitions of Coding Theory. Coding Theory is indeed a fascinating field of Mathematics and Computer Science that deals with designing error-correcting codes for reliable data transmission and data storage.

2.1.4.

1. Code [9]**:** A set of codewords $\{x_1, x_2, x_3, \ldots x_n\}$can be used for transmitting messages. Each message is then mapped to a codeword. A finite set of symbols (e.g. binary alphabet: {0, 1}) as well as nonzero symbols can also be used to construct codewords.
2. Hamming Distance: For any two codewords l and m, distance is the no. of those nonzero coordinates where the codewords differ. For Example;

   If $l = (5,7,8)$ and $m = (3,4,8)$, Then $d(l, m) = 2$. A Code C with parameters (l, N, d); indeed has :

(a) Length (l): Every codeword in the code has a fixed length of l symbols.

(b) Number of codewords (N): The code consists of N codewords.

(c) Minimum Hamming Distance (d):The minimum Hamming distance between any two distinct codewords in the code is d.

3. Collusion Attack: The Problem of copying the digital Data has become the big and challenging problem of the world. if any two dishonest users can indeed collaborate, match up to their copies and craft a new codeword by exploiting the difference between their marks, and it could lead to framing of an innocent user. So, we represent a general fingerprinting scheme which is secure in the context of the collusion.

4. Frameproof Code: Frameproof codes are designed to prevent a group of users from creating a new codeword that could frame another users. In [12], the sufficient condition provides a way to construct codes that are resistant to framing attacks. This work has really been influential in the field of Digital Fingerprinting and Traitor Tracing.

2.1.5. Theorem [12]: A code T, formed by replacing each symbol of a c-frameproof code M with a codeword from an error-correcting code N, is itself c-frameproof if the minimum distance d of N satisfies the condition: d> (1 - 1/c)l, where l is the length of the codewords in N and c is a positive integer greater than 1." i.e. T is a c −frameproof code if $d > (1 - \frac{1}{c})l$; $c = 2,3,4$ ...

Message Words	Codewords				
(a, b)	a	$a\alpha + b$	$a\alpha + b\alpha$	$a + b\alpha$	b
(0,0)	0	0	0	0	0
(0,1)	0	1	α	α	1
$(0, \alpha)$	0	α	α^2	α^2	α
$(0, \alpha^2)$	0	α^2	1	1	α^2
(1,0)	1	α	α	1	0
(1,1)	1	α^2	0	α^2	1
$(1, \alpha)$	1	0	1	α	α
$(1,\alpha^2)$	1	1	α^2	0	α^2
$(\alpha, 0)$	α	α^2	α^2	α	0
$(\alpha, 1)$	α	α	1	0	1
(α, α)	α	1	0	1	α
(α, α^2)	α	0	α	α^2	α^2
$(\alpha^2, 0)$	α^2	1	1	α^2	0
$(\alpha^2, 1)$	α^2	0	α^2	1	1
(α^2, α)	α^2	α^2	α	0	α
(α^2, α^2)	α^2	α	0	α	α^2

In this theorem, N is a code defined over the set S_q={1,2,3,4……q}of q symbols. Code M is a c-frameproof code defined over a binary field (smaller size). The composition T is formed by replacing each symbol of N with a

codeword from M and the goal of theorem is to increase the numeral of codewords in T (the resulting c-frameproof code) by increasing the length of the code. Here we give an example of 2-frameproof code. Since for this problem length of the codeword is $n = 5$ and distance $d = 4$.

Therefore $d > (1 - \frac{1}{3})n$, i.e. the condition simplifies to $d > \frac{2n}{3}$ .it clearly confirms that the code C is 3-FP Code.

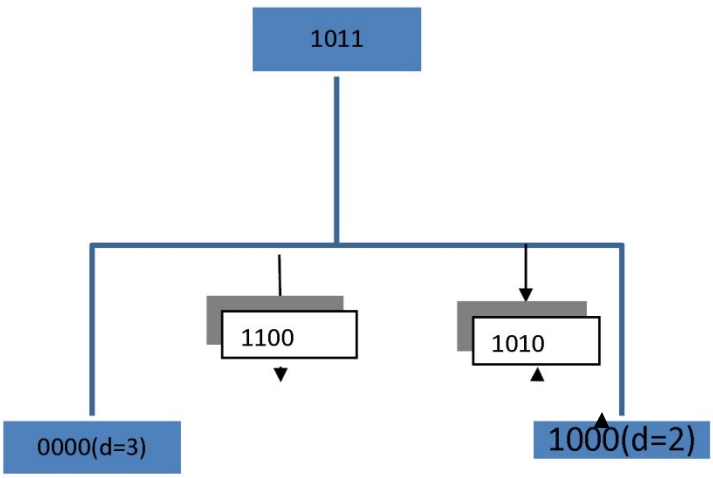
5. Identifiable Parent Property Code: Let $C \subseteq \sum^n$ be a code over some alphabet $\sum$.

