Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 47 (2026), No. 6, pp. 2245–2261
DOI : 10.47974/JIOS-2120

A multicriteria optimization approach for maximizing flow in multiterminal networks

Bishwa Raj Adhikari †
Department of Mathematics
Prithvi Narayan Campus
Tribhuvan University
Pokhara 33700
Nepal

Hari Nandan Nath *
Department of Mathematics
Bhaktapur Multiple Campus
Tribhuvan University
Bhaktapur 44800
Nepal

Tanka Nath Dhamala §
Central Department of Mathematics
Tribhuvan University
Kathmandu 44600
Nepal

Abstract

Many real-world applications—such as optimizing traffic flow, power distribution, communication networks, and financial transactions—frequently rely on network flow models. These problems often involve conflicting objectives, such as maximizing flow while minimizing cost or egress time. An effective strategy is to pose such problems as multi-objective optimization models that provide a set of optimal solutions and allow trade-offs between objectives. In this study, we focus on maximizing flow from a single source to multiple sinks in a multi-sink network. The problem is modeled as a multicriteria

† *E-mail:* **bishwa208@gmail.com**
* *E-mail:* **hari672@gmail.com (Corresponding Author)**
§ *E-mail:* **tanka.nath.dhamala@gmail.com**

optimization problem, and an algorithm based on the ϵ-constraint method is proposed. The results are compared with solutions obtained using the weighted sum method.

Subject Classification: *90B10, 90C27, 68Q25, 90B06, 90B20, 90B50, 90C05, 90C10.*

Keywords: *Pareto solution, Trade-off, Network flow, Multi-sink, Multicriteria optimization.*

1. Introduction

Motivation. Network flow problems are fundamental in operations research, extensively studied and applied as powerful tools to tackle a variety of real-world challenges. These problems play crucial roles within complex models and benefit from specialized algorithms designed to efficiently resolve even large instances within practical computational limits. Moreover, similar to many real-world issues, network flow problems inherently require consideration of multiple objectives. For example, in transportation planning, minimizing costs on specific routes, reducing goods deterioration, and maximizing safety, these objectives often contradict one another. It's rare to find a solution that optimizes all objectives simultaneously. Hence, the focus lies on finding solutions where improving one objective necessitates a trade-off with another. Finding all or a suitable subset of these Pareto or efficient solutions is the goal of multiple objective optimization.

Literature review

The objective of maximizing flow in a capacitated network involves optimizing the transmission of flow between two designated nodes, namely a source s and a sink t, while ensuring that the flow through each edge does not exceed its capacity. The maximum flow problem focuses on maximizing throughput under capacity constraints, without considering associated costs. One of the most common approaches to solving the max-flow problem in polynomial time is the labeling algorithm of Fulkerson and Ford [1]. In the worst-case scenario, the computational complexity can become significant for large capacities. Although the algorithm remains polynomial when capacities are bounded by integers, it may require many iterations, especially when edge capacities are large. Moreover, if some arc capacities are irrational, the algorithm may fail to terminate in finite time. To ensure finite convergence in such cases, modified algorithm, as proposed in Johnson [2], is employed.

The Edmonds–Karp algorithm, a refinement of the Fulkerson and Ford [1] method, distinguishes itself by always selecting the shortest augmenting path in the residual graph [3]. On the other hand, Dinitz [4] algorithm is considered both efficient and practical. The capacity scaling algorithm improves performance by initially pushing flow through edges with higher capacities and gradually considering edges with smaller capacities, thereby accelerating convergence to the maximum flow [5]. In the preflow-push algorithm, the presence of an active node indicates infeasibility; the algorithm iteratively selects such nodes and pushes excess flow to neighboring nodes until all excess is eliminated, resulting in a maximum flow [6]. Furthermore, Minieka [7] extended the Fulkerson and Ford [1] framework to networks with multiple sources and sinks, introducing the concept of lexicographic network flow based on priority ordering.

