Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 47 (2026), No. 4, pp. 1277–1300
DOI : 10.47974/JIOS-1393

$(\in, \in \vee q)$-Hesitant anti-intuitionistic fuzzy soft ideals of BCK-algebras

Aiyared Iampan *
Department of Mathematics
School of Science
University of Phayao
Mae Ka Mueang
Phayao 56000
Thailand

M. Balamurugan †
Department of Mathematics
Sri Vidya Mandir Arts and Science College (Autonomous)
Uthangarai 636902
Tamil Nadu
India

V. Govindan §
Department of Mathematics
Dmi St John The Baptist University
Mangochi
Central Africa 409
Malawi

Abstract

The class of algebras known as algebras of logic includes a variety of algebraic structures. The logical algebras BCK and BCI belong to two different classes. They were introduced by Imai and Iséki [13, 14] in 1966 and have been extensively investigated by many researchers. The concept of fuzzy soft sets (FSSs) is introduced in [21] to generalized standard soft sets [25]. The concept of intuitionistic fuzzy soft sets (IFSSs) is introduced by Maji et al. [22], which is based on a combination of the intuitionistic fuzzy set (IFS) [4] and soft set models. In this paper, we apply the idea of hesitant intuitionistic fuzzy soft sets (HIFSSs) for dealing with some types of theories in BCK-algebras. The views of hesitant anti-intuitionistic fuzzy soft commutative ideals (HAIFSCIs) and hesitant anti-intuitionistic fuzzy soft prime

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* *E-mail:* **aiyared.ia@up.ac.th (Corresponding Author)**
† *E-mail:* **balamurugansvm@gmail.com**
§ *E-mail:* **govindoviya@gmail.com**

ideals (HAIFSPIs) of BCK-algebras are presented and interrelated properties are examined. Relations between hesitant anti-intuitionistic fuzzy soft ideals (HAIFSIs) and HAIFSCIs are studied. Furthermore, the characterization of HAIFSPI is a HAIFSI. Finally, the concept of $(\in, \in \vee q)$ -intuitionistic fuzzy prime ideals (IFPIs) of BCK-algebras are introduced and several properties are investigated.

Subject Classification: *06F35, 08A72.*

Keywords: *BCK-algebra, Hesitant fuzzy soft set, Hesitant intuitionistic fuzzy soft set, Hesitant intuitionistic fuzzy soft ideal, Hesitant intuitionistic fuzzy soft commutative ideal, Hesitant intuitionistic fuzzy soft prime ideal and* $(\in, \in \vee q)$ *-hesitant intuitionistic fuzzy soft prime ideal.*

1. Introduction

Zadeh [31] developed the notion of a fuzzy set (FS) of a set as a technique for conveying uncertainty in the real world. Atanassov [4] developed the idea of IFS, which generalizes the notion of FSs. FSs provide a degree of membership for each element in the set, whereas IFSs provide both a degree of membership and a degree of non-membership. Imai et al. [13, 14] established a novel algebra induced by the BCI-system of propositional calculus.

The notion of $(\in,\in\vee q)$-intuitionistic fuzzy ideals of BG-algebras was put up by Barbhuiya in [8]. The notion of doubt fuzzy ideals of BG-algebras was introduced by Barbhuiya [9]. The idea of $(\in,\in\vee q)$-fuzzy ideals of d-algebras was put up by Barbhuiya et al. in [10]. Basnet et al. [11] developed the concept of $(\in,\in\vee q)$-fuzzy ideals of BG-algebras. Bhakat et al. [12] introduced the notion of $(\in,\in\vee q)$-fuzzy subgroups/subrings using the relationship of belongs to and quasi coincident with between fuzzy point and FS. The idea of generalising $(\in,\in\vee q)$-fuzzy subalgebras in BCK/BCI-algebras was developed by Jun [13].