   Desc(S) = the descendant set of a set S $\subseteq C$, i.e. the code consists of all the codewords that can be created by combining symbols from members of S.

   Then C is $t - IPP$ if for every $x \in desc(S)$ with $|S| \leq t$.There exists $c \in S$ such that $c \in \cap \{S' \subseteq C;\ x \in desc(S'), |S'| \leq t\}$, $i.e.$ atleast one parent is identifiable from any possible group of size $\leq t$ that could have generated x.

6. Traceable Code: Traceable Codes are proposed by Chor, Fiat and Naor in 1994.These codes enable us to trace atleast one user from coalition of traitors. Traceable codes enhance IPP codes with an effective, deterministic identification algorithm. They're a cornerstone of collusion-secure digital fingerprinting, first rigorously studied in 1994. Their significance lies in the guarantee of tracing, not just the structure of the code

2.1.6. Definition [8]: Let $C \subseteq \sum^n$ be any code of size $Mand$ minimum Hamming Distance d. if $d > (1 - \frac{1}{m^2})n$ .Then C is a $m-tracea$ bility code $(m-TA)$code for any integer m≥ 2. Example is;



For the 1st codeword ; $d(1011, 0000) = 3$, 2nd codeword; $d(1011,1100) = 3$, 3rd codeword ; $d(1011,1010) = 1$; 4th codeword $d(1011,1000)=2$.

2.2. *Relation between Traceable and Frameproof Codes*

Traceable codes are indeed a stronger form of codes with IPP. Every traceable code is also frameproof, but not vice versa. This is because if a code allows the tracing of at least one traitor, it naturally prevents framing of non-colluders. However, a code that prevents framing does not necessarily enable traitor identification.

Taking into account the foundational relationship between frame proof codes and traceable codes, the properties of BIBDs, such as balance and uniformity, can be leveraged to create codes with desirable properties, like frameproofness. In earlier Constructions, BIBDs have been utilized to derive frameproof codes due to their combinatorial properties. So here in this study we offer corrected results by redefining necessary conditions on the parameters of BIBDs that ensure a code derived from them is not just frameproof but also satisfies the condition of 2-Traceability.

3. Equidistant Constant Weight Code [9]

A code A is called equidistant constant weight code if all the codewords have same weight and distance between every pair of codewords is exactly same.

Example: A = {0000, 2101, 1202, 1110};

3.1. *Result*

We claim that incidence matrix of (t, b, k, l, m) is an Equidistant Constant Weight Code.

3.2 *Incidence Matrix [13]*

Representing a Balanced Incomplete Block Design (BIBD) using an Incidence matrix is really a standard and very effective method, especially for computational purposes.

An incidence matrix is a $t \times b$ binary matrix $M = [m_{i,j}]$ that represents the relationship between elements and blocks, where

$m_{i,j} = 1$ if the i^{th}element belongs to j^{th} block

$m_{i,j} = 0$ Otherwise.

For Example,; in case of (7, 7, 3, 3, 1)-BIBD Fano Plane

Var./Blocks	a	b	c	d	e	f	g
t_1	1	1	0	0	1	0	0
t_2	1	0	0	1	0	0	1
t_3	1	0	0	1	0	0	1
t_4	0	1	1	0	0	0	1
t_5	0	1	0	1	0	1	0
t_6	0	0	1	1	1	0	0
t_7	0	0	0	0	1	1	1

3.3. *Representation of Incidence Matrix as Equidistant Code*

The incidence matrix M of (t, b, k, l, m) −BIBD always satisfies the following properties;

- Every column of M consists of exactly l $1's$.
- Every row of M consists of exactly k1's.
- Two distinct rows of M always contain 1's in exactly $m$columns.
- If we form a code by representing all rows of M as the different codewords, then we find against the definition discussed above, C represents an Equidistant Constant Weight Code, for which length of every codeword is $b = \frac{tk}{l}$. The total number of codewords are t and weight of every codeword is k and distance between any two codewords is $(t - l)$. The incidence matrix of a BIBD corresponds to an Equidistant Constant Weight Code with parameters $(b, t, t - l)$.