In practical applications, Pyakurel, Nath, and Dhamala [8] developed an efficient algorithm for quickest evacuation planning by incorporating time-varying parameters into a dynamic contraflow model for two-terminal networks. Pyakurel and Dempe [9] proposed polynomial-time algorithms for both static and dynamic flow problems that allow intermediate storage and contraflow configurations. Extending this work, Pyakurel, Khanal, and Dhamala [10] applied these concepts to abstract networks for evacuation planning. Additionally, Dhamala, Wagle, and Pyakurel [11] explored facility location problems using similar principles, while Dhamala et al. [12] investigated contraflow configurations in lossy networks considering both symmetric and asymmetric travel times.

In the context of multiobjective network flow problems, Aneja and Nair [13] utilized the chord rule to identify efficient extreme points in bicriteria transportation problems, where the weighted sum method [14] is applied iteratively. Among constraint-based approaches, the

ϵ-constraint method [15] optimizes one objective while treating others as constraints, enabling the generation of Pareto-optimal solutions for both convex and non-convex problems, although it may produce weakly efficient solutions and requires solving multiple single-objective problems. Isermann [16] addressed multicriteria transportation problems by characterizing the efficient region using graph-theoretic properties of basic feasible solutions. The multiobjective simplex method, originally developed by Yu and Zeleny [17], was extended to network problems by Klingman and Mote [18].

Malhotra and Puri [19] proposed a method to obtain all efficient solutions for bicriteria network problems using a modified Out-of-Kilter algorithm. Lee and Pulat [20] developed an approach integrating bicriteria programming with the Out-of-Kilter method for continuous cases, while Pulat, Huarng, and Lee [21] combined this approach with the network simplex method for parametric analysis. Lee and Pulat [22] further extended the model to integer flows, utilizing the unimodularity property of the incidence matrix. Calvete and Mateo [23] introduced a lexicographic optimization approach based on preemptive priorities. Sedeño-Noda and González-Martín [24] proposed an alternative adjacency-based method using network simplex techniques, whereas Przybylski, Gandibleux, and Ehrgott [25] demonstrated its limitations and proposed a revised adjacency concept to ensure connectivity in multiobjective minimum cost integer flow problems. Hamacher, Pedersen, and Ruzika [26] showed that efficient flows can be generated through incremental circulations while preserving optimality, highlighting the structural robustness of efficient solutions.

In recent years, multicriteria optimization has been increasingly applied to network flow problems to address real-world complexities. Mixed-integer linear programming models have been used for partner selection in virtual organizations, incorporating costs, risks, and interdependencies [27]. Similarly, evolutionary multi-objective approaches have been applied to stochastic environments, such as air traffic network flow optimization, to generate Pareto-optimal solutions balancing efficiency and delay [28]. Classical minimum cost flow models assume deterministic parameters, limiting their performance under real-world uncertainty. Recent studies incorporate neutrosophic theory using Single-Valued Triangular Neutrosophic (SVTN) numbers, which represent uncertain parameters through triangular truth–indeterminacy–falsity structures, enabling more robust and realistic optimization via linear programming techniques [29]. Furthermore, multicommodity and multiterminal flow models with local constraints extend classical max-flow formulations by ensuring demand satisfaction while maximizing overall flow [30].

Despite these advancements, many methods for finding all efficient solutions remain primarily theoretical. Ruhe [31] showed that the number of non-dominated extreme points can grow exponentially with network size. Eusébio and Figueira [32] proposed an algorithm for identifying all non-dominated solutions in biobjective integer network flow problems using ϵ-constraint techniques. Raith and Ehrgott [33] developed a two-phase parametric network simplex method for solving bi-objective minimum cost flow problems. Eusébio, Figueira, and Ehrgott [34] further proposed methods for generating representative subsets of non-dominated solutions. Nath, Dhamala, and Dempe [35] applied bicriteria optimization to minimize time while preserving specific paths, and later extended their work to maximize dynamic flow with arc reversals under shortest-path constraints [36].

Research Gap. For optimizing flow in multi-terminal networks, multicriteria optimization is pulled since the objectives of optimization conflict one another when they are to be handled simultaneously. While the weighted sum and ϵ-constraint methods are commonly used to address such challenges, there is a absence in the literature of these methods being applied to solve multiobjective network flow problems. Therefore, our goal is to incorporate these techniques into the solution of network flow problems by developing algorithms that are well-suited for cases involving three or more objectives.