Molodstov [25] developed the concept of the soft set as a new mathematical instrument for dealing with uncertainties that are free of the problems that have plagued the conventional theoretical approach. Maji et al. [20, 22, 23, 24] were necessary to finish a large number of works on IFSSs. Jun et al. [15, 16, 17, 18, 19] planned to finish a lot of work on FSS idea utilizing BCK/BCI-algebras. Balamurugan et al. [6] introduced the concept of intuitionistic fuzzy soft ideals in BCK/BCI-algebras. Balasubramanian et al. [7] investigated extensions of $(\in,\in\vee q)$-anti intuitionistic fuzzy soft subalgebras of BG-algebras. Torra [30] developed the notion of the hesitant fuzzy set (HFS) as one of the expansions of Zadeh's FS. According to Babitha et al. [5], another notable soft set is known as hesitant fuzzy soft sets (HFSSs). Abdullah [1] proposed the notion of IFPIs of BCK-algebras. Alshehri et al. [2, 3] presented and researched the hesitant fuzzy soft implicative/positive implicative/ commutative ideals in BCK-algebras. Alshehri et al. [28] proposed the concept of hesitant anti-fuzzy soft commutative ideals in BCK-algebras. Sahu and Gupta [28] developed the method with incomplete hesitant fuzzy preference relations. Pan and Zhan [27] defined $AND-(s,t)$-norm and $OR-(s,t)$-norm products of intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets. Seyedjoula et al. [29] studied BCK-sequences and finitely presented BCK-modules.

We present HAIFSCIs with examples and show that every HAIFSCI is a HAIFSI. In addition, we define HAIFSPIs and demonstrate that a HAIFSPI is a HAIFSI of a commutative BCK-algebra. In addition, the need and possibility operator of HAIFSPIs is a HAIFSPI. Finally, the notion of $(\in,\in\vee q)$-IFPIs of BCK-algebras is presented, and numerous features of these prime ideals are examined.

2. Preliminaries

If an algebra $(\aleph;*,0)$ meets the following axioms for every $\eta,\varkappa,\varsigma\in\aleph$, then it is said to be a *BCK-algebra* [13].

1. $((\eta*\varkappa)*(\eta*\varsigma))*(\varsigma*\varkappa)=0.$
2. $(\eta*(\eta*\varkappa))*\varkappa)=0.$
3. $\eta*\eta=0.$
4. $0*\eta=0.$
5. $\eta*\varkappa=0$ and $\varkappa*\eta=0$ imply $\eta=\varkappa.$

Let $(\aleph;*,0)$ or simply $\aleph$ mean a BCK/BCI-algebras unless specified.

Define a binary relation $\leq$ on $\aleph$ by letting $\eta\leq\varkappa \Leftrightarrow \eta*\varkappa=0$. Then $(\aleph;\leq)$ is a partially ordered set with the least element 0.

A BCK/BCI-algebra $\aleph$ satisfies the following conditions for each $\eta,\varkappa,\varsigma\in\aleph$,

1. $(\eta*\varkappa)*\varsigma=(\eta*\varsigma)*\varkappa$
2. $\eta*\varkappa\leq\eta.$
3. $\eta*0=\eta.$
4. $(\eta*\varsigma)*(\varkappa*\varsigma)\leq\eta*\varkappa.$
5. $\eta*(\eta*(\eta*\varkappa))=\eta*\varkappa.$
6. $\eta\leq\varkappa$ implies that $\eta*\varsigma\leq\varkappa*\varsigma$ and $\varsigma*\varkappa\leq\varsigma*\eta.$

A nonempty subset $\mathcal{I}$ of $\aleph$ is called an *ideal* of $\aleph$ if $(\mathcal{I}_1)\, 0 \in \mathcal{I}$ and $(\mathcal{I}_2)\, \eta * \varkappa \in \mathcal{I}$ and $\varkappa \in \mathcal{I}$ imply that $\eta \in \mathcal{I}$.

A nonempty subset $\mathcal{I}$ of $\aleph$ is called a *commutative ideal* of $\aleph$ if it satisfies $(\mathcal{I}_1)$ and $(\mathcal{I}_3)$ $(\eta * \varkappa) * \varsigma$ in $\mathcal{I}$ and ς in $\mathcal{I}$ imply $\eta * (\varkappa * (\varkappa * \eta))$ in $\mathcal{I}$.