4. From BIBDs to 2-Traceable Codes: A Combinatorial Approach

In their foundational work, Stinson and Wei explored the use of Balanced Incomplete Block Designs (BIBDs) for the construction of c-frameproof codes**,** deriving specific combinatorial conditions under which these codes prevent framing by colluding users. Their approach linked the structural properties of BIBDs—such as block size, incidence, and intersection numbers—to the security guarantees offered by frameproof codes. In the present study, we revisit and redefine these established conditions in the context of traceable codes**,** particularly focusing on 2-traceable codes.

4.1. Theorem [6]: *For a (v, b, r, k, λ)-BIBD - There exist frameproof codes as follows;*

1. *There always exists 2-FPC $(v, \frac{v(v-1)}{6})$ for all $v \equiv 1,3(mod\ 6)$*
2. *There always exists 3-FPC $(v, \frac{v(v-1)}{12})$ for all $v \equiv 1,4(mod\ 12)$*
3. *There always exists 4-FPC $(v, \frac{v(v-1)}{20})$ for all $v \equiv 1,5(mod\ 20)$"*

4.2. Theorem [1]: *An equidistant code with distance $d > \frac{2n}{3}$ is always a 2-TA Code.*

4.3. Theorem [7]: *If a $(v, k, \lambda) - BIBD\ exists$ then $v \equiv 1, k(mod(k^2 - k)$*

4.4 Result: We claim that

(a) If $v \equiv 1(mod\ 5)$; Then there exists 2-TA code of size $(v, \frac{v(v-1)}{30})$ for $k = 6$.

(b) if $v \equiv 1(mod\ 6)$; Then there exists a code of size $(v, \frac{v(v-1)}{42})$ for $k = 7$

(c) if $v \equiv 1(mod\ 7)$; Then there exists a code of size $(v, \frac{v(v-1)}{56})$ for $k = 8$

Proof: An equidistant code with distance $d > \frac{2n}{3}$ is always a 2-TA code and given the Definition of BIBD and Equidistant Constant Weight Code discussed above, we establish the following relationship; i.e. $d = n - k$,

$$n = b.$$

As a (v, b, r, k, λ)-BIBD is 2-$TA\ if$

$$\rightarrow (v-k) > \frac{2b}{3} \quad \ldots(1)$$

$$\Rightarrow (v-k) > \frac{2v(v-1)}{3k(k-1)} \quad \ldots(2)$$

$$\Rightarrow (v-k)\,3k(k-1) > 2v(v-1)$$

$$\Longrightarrow 2v(v-1) < (v-k)\,3k(k-1)$$

it becomes; $2v(v-1) < 3.6.5(v-6)$ (For $k = 6$)

$$\rightarrow\ v(v-1) < 45(v-6)$$

$$\rightarrow\ v^2 - v < 45v - 270$$

$$\rightarrow\ v^2 + 270 < 46v$$

For this equation v definitely can be 7,8,9,10….. and according to condition of integers for the value of r ; $k-1/v-1$ (must). By the rule of divisibility; 5/10, 5/15, 5//20, 5/25, 5/30…….

$$\rightarrow v - 1 = 10\ \rightarrow v = 11$$

$$\rightarrow v - 1 = 15$$

$$\rightarrow v = 16, 21, 26, 31\ldots..(v \text{ can take these values})$$

For the BIBD; [11 ,6, 1] According to the equation Here $r = 2$ and $b = \frac{vr}{k}$;

Therefore $b = \frac{11}{3}$(not an integer)

$$\rightarrow [11,6,1];\ \text{BIBD does not exist.}$$

For the BIBD; [16,6,1] Here $r = 3$ and $b = \frac{vr}{k} \rightarrow b = 8$

$$\rightarrow [16,6,1] - \text{BIBD}\quad \text{exists}\quad \text{and it gives us 2-TA code.}$$

$$\rightarrow \text{For the BIBD}[21,6,1];\ r = 4 \text{ and } b = \frac{vr}{k} \rightarrow b = 14$$

$$\rightarrow [21, 6, 1]\text{-BIBD exists leading to 2-TA code.}$$

For the BIBD;$[26, 6, 1]$; $r = 5$ and $b = \frac{vr}{k} \rightarrow b\ \neq an\ integer.$

$$\rightarrow [26,6,1] - \text{BIBD does not exist.}$$

if for $(v, k, \lambda) - [31, 6, 1] - \text{BIBD};\rightarrow r = 6$ and $b = 31.$

$$\rightarrow [31,6,1] - \text{BIBD exists leading to 2-TA Code.}$$

For [36, 6, 1]; $r = 7$ and $b = \frac{vr}{k} \rightarrow b = 42$

→ [36, 6, 1]- BIBD exists leading to 2-TA Code.