Our Contribution. We formulated multiobjective network flow problem aimed at determining non-dominated and efficient solutions for maximizing flows across multiple objectives. We designed a multiobjective algorithm for ϵ-constraint approach and applied it to solve a triobjective maximization problem. We further solved the problem using weighted sum method and compare the results.

Organisation of the Paper. This paper is organised as follows. Section 2 contains concepts and definitions. Section 3 presents mathematical model and solution techniques with examples. Section 4 reports of some theorems, computational results and discussion. Finally, concluding remarks and avenues for future research are provided in Section 5.

2. Basic Terminologies

2.1 *Maximum Flow Problem*

A network is directed graph G = (N, A), where N is a collection of n nodes and $A \subseteq N \times N$ is a collection of m arcs. Each arc (i, j) is also considered to have a non-negative capacity u_{ij}. For each $i \in N$ we let $A(i) = \{k \in N \,|(i, k) \in A)\}$, the set of nodes after i and $B(i) = \{l \in N \,|(l, i) \in A\}$, the set of nodes before i.

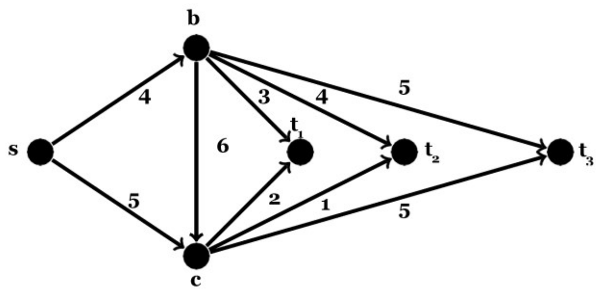

Figure 1
Diagrammatic representation of network G with arc capacities

Example 1: Consider a network G shown in Figure 1. In this network, N $=\{s, b, c, t_1, t_2, t_3\}$, A = $\{(s, b), (s, c), (b, c), (b, t_1), (b, t_2), (b, t_3), (c, t_1),(c, t_2), (c, t_3)\}$. In the diagram, the numbers against arcs represent capacities which are denoted by u_{ij} for $(i, j) \in A$.

In network flow problems, we consider an entity called flow to be generated at some special nodes called the source nodes. The flow reaches ultimately to some special nodes called sink nodes. We denote the set of source nodes by S, that of sink nodes by T and denote the network by G = (N, A, u, S, T). A network with multiple sources or multiple sinks is known as a multi-terminal network. If $S = \{s\}$, $T = \{t\}$, we denote the network as G = (N, A, u, s, t). It is also known as the two-terminal network. If not stated otherwise, we consider $B(s) = A(t) = \emptyset \ \forall s \in S, t \in T$. Formally, a flow in G is a function $z : A \to R_{\geq 0}$ that satisfies,

$$\sum_{k\in A(i)} z_{ik} - \sum_{l\in B(i)} z_{li}=0 , \forall \, i \in N\text{-}\{S \cup T\} \tag{1}$$

$$0\leq z_{ij} \leq u_{ij}, \forall (i, j) \in A \tag{2}$$

The value of the flow z is given by

$$v(z) = \sum_{s\in S} \sum_{k\in A(s)} z_{sk} = \sum_{t\in T} \sum_{l\in B(t)} z_{lt} \tag{3}$$

The maximum flow problem is to maximize v(z) subject to (1) and (2).

In the network G = (N, A, u, s, t), the maximum flow problem reduces to:

$$\text{maximize } \sum_{k\in A(s)} z_{sk} \tag{4}$$

$$\text{subject to } \sum_{k\in A(i)} z_{ik} - \sum_{l\in B(i)} z_{li} = 0 \text{ , } \forall i\in N\text{-}\{s, t\} \tag{5}$$

$$0 \leq z_{ij} \leq u_{ij}, \forall (i, j) \in A \tag{6}$$

by adding nodes s^*, t^* in G, arcs (s^*, s) $\forall s \in S$ such that $u_{s^* s} = \infty$, and (t^*, t) $\forall t \in T$ such that $u_{tt^*} = \infty$. Solving the single-source single-sink problem gives the solution to the multi-source-multi-sink case which maximizes the sum of the flow accumulated at all sinks which is equal to the sum of the flow generated at all the sources. Given a network G= (N, A, u, S, T), the problem of maximizing the flow at each source or each sink (in spite of maximizing the sum of the flow there) is a problem with multiple objectives. This requires the concept of multicriteria optimization.