An ideal $\mathcal{I}$ of commutative BCK-algebra $\aleph$ is said to be *prime ideal* of $\aleph$ if it satisfies $(\mathcal{I}_1)$ and $(\mathcal{I}_4)\, \eta \in \mathcal{I}$ and $\varkappa \in \mathcal{I}$ imply $\eta \wedge \varkappa \in \mathcal{I}$.

Definition 2.1: A hesitant fuzzy set (HFS) on $\aleph$ is a function from a fixed set $\aleph$ to a power set of the unit interval.

The HFS can be expressed by a mathematical symbol:

$$\mathfrak{H} := \{(\eta, \hbar_{\mathfrak{H}}(\eta)) \mid \eta \in \aleph\},$$

where h_H is the set of some values in $[0,1]$.

Definition 2.2: [26] Denote by $HF(\aleph)$ the set of all HFSs. A couple $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is called a *hesitant fuzzy soft set* (HFSS) over a reference set $\aleph$, where $\tilde{H} : \mathfrak{A} \to HF(\aleph)$.

Definition 2.3: [26] A HFS $\mathfrak{H} := \{(\eta, \hbar_{\mathfrak{H}}(\eta)) \mid \eta \in \aleph\}$ in $\aleph$ is called *hesitant anti-fuzzy prime ideal* (HAFPI) of $\aleph$ if for every for each $\eta, \varkappa, \varsigma \in \aleph$ satisfies the following condition:

$$\hbar_{\mathfrak{H}}(\eta \wedge \varkappa) \geq \hbar_{\mathfrak{H}}(\eta) \wedge \hbar_{\mathfrak{H}}(\varkappa).$$

Definition 2.4: Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a HFSS over $\aleph$ based on $\rho' \in \mathfrak{A} \subseteq \mathfrak{E}$. We say that $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is called a *hesitant anti-fuzzy soft ideal* (HAFSI) over $\aleph$ if the HFS $\tilde{\mathfrak{H}}[\rho'] := \{(\eta, \hbar_{\tilde{\mathfrak{H}}[\rho']}(\eta)) \mid \eta \in \aleph\}$ is a hesitant anti-fuzzy positive implicative ideal over $\aleph$.

Definition 2.4: Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a HFSS over $\aleph$ based on $\rho' \in \mathfrak{A} \subseteq \mathfrak{E}$. We say that $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is called a *hesitant anti-fuzzy soft prime ideal* (HAFSPI) over $\aleph$ if the HFS $\tilde{\mathfrak{H}}[\rho'] := \{(\eta, \hbar_{\tilde{\mathfrak{H}}[\rho']}(\eta)) \mid \eta \in \aleph\}$ is a HAFPI over $\aleph$.

3. Hesitant anti-intuitionistic fuzzy soft commutative/prime ideals

In this section, HAIFSCIs are defined with few results investigated.

Definition 3.1: Given a set $\aleph$. Suppose $\hbar_{1_{\mathfrak{H}}} : \aleph \to [0,1]$ and $\hbar_{2_{\mathfrak{H}}} : \aleph \to [0,1]$. Then the *hesitant intuitionistic fuzzy set* (HIFS) on $\aleph$ is a set $\mathfrak{H} := \{(\eta, \hbar_{1_{\mathfrak{H}}}(\eta), \hbar_{2_{\mathfrak{H}}}(\eta)) \mid \eta \in \aleph\}$.

The values $\hbar_{1_{\mathfrak{H}}}$ and $\hbar_{2_{\mathfrak{H}}}$ denote the possible existence and non-existence values of the element $\eta \in \aleph$ to the set $\mathfrak{H}$ respectively, which satisfy

$$\max \hbar_{1_{\mathfrak{H}}}(\eta) + \min \hbar_{2_{\mathfrak{H}}}(\eta) \leq 1$$

and

$$\min \hbar_{1_{\mathfrak{H}}}(\eta) + \max \hbar_{2_{\mathfrak{H}}}(\eta) \leq 1.$$

Definition 3.2: [26] Denote by $HIF(\aleph)$ the set of all HIFSs. A couple $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is called a *hesitant intuitionistic fuzzy soft set* (HIFSS) over a reference set $\aleph$, where $\tilde{H} : \mathfrak{A} \to HIF(\aleph)$.