4.5 Result: We also claim that if v is of the form $5n + 1$ i.e. $v \equiv 1(\mathrm{mod}\ 5)$. Then there exists 2-TA code with $k = 6$. it completes the proof of the above result.

Now we discuss the above case for taking large block size $k = 7$. In that case the above equation becomes

$$2v(v-1) < (v-k)\ 3k(k-1) \text{ and it leads to} \quad \ldots(1)$$

$$\rightarrow 2.v(v-1) < 21.(6)(v-7) \quad \ldots(2)$$

$$\rightarrow v(v-1) < 21.3(v-7) \quad \ldots(3)$$

$$\rightarrow v(v-1) < 63v - 201 \quad \ldots(4)$$

$$\rightarrow v^2 - v < 63v - 201 \quad \ldots(5)$$

$$\rightarrow v^2 + 201 < 64v \quad \ldots(6)$$

Since $v(>7)$ and $(^{v-1}/_{k-1})$ should be an integer. So by the rule of divisibility, 6/12 , 6/18 , 6/24 , 6/30 For $v - 1 = 12 \rightarrow v = 13$and v can take the values 19, 25, 31, 37……

Here if we start with the lowest value; For the BIBD, [13,7, 1]

→ Here $r = 2$ and b is not an integer.

Therefore [13, 7, 1]- BIBD does not exist. It cannot form 2-Traceable Code.

(ii) For the BIBD, [19, 7, 1]→ Here $r = 3$ and b is not an integer.

→ [19, 7, 1]- BIBD does not exist. It cannot form 2-TA Code.

(iii) For the BIBD, [25, 7, 1]→Here $r = 4$ and b is not an integer.

Therefore, this BIBD does not exist and there does not exist 2-TA code.

(iv) In the BIBD, [31, 7, 1]→Here $r = 5$and b is not an integer.

BIBD does not exist and it does not form 2-TA Code.

(v) For the BIBD, [43, 7, 1]; Here $r = 7$ and $b = 43$. So if [43, 7, 1]– BIBD exists then it leads to 2-TA Code.

(vi) For the BIBD, [49, 7 ,1]; Here $r = 8$ and $b = 56$. So if [49, 7, 1]– BIBD exists then it leads to 2-TA Code.

Result 2: If v is of the form $6n + 1$, Then it leads to 2-TA Code for $k = 7$ and the size of the code is $(v, \frac{v(v-1)}{42})$

Result 3: Existence of 3-TA Code; In this result we show that there is no value of v for which there does not exist 3-TA Code.

Proof: For a code C to be 3-TA Code; $d > (1 - \frac{1}{m^2}).n$

i.e. $$d > (\frac{8n}{9})$$

$$\rightarrow (v-k) > \frac{8(vr)}{9k} \quad \ldots(1)$$

$$\rightarrow 9k(v-k) > \frac{8v\lambda(v-1)}{k(k-1)} \quad \ldots(2)$$

$$\rightarrow 9k(k-1)(v-k) > 8v(v-1)(Taking\ \lambda = 1)$$

For $k = 2$;Theabove equation becomes;

$$\rightarrow 9.2.1.(v-2) > 8v(\ v-1\ ) \quad \ldots(3)$$

$$\rightarrow 18(v-2) > 8v^2 - 8v \quad \ldots(4)$$

$$\rightarrow 9(v-2) > 4v^2 - 4v \quad \ldots(5)$$

$$\rightarrow 9v - 18 > 4v^2 - 4v \quad \ldots(6)$$

$$\rightarrow 4v^2 + 18 < 13v \quad \ldots(7)$$

There is no value of v for which above result will be satisfied. If we take $k = 3$;Then the above result will be,

$$\rightarrow 9.k.(k-1)(v-k) > 8v(v-1)$$

$$\rightarrow 9.3.2.(v-3) > 8v(v-1)$$

$$\rightarrow 54(v-3) > 8v^2 - 8v$$

$$\rightarrow 54v - 162 > 8v^2 - 8v$$

$$\rightarrow 8v^2 - 62v < -162 \quad \ldots(8)$$

For equation (8) There is no value of v for which above result will be satisfied. Therefore, there does not exist

3-TA code for the value of v .

5. Conclusion

Here in this paper we have revisited and improved the results of BIBD in terms of 3-frameproof and 2-Traceability for the higher values of k in caseof $(v, b, r, k, \lambda) - BIBD$. As 2-TA codes and 3-FP codes are stronger than frameproof codes. Therefore these conditions enhance the importance and Applications]of BIBDs. In future we would like to define the traceability and frameproof conditions of BIBDs for other higher values of k.

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