2.2 *Multicriteria Optimization*

Multicriteria optimization, or multiple objective decision-making, refers to the process of seeking optimal solutions when there are multiple conflicting objectives.

Consider a function $h : R^m \rightarrow R^p$ defined by $h(z) = (h_1(z), h_2(z), \cdots, h_p(z))$, $\forall z \in R^m$ where $h_q(z) : R^m \rightarrow R$. A multicriteria optimization problem [37] can be formulated as,

$$\max_{z\in Z} \ h(z) \tag{7}$$

where $Z \subseteq R^m$ is called a feasible set of decision vectors z, determined by w constraints $\phi(z) \geq \mathbf{0}$, with $\phi : R^m \rightarrow R^w$. The vector $y = h(z)$ for some $z \in Z$ is an outcome vector. The set of outcome vectors $Y = \{h(z) : z \in Z\}$ is termed as the objective space.

2.3 *Pareto Optimality*

Pareto-optimality is a concept in multicriteria optimization where, a solution is considered optimal if no objective can be improved without worsening at least one other objective. Without loss of generality, assume that all objective functions h_q are of maximization type. A feasible solution z is efficient, if there is no feasible solution $\bar{z}$ such that $h_q(\bar{z}) \geq h_q(z)$ for q = 1, ..., p, with a strict inequality for at least one index q. A solution z is weakly efficient if there is no feasible $\bar{z}$ such that $h_q(\bar{z}) > h_q(z)$ for q = 1, 2, ..., p. The set of all (weakly) efficient solutions in Z is denoted by $(Z_{wE})Z_E$. The point $y^* = h(z^*)$, $z^* \in$

$(Z_{wE})Z_E$ is (weakly) non-dominated and the set of all such solutions in $Y = h(Z)$ is denoted by $(Y_{wN})Y_N$. The full collection of Pareto optimal plans, typically an infinite set and is refereed to as the Pareto frontier (Pareto surface).

2.4 *Methods of Generating Pareto Optimal Solution*

The common methods of obtaining Pareto optimal solutions are weighted sum and ϵ- constraint methods.

2.4.1 Weighted Sum Method

In this method, a scalar weight is allocated against each objective function that compares the relative importance of the objectives. Then, the multi-objective problem is transformed into a single-objective problem by using the weighted sum [14] of the original multiple objectives as,

$$\text{maximize } \{\textstyle\sum_{q=1}^{p} w_q h_q(z): z \in Z, \text{ where } 0 \leq w_q \leq 1, \sum_{q=1}^{p} w_q = 1\}$$
$$\text{subject to } \emptyset(z) \geq 0 \tag{8}$$

If $w_q \geq 0$ (q = 1, · · ·, p) and $\tilde{z}$ is the optimal solution of problem (8), then $\tilde{z} \in Z_{wE}$ otherwise if $w_q > 0$, then $\tilde{z} \in Z_E$. If the feasible region is convex, Pareto-optimal solutions can be obtained by suitable choices of weighting coefficients beginning with different initial solutions [14].

2.4.2 ϵ-Constraint Method

A method suitable for both convex and non-convex feasible region is ϵ-constraint method [32]. In this technique, one of the objectives is optimized while all others are constrained by ϵ. Let $\epsilon \in R^p$, then ϵ-constraint method is the optimization problem

$$\text{maximize } \{h_k(z) : h_q(z) \geq \epsilon_q, \text{ for } q \neq k, z \in Z\} \tag{9}$$

where ϵ_q is the limiting value of h_q desired by the decision-maker. Choosing different initial points and different values of ϵ_q, Pareto-optimal solutions are obtained. If $\bar{z}$ is an optimal solution of (9) for some q, then $\bar{z}$ is weakly efficient.