Definition 3.3: [26] Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a HIFSS of $\aleph$ and $A \subset E$. Then $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a hesitant anti-intuitionistic fuzzy soft ideal (HAIFSI) of $\aleph$ if $\tilde{\mathfrak{H}}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph$ and $\rho' \in \mathfrak{A}\}$ is a hesitant intuitionistic fuzzy soft ideal of $\aleph$ satisfies for every $\eta, \varkappa \in \aleph$:

1. $\hbar_{1\tilde{\mathfrak{H}}[\rho']}(0) \geq \hbar_{1\tilde{\mathfrak{H}}[\rho']}(\eta)$ and $\hbar_{2\tilde{\mathfrak{H}}[\rho']}(0) \leq \hbar_{2\tilde{\mathfrak{H}}[\rho']}(\eta)$.
2. $\hbar_{1\tilde{\mathfrak{H}}[\rho']}(\eta \wedge \varkappa) \geq \hbar_{1\tilde{\mathfrak{H}}[\rho']}(\eta) \wedge \hbar_{1\tilde{\mathfrak{H}}[\rho']}(\varkappa)$.
3. $\hbar_{2\tilde{\mathfrak{H}}[\rho']}(\eta \vee \varkappa) \leq \hbar_{2\tilde{\mathfrak{H}}[\rho']}(\eta) \vee \hbar_{2\tilde{\mathfrak{H}}[\rho']}(\varkappa)$.

Now, HAIFSPIs are defined with few results discussed. Unless otherwise, $\aleph$ denotes a commutative BCK-algebra.

Definition 3.4: Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a hesitant anti-intuitionistic fuzzy soft set of $\aleph$ and $A \subset E$. Then $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a hesitant anti-intuitionistic fuzzy soft prime ideal (HAIFSPI) of $\aleph$ if $\tilde{\mathfrak{H}}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph$ and $\rho' \in \mathfrak{A}\}$ is a HAIFPI (HAIFPI) of $\aleph$ satisfies for each $\eta, \varkappa, \varsigma \in \aleph$,

1. $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)$.
2. $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)$.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

$\aleph$	0	1	2	3
$\tilde{H}[\rho_1']$	[0,1]	[0.1, 0.9]	[0.5, 0.5]	[1, 0]
$\tilde{H}[\rho_2']$	[0.1,0.9]	[0.2, 0.8]	[0.4, 0.4]	[0.9, 0.1]

Example 3.5: Consider a commutative BCK-algebra $\aleph = \{0,1,2,3\}$ with Cayley table: Let $(\tilde{H}, A)$ be a hesitant anti-intuitionistic fuzzy set in $\aleph$ defined as Then $\tilde{H}[\rho_1']$ and $\tilde{H}[\rho_2']$ are HAIFPIs of $\aleph$. Therefore, $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a HAIFSPI of $\aleph$.

Theorem 3.6: *Any HAIFSPI of $\aleph$ is a HAIFSI of $\aleph$.*

Proof: Trivial.

Theorem 3.7: *Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a hesitant anti-intuitionistic fuzzy soft set of. Then $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a HAIFSPI of $\aleph \Leftrightarrow \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$ and $h^C_{2_{\tilde{H}[\rho']}}(\eta)$ are HAFSPIs of $\aleph$, where $h^C_{2_{\tilde{H}[\rho']}}(\eta) = 1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$.*

Proof: Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a HAIFSPI of. Since by Theorem 7, $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a HAIFSI and so $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$ and $h^C_{2_{\tilde{H}[\rho']}}(\eta)$ are HAFSIs of $\aleph$. For every $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$,

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)$$

and

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa).$$

Now,

$$\begin{aligned} & 1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq 1 - \{\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\} \\ & \Rightarrow h^C_{2_{\tilde{H}[\rho']}}(\eta \wedge \varkappa) \geq \{1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\} \wedge \{1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\} \\ & \Rightarrow h^C_{2_{\tilde{H}[\rho']}}(\eta \wedge \varkappa) \geq h^C_{2_{\tilde{H}[\rho']}}(\eta) \wedge h^C_{2_{\tilde{H}[\rho']}}(\varkappa). \end{aligned}$$

Hence, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa)$ and $h^C_{2_{\tilde{H}[\rho']}}(\eta)$ are HAIFPIs of $\aleph$. The contrary is easy, we neglect the proof.