3. Multiobjective Maximum Flow Model with Single Source and Multiple Sinks

In a network G = (N, A, u) modeling a static flow problem with a specified source node s and multiple sink nodes $t_1, t_2, \ldots, t_p$, we define distinct objectives at each sink aimed at maximizing the flow allocation directed towards them. Define

$$h_q(z) = \textstyle\sum_{l \in B(t_q)} z_{lt_q}, \quad q = 1, 2, \ldots, p \tag{10}$$

so, the multicriteria model that maximizes the flow at each sink takes the form,

$$\max_z h(z) = (h_1(z), h_2(z), \ldots, h_p(z)) \tag{11}$$

$$\text{subject to } \sum_{k\in A(i)} z_{ik} - \sum_{l\in B(i)} z_{li} = 0 \text{ , } \forall \text{ i} \in N\text{- \{s, } t_1, t_2, t_3, ..., t_p\} \quad (12)$$

$$0 \leq z_{ij} \leq u_{ij}, \forall (i, j) \in A \quad (13)$$

where decision variable z = z_{ij} denotes static flow and parameter $u_{ij} \in Z_{>0}$ represents capacity of an arc (i, j).

3.1 *Multiobjective Formulation in Weighted Sum Method*

Selecting weights w_q for each objective in (10) we obtain weighted sum formulation as, maximize

$$\{\sum_{q=1}^{p} w_q h_q (z): z \in Z, \text{ where } 0 \leq w_q \leq 1, \sum_{q=1}^{p} w_q = 1\} \quad (14)$$

subject to constraints (12) and (13).

Theorem 1: *When all arc capacities are integer, the weighted sum method gives integer solution.*

Proof: In the network G, connect each sink t_q with source s having arc capacities $v^*(t_q)$, q = 1, 2, ..., p, where $v^*(t_q)$ is the value of the maximum flow taking t_q as the single sink . The constraint matrix M of new graph G' constructed from the weighted sum formulation (12) − (14), exhibits a distinctive characteristic: each column contains precisely one entry of 1, one entry of -1, and all other entries are 0. This pattern arises from the representation of arcs in the network: a 1 signifies an arc emanating from a node, a -1 signifies an arc approaching a node, and 0 indicates absence. Essentially, this generates a matrix of order (n − p − 1) × (m + p) of the modified network and exhibits the network property, hence its determinant falls within the set {1, −1, 0}. So, M is unimodular by [38]. Moreover, the matrix corresponding to capacity constraints (13) including capacity constraints to p new arcs forms identity matrix of order m + p.

$$I_{(m+p)\times(m+p)} = \begin{bmatrix} 1\,0\,0\,\ldots\,0 \\ 0\,1\,0\,\ldots\,0 \\ 0\,0\,1\,\ldots\,0 \\ \ldots\,\ldots\,\ldots\,\ldots\,\ldots \\ \ldots\,\ldots.\ldots\,\ldots\,\ldots \\ \\ 0\,0\,0\,\ldots\,1 \end{bmatrix}$$

Also, when M is totally unimodular and I is the identity matrix, by property of unimodularity, matrix B= $\begin{bmatrix} M \\ I \end{bmatrix}$, preserves unimodularity. Thus, by solving equations (12)-(14), using linear programming problem with simplex method [39] taking arc capacities u_{ij} all integer, results flow value at each sink t_q with integer components. When arc capacities are taken real, the results also become real.

Example 2: Consider the tri-sink network as in Figure 1, having nodes {s, b, c, t_1, t_2, t_3}, arcs {(s, b),(s, c),(b, c),(b, t_1),(b, t_2),(b, t_3),(c, t_1),(c, t_2),(c, t_3)}, arc capacities {4, 5, 6, 3, 4, 5, 2, 1, 5} respectively. We aim to determine maximum flow values from source node s to sink nodes t_1, t_2 and t_3. We select weights w_1, w_2, w_3 such that w1 + w2 + w3 = 1, then equation (14) takes the form,

$$h = h_1 \times w_1 + h_2 \times w_2 + h_3 \times w_3 \tag{15}$$

subject to constraints (12)-(13) for case p = 3. The problem is solved using Matlab 'linprog' solver. Each weight w_q is selected within the interval 0 to 1 with increments of 0.1. Flow values of the functions $h_1(z)$, $h_2(z)$ and $h_3(z)$ are depicted in the Table 1 while the Pareto frontier is illustrated in Figure 2.

Table 1

(Weakly) Non-dominated solutions in tri-objective weighted sum method

v_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_2	0	0	0	0	0	0	5	5	5	5	5	0	0	0	0	0	5	5	5
v_3	9	9	9	9	9	9	4	4	4	4	0	9	9	9	9	9	4	4	4
v_1	0	2	2	0	0	0	0	0	0	0	2	2	2	2	0	0	0	0	0
v_2	5	5	5	0	0	0	0	0	5	5	5	5	5	5	0	0	0	0	5
v_3	4	0	0	9	9	9	9	9	4	4	2	0	0	0	9	9	9	9	4
v_1	2	2	2	2	2	2	0	0	0	5	2	2	2	2	2	2	2	0	5
v_2	5	5	5	5	5	5	0	0	0	0	5	5	5	5	5	5	5	0	0
v_3	2	2	0	0	0	0	9	9	9	4	2	2	0	0	0	0	0	9	4
v_1	5	5	5	2	2	2	2	2	2	5	5	5	5	5	5	2	2	2	2
v_2	0	2	2	5	5	5	5	5	5	0	0	0	2	2	2	5	5	5	5
v_3	4	2	2	0	0	0	0	0	0	4	4	4	2	0	0	0	0	0	0
v_1	2	5	5	5	5	5	5	5	2	2	2	2	5	5	5	5	5	5	5
v_2	5	0	0	2	2	2	2	2	5	5	5	5	0	0	2	2	2	2	2
v_3	0	4	4	2	0	0	0	0	0	0	0	0	4	4	0	0	0	0	0
v_1	5	2	2	2	5	5	5	5	5	5	5	5	5	2	2	5	5	5	5
v_2	2	5	5	5	0	2	2	2	2	2	2	2	2	5	5	0	2	2	2
v_3	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_1	5	5	5	5	5	5	2												
v_2	2	2	2	2	2	2	5												
v_3	0	0	0	0	0	0	0												

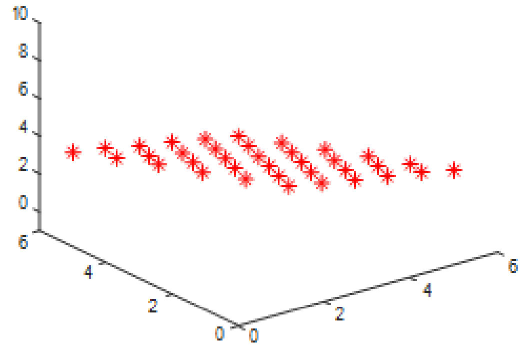

Figure 2
(Weakly) Non-dominated solution generated by weighted sum method

3.2 *Multiobjective Formulation in ϵ-Constraint Method*

The ϵ-constraint formulation of the problem is,

$$h_r(z) = \sum_{l\in B(t_r)} z_{lt_r}, \text{ for some } r \tag{16}$$

$$\text{subject to } \sum_{l\in B(t_q)} z_{lt_q} \geq \epsilon_q, \exists\, \epsilon_q \geq 0, q=1, 2, \ldots, p, q\neq r \tag{17}$$

$$\sum_{k\in A(i)} z_{ik} - \sum_{l\in B(i)} z_{li} = 0, \forall\, i \in N\text{-}\{s, t_1, t_2, t_3, \ldots, t_p\} \tag{18}$$

$$0 \leq z_{ij} \leq u_{ij}, \forall (i, j) \in A \tag{19}$$

Theorem 2: *When all arc capacities are integer, the ϵ-constraint method gives integer solution provided each ϵ_q, $q = 1, 2, \ldots, p$ is integer.*