Theorem 3.8: *If $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a HAIFSI $\aleph$, then $\Box(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a HAIFSPI of $\aleph$, where $\Box\tilde{H}[\rho'] = \{(\eta,\ h^C_{1_{\tilde{H}[\rho']}}(\eta),\ \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph$ and $\rho' \in \mathfrak{A}\}$ and $h^C_{1_{\tilde{H}[\rho']}}(\eta) = 1 - \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$.*

Proof: Let $(\tilde{\mathfrak{H}},\mathfrak{A})$ be a HAIFSPI of $\aleph$. Since $(\tilde{\mathfrak{H}},\mathfrak{A})$ is a HAIFSI $\aleph$, we have $\square(\tilde{\mathfrak{H}},\mathfrak{A})$ is a HAIFSI of $\aleph$. For any $\eta,\varkappa\in\aleph$ and $\rho'\in\mathfrak{A}$,

$$\begin{aligned}
&\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\\
&\Rightarrow 1-\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\leq 1-\{\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\}\\
&\Rightarrow h^{C}_{1_{\tilde{H}[\rho']}}(\eta\wedge\varkappa)\leq\{1-\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\}\vee\{1-\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\}\\
&\Rightarrow h^{C}_{1_{\tilde{H}[\rho']}}(\eta\wedge\varkappa)\leq h^{C}_{1_{\tilde{H}[\rho']}}(\eta)\vee h^{C}_{1_{\tilde{H}[\rho']}}(\varkappa).
\end{aligned}$$

Therefore, $\square\tilde{H}[\rho']$ is a HAIFPI of $\aleph$. Hence, $\square(\tilde{\mathfrak{H}},\mathfrak{A})$ is a HAIFSPI of $\aleph$.

Theorem 3.9: *If* $(\tilde{\mathfrak{H}},\mathfrak{A})$ *is a HAIFSPI of* $\aleph$, *then* $\lozenge(\tilde{\mathfrak{H}},\mathfrak{A})$ *is a HAIFSPI of* $\aleph$, *where* $\lozenge\tilde{H}[\rho'] = \{(\eta,\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta),\ h^{C}_{2_{\tilde{H}[\rho']}}(\eta))\ |\ \eta\in\aleph$ *and* $\rho'\in\mathfrak{A}\}$ *and* $h^{C}_{2_{\tilde{H}[\rho']}}(\eta) = 1-\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$.

Proof: Let $(\tilde{\mathfrak{H}},\mathfrak{A})$ be a HAIFSPI of $\aleph$. Since $(\tilde{\mathfrak{H}},\mathfrak{A})$ is a HAIFSI of $\aleph$, we have $\lozenge(\tilde{\mathfrak{H}},\mathfrak{A})$ is a HAIFSI of $\aleph$. For any $\eta,\varkappa\in\aleph$ and $\rho'\in\mathfrak{A}$,

$$\begin{aligned}
&\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\leq\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\vee\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\\
&\Rightarrow 1-\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq 1-\{\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\vee\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\}\\
&\Rightarrow h^{C}_{2_{\tilde{H}[\rho']}}(\eta\wedge\varkappa)\leq\{1-\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\}\wedge\{1-\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\}\\
&\Rightarrow h^{C}_{2_{\tilde{H}[\rho']}}(\eta\wedge\varkappa)\geq h^{C}_{2_{\tilde{H}[\rho']}}(\eta)\vee h^{C}_{2_{\tilde{H}[\rho']}}(\varkappa).
\end{aligned}$$

Therefore, $\lozenge\tilde{H}[\rho']$ is a HAIFPI of $\aleph$. Hence, $\lozenge(\tilde{\mathfrak{H}},\mathfrak{A})$ is a HAIFSPI of $\aleph$.

4. $(\in,\in\vee q)$-Hesitant anti- intuitionistic fuzzy soft prime ideals

In this section, $(\in,\in\vee q)$-HAIFSPIs of BCK-algebras are defined with some results studied.

Definition 4.1: A HFSS $\hbar_{\tilde{\mathfrak{H}}[\rho']}$ of the form

$$\hbar_{\tilde{\mathfrak{H}}[\rho']}(\varkappa)=\begin{cases}t, & \text{if } \varkappa=\eta,\, t\in(0,1]\\ 0, & \text{if } \varkappa\neq\eta\end{cases}$$

is called a *hesitant fuzzy soft point* with support η and value t and is denoted by η_t.

Definition 4.2: A hesitant fuzzy soft point η_t is said to belong to a HFSS $\hbar_{\tilde{\mathfrak{H}}[\rho']}$ written as $\eta_t \in \hbar_{\tilde{\mathfrak{H}}[\rho']}$ or $\eta_t q \hbar_{\tilde{\mathfrak{H}}[\rho']}$ if $\hbar_{\tilde{\mathfrak{H}}[\rho']} \geq t$ or $\hbar_{\tilde{\mathfrak{H}}[\rho']}(\eta) + t > 1$. If $\eta_t \in \hbar_{\tilde{\mathfrak{H}}[\rho']}$ or $\eta_t q \hbar_{\tilde{\mathfrak{H}}[\rho']}$, then $\eta_t \in \vee q \hbar_{\tilde{\mathfrak{H}}[\rho']}$.

Definition 4.3: An $(\in, \in \vee q)$-HAFSI is said to be an $(\in, \in \vee q)$-HAFSPI of $\aleph$ if $(\eta \wedge \varkappa)_t \in \hbar_{\tilde{\mathfrak{H}}[\rho']} \Rightarrow \eta_t \in \vee q \hbar_{\tilde{\mathfrak{H}}[\rho']}$ and $\varkappa_t \in \vee q \hbar_{\tilde{\mathfrak{H}}[\rho']}$ for each $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$, where $t \in (0,1]$.

Definition 4.4: An $(\in, \in \vee q)$-HAIFSI is said to be an $(\in, \in \vee q)$-HAIFSPI of $\aleph$ if

$$\tilde{\mathfrak{H}}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph \text{ and } \rho' \in \mathfrak{A}\}$$

satisfies the following conditions:

1. $\eta_t, \varkappa_s \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \Rightarrow \eta_{t \vee s} \in \vee q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$, i.e., $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq s \Rightarrow \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq t \wedge s$ or $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + t \wedge s > 1$, for each $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$, where $t \wedge s = \min\{t, s\}$.
2. $\eta_t, \varkappa_s \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \Rightarrow \eta_{t \vee s} \in \vee q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$, i.e., $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \leq t$, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \leq s \Rightarrow \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq t \vee s$ or $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + t \vee s < 1$, for each $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$, where $t \vee s = \max\{t, s\}$ and $t, s \in (0,1]$.

Example 4.5: Consider a BCK-algebra $\aleph = \{0,1,2,3\}$ with the following Cayley tables

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	1	2	0
3	0	0	0	3

$\aleph$	0	1	2	3
$\tilde{H}[\rho']$	[0.1,0.5]	[0.2, 0.4]	[0, 0.6]	[0.14, 0.46]

Let $(\tilde{H}, A)$ be a HIFS in $\aleph$ defined as Then $\tilde{H}[\rho']$ is an $(\in, \in \vee q)$-HAIFPI of $\aleph$, but not a HAIFPI of $\aleph$ because

$$\begin{aligned} \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(1 \wedge 3) &\geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(1) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(3) \\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(0) &\geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(1) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(3) \\ 0.1 &\geq 0.2 \wedge 0.14 \\ 0.1 &\ngeq 0.14 \end{aligned}$$

and

$$\begin{aligned} \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(1 \wedge 3) &\leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(1) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(3) \\ \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(0) &\leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(1) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(3) \\ 0.5 &\leq 0.4 \vee 0.46 \\ 0.5 &\nleq 0.46. \end{aligned}$$

Theorem 4.6: *A HAIFSI is a HAIFSPI of $\aleph$ if and only if $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in)$-HAIFSI of $\aleph$.*

Proof: Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be an HAIFSPI of $\aleph$. Then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)$$

and

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa).$$

Let $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$ be such that $\eta_t,\ \varkappa_s \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$, where $t, s \in (0,1)$. Then $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t$, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq s$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \leq t$, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \leq s$.

Now, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq t \wedge s \Rightarrow \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \leq t \vee s \Rightarrow (\eta \wedge \varkappa)_{t \vee s} \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$. Therefore, $\tilde{H}[\rho']$ is an $(\in, \in)$-HAIFPI of $\aleph$. Hence, $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in)$-HAIFSPI of $\aleph$.

Conversely, let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be an $(\in, \in)$-HAIFSPI of $\aleph$. To prove that $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is a HAIFSPI of $\aleph$.

Let $\eta, \varkappa \in \aleph$ and $t = \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), s = \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)$. Then

$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq s$

$\Rightarrow \quad \eta_t \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}, \varkappa_s \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$

$\Rightarrow\ (\eta \wedge \varkappa)_{t\wedge s} \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ [since $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in)$-HAFSPI of $\aleph$]

$\Rightarrow\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq t \wedge s.$

Thus

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa). \tag{1}$$

Again, let $\eta, \varkappa \in \aleph$ and $t = \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)$, $s = \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)$. Then

$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \leq t$, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \leq s$

$\Rightarrow\ \eta_t \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}, \varkappa_s \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$

$\Rightarrow\ (\eta \wedge \varkappa)_{t\vee s} \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$ [since $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in)$-HAFSPI of $\aleph$]

$\Rightarrow\ \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq t \vee s.$

Thus

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa). \tag{2}$$

Hence, (1) and (2) $\Rightarrow \tilde{\mathfrak{H}}[\rho'] = \{(\eta,\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta),\ \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph$ and $\rho' \in \mathfrak{A}\}$ is a HAIFPI of $\aleph$. Hence, $(\tilde{\mathfrak{H}},\mathfrak{A})$ is a HAIFSPI of $\aleph$.

Theorem 4.7: *If $(\tilde{\mathfrak{H}},\mathfrak{A})$ is a (q,q)-HAIFSPI, then $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in)$-HAIFSPI.*

Proof: Let $(\tilde{\mathfrak{H}},\mathfrak{A})$ be a (q,q)-HAIFSPI of $\aleph$. Let $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$ be such that $\eta_t, \varkappa_s \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$. Then

$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq s$

$\Rightarrow\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + \varepsilon > t$ and $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) + \varepsilon > s$, where ε is any arbitrarily small positive number.

$\Rightarrow\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + \varepsilon - t + 1 > 1$ and $\sigma_{\tilde{H}[\rho']}(\varkappa) + \varepsilon - s + 1 > 1$

$\Rightarrow\ (\eta)_{\varepsilon-t+1} q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ and $(\varkappa)_{\varepsilon-s+1} q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$.

Since $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in)$-HAIFSPI of $\aleph$, we have

$(\eta \wedge \varkappa)_{(\varepsilon-t+1)\wedge(\varepsilon-s+1)} q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$

$\Rightarrow\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + (\varepsilon - t + 1) \wedge (\varepsilon - s + 1) > 1$

$\Rightarrow\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + \varepsilon - (t \vee s) + 1 > 1$

$\Rightarrow\ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) > (t \vee s) - \varepsilon$ [since ε is an arbitrary]

$\Rightarrow \;\; \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) > t \vee s$

$\Rightarrow \;\; \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) > t \vee s > t \wedge s$

$\Rightarrow \;\; (\eta \wedge \varkappa)_{t \wedge s} \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$

Therefore,

$$\eta_t, \varkappa_s \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}} \Rightarrow (\eta \wedge \varkappa)_{t \wedge s} \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}. \tag{3}$$

Again, let $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$ be such that $\eta_t, \varkappa_s \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$. Then

$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \le t, \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \le s$

$\Rightarrow \;\; \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) - \varepsilon < t$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) - \varepsilon < s$, where ε is any arbitrarily small positive number.

$\Rightarrow \;\; \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + 1 - \varepsilon - t < 1$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) + 1 - \varepsilon - s < 1$

$\Rightarrow \;\; (\eta)_{1-\varepsilon-t} q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$ and $(\varkappa)_{1-\varepsilon-s} q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$.

Since $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in)$-HAIFSPI of $\aleph$, we have

$(\eta \wedge \varkappa)_{(1-\varepsilon-t) \vee (1-\varepsilon-s)} q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$

$\Rightarrow \;\; \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + (1-\varepsilon-t) \vee (1-\varepsilon-s) > 1$

$\Rightarrow \;\; \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + 1 - \varepsilon - (t \wedge s) < 1$

$\Rightarrow \;\; \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) < (t \wedge s) + \varepsilon$ [since ε is an arbitrary]

$\Rightarrow \;\; \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) < t \wedge s$

$\Rightarrow \;\; \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) < t \wedge s < t \vee s$

$\Rightarrow \;\; (\eta \wedge \varkappa)_{t \vee s} \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}.$

Therefore,

$$\eta_t, \varkappa_s \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} \Rightarrow (\eta \wedge \varkappa)_{t \vee s} \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}. \tag{4}$$

Hence, (3) and (4) $\Rightarrow$ $\tilde{\mathfrak{H}}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph$ and $\rho' \in \mathfrak{A}\}$ is an $(\in, \in)$-HAIFPI of $\aleph$. Hence, $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in)$-HAIFPI of $\aleph$.

Theorem 4.8: *A HIFSS $(\tilde{\mathfrak{H}}, \mathfrak{A})$ of $\aleph$ is an $(\in, \in \vee q)$-HAIFSPI of $\aleph$ if and only if*

1. $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5$,
2. $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\leq\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\leq\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\vee\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\vee 0.5$

for each $\eta,\varkappa\in\aleph$ and $\rho'\in\mathfrak{A}$.

Proof: (1) First, let $(\tilde{\mathfrak{H}},\mathfrak{A})$ be an $(\in,\in\vee q)$-HAIFSPI of $\aleph$.

Case 1: Let $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)<0.5$ for each $\eta,\varkappa\in\aleph$ and $\rho'\in\mathfrak{A}$. Then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5=\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa).$$

If possible, let $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)<\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)$.

A real number t should be chosen such that

$$\begin{aligned}&\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)<t<\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\\ \Rightarrow\ &\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)>t,\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)>t\\ \Rightarrow\ &\eta_t\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}},\varkappa_t\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.\end{aligned}$$

But $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)<t$

$$\begin{aligned}\Rightarrow\ &(\eta\wedge\varkappa)_t\notin\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\text{ and }\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t<2t\\ \Rightarrow\ &\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t<2(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa))<2\times 0.5=1\\ \Rightarrow\ &\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t<1,\end{aligned}$$

which is a contradiction.

Therefore, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)=\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5$.

Case 2: Let $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5$ for each $\eta,\varkappa\in\aleph$ and $\rho'\in\mathfrak{A}$. Then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)=0.5.$$

If possible, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5=0.5$. Then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\geq 0.5\text{ and }\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\geq 0.5.$$

Therefore, $\eta_{0.5},\varkappa_{0.5}\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$.

But $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < 0.5,$ therefore, $\eta_{0.5} \notin \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ and $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + 0.5 < 0.5 + 0.5 = 1,$ which is a contradiction. Hence, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq 0.5 = \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5.$

Conversely, let

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5. \tag{5}$$

Let $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$ be such that $\eta_t, \varkappa_s \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$

Then $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t$ and $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq s.$ Therefore,

$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq t \wedge s.$ By (5), $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq t \wedge s \wedge 0.5.$

Now, if $t \wedge s \leq 0.5,$ then $t \wedge s \wedge 0.5 = t \wedge s.$

Therefore, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq t \wedge s.$ Thus

$$(\eta \wedge \varkappa)_{t \wedge s} \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}. \tag{6}$$

Again, if $t \wedge s > 0.5,$ then $t \wedge s \wedge 0.5 = 0.5.$

Therefore, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq t \wedge s \wedge 0.5 = 0.5,$ so $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + t \wedge s > 0.5 + 0.5 = 1.$ Thus

$$(\eta \wedge \varkappa)_{t \wedge s} q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}. \tag{7}$$

From (6) and (