Proof: In the network G connect each sink t_q, $q = 1, 2, \ldots, p, q \neq r$ to source s, In each of these arcs choose arc capacities $\epsilon_q = v^*(t_q)$. The constraint matrix M formed by ϵ-constraint formulation (17)-(19) demonstrates a unique trait: every column comprises exactly one element of 1, one element of -1, and all other elements are 0. This structure is a result of how arcs are represented in the network: a 1 indicates an arc originating from a node, a -1 indicates an arc leading to a node, and 0 denotes the lack of such an arc. Since flow value to one of the sinks has to be determined and is taken as objective function, this matrix M has order $(n - 2) \times (m + p)$ and satisfy the network property. It means $\det(M) \in \{1, -1, 0\}$. So, M is unimodular by [38]. Moreover, the matrix corresponding to capacity constraints (19) including capacity constraints to new arcs forms identity matrix of order $(m + p\text{-}1)$,

$$I_{(m+p-1)\times(m+p-1)} = \begin{bmatrix} 1\,0\,0 \,\ldots\, 0 \\ 0\,1\,0 \,\ldots\, 0 \\ 0\,0\,1 \,\ldots\, 0 \\ \ldots\;\ldots\;\ldots\;\ldots\;\ldots \\ \ldots\;\ldots.\ldots\;\ldots\;\ldots \\ \\ 0\,0\,0 \,\ldots\, 1 \end{bmatrix}$$

Also, when M is totally unimodular matrix and I is the identity matrix, by property of unimodularity, matrix B=$\begin{bmatrix} M \\ I \end{bmatrix}$, is also unimodular. The equations (16)-(20) are solved by linear programming simplex method with the arc capacity and ϵ_q choosen to be integer. We obtain maximum flow value v_q at each sink t_q with integer components. For more detail refer [39]. But when arc capacities and/or ϵ_q are choosen real, the solution consists real values.

Algorithm 1: Determination of (weakly) efficient solutions using ϵ-constraint method

Require: A directed network $G = (N, A, u, s, T)$ with $u \in Z_{\geq 0}$
Ensure: Set of (weakly) efficient and (weakly) non-dominated integral solutions
1: sList ← {}, vList ← {}
2: **for** q = 1 to $p-1$ **do**
3: $v^*(t_q) \leftarrow$ the value of the maximum flow taking t_q as the single sink
4: $E_q \leftarrow \{0, \cdots, v^*(t_q)\}$
5: **end for**
6: **for** $\epsilon \in E_1 \times \cdots \times E_{p-1}$ **do**
7: z ← a solution of (16)–(19)
8: $v_p \leftarrow$ value of z, $(v_q)_{q=1}^{p-1} \leftarrow [\sum_{l\in B(t_q)} z_{lt_q}]_{q=1}^{p-1}$
9: Add z to sList, $(v_1, \cdots v_p)$ to vList
10: end for
11: return vList, sList

Example 3: Determination of (weakly) non-dominated solutions in tri-objective network flow model using ϵ constraint method.

We use the same data from Example 2. Consider nodes t_1, t_2 and t_3, as sinks and node s as source. We use Algorithm 1 to find the efficient solutions. The value of the maximum flow taking t_2 and t_3 as the single sink is $v^*(t_2) = 5$ and $v^*(t_3) = 9$. Selecting ϵ_2 and ϵ_3 in the limit $0 \leq \epsilon_2 \leq 5$ and $0 \leq \epsilon_2 \leq 9$ for the second and third objective and using Matlab 'linprog' solver, the flow values of the problem (16)-(19) are listed in Table 2, and set of (weakly) non-dominated solutions are given in Figure 3. We can also determine efficient solutions in feasible space and path corresponding to the maximum flow values.

Table 2

(Weakly) Non-dominated solutions in tri-objective ϵ-constraint method

v_1	0	1	2	3	4	5	5	5	5	5	0	1	2	3	4	5	5	5	5
v_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
v_3	9	8	7	6	5	4	3	2	1	0	8	7	6	5	4	3	2	1	0
v_1	0	1	2	3	4	5	5	5	0	1	2	3	4	4	4	0	1	2	3
v_2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4
v_3	7	6	5	4	3	2	1	0	6	5	4	3	2	1	0	5	4	3	2
v_1	3	3	0	1	2	2	2												
v_2	4	4	5	5	5	5	5												
v_3	1	0	4	3	2	1	0												

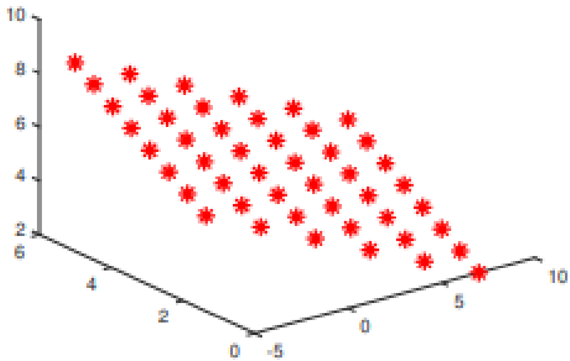

Figure 3
(Weakly) Non-dominated solutions generated by ϵ-constraint method

4. Results and Discussion

We developed a multi-criteria optimization framework aimed at maximizing flows in a network with multiple sinks originating from a single source. Our approach utilized the ϵ-constraint method for handling multiobjective scenarios, specifically addressing the problem with three objectives. Additionally, we conducted a comparative analysis against results obtained using the weighted sum method.

Our analysis revealed that while some solutions obtained through the weighted sum method are weakly non-dominated (e.g., (2, 5, 0), (5, 2, 0)), others, such as (0, 0, 9) and (2, 5, 2), are non-dominated. Moreover, the method tends to produce identical solutions for different weight configurations, as evidenced by the repetition of (0, 0, 9) and (2, 5, 2) multiple times. We also encountered instances where non-dominated solutions were missing, exemplified by (1, 3, 5) and (1, 4, 4). Consequently, the outcomes yielded by this approach were deemed unsatisfactory. This inadequacy suggests a need for computationally efficient algorithms, since the selection of weights must be made a priori without a full understanding of the problem.

Based on the results of the ϵ-constraint method, where we employed the approximate minimum flow value or zero as the lower bound and the maximum flow value as the upper bound for each objective, we generated all non-dominated solutions, including some additional weakly non-dominated solutions, exemplified by (5, 0, 3), (5, 0, 2), and (5, 1, 2). From Theorems 1 and 2, it is evident that the weighted sum technique can produce integer solutions despite the fractional nature of flow variables and weighting factors. Similarly, within the ϵ-constraint method, we obtain integer solutions while maintaining integer arc capacities and choosing integer ϵ values. Hence, the inherent characteristics of these methods ensure their applicability across both continuous and discrete variables.

In the literature, it has been observed that the number of non-dominated solutions increases rapidly as the network's vertex count expands. The time complexity of the problem escalates significantly due to its dependence on various factors, such as the number of objectives, arcs, and, occasionally, arc capacities. Additionally, it depends on the choice of weights ranging from 0 to 1 for each weight w_q in the weighted sum method and on the selection of ϵ in the ϵ-constraint method. Consequently, achieving all efficient solutions proves challenging, particularly in complex networks with more than two objectives, owing to the absence of efficient algorithms. In a tri-objective network with six vertices and nine arcs, the runtime of the weighted sum method increases to approximately 1.63 seconds for each increment of 0.1 in the weighting factor. Similarly, in the same network, the ϵ-constraint method achieves a runtime of approximately 0.68 seconds.

5. Conclusions

We successfully formulated and solved a multiobjective network flow model to find the maximum possible flow in a single-source, multi-sink network using the weighted sum and ϵ-constraint methods, with a focus on scenarios involving more than two objectives. It has been observed that the weighted sum method allows for obtaining all non-dominated points only through full optimization of the weighted objectives, which presents significant computational challenges due to its exhaustive nature. In contrast, the ϵ-constraint method, with upper and lower bound approximations, identifies all non-dominated solutions. Utilizing these bounds overcomes the weaknesses of the weighted sum method and proves superior in terms of convergence, uniformity, cardinality, and computational efficiency.

While the ϵ-constraint method can identify all non-dominated solutions, a notable drawback is its tendency to yield weakly efficient solutions due to the inflexibility of boundary value inequalities. It also requires considerable computational time even for small network models. Further research is needed to develop novel and computationally efficient algorithms for obtaining efficient (Pareto) solutions in multi-source, multi-sink networks, addressing both continuous and discrete domains.

Acknowledgements

The first author expresses gratitude to the University Grants Commission Nepal for providing "PhD Fellowship and Research Support" in his PhD research.

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Received March, 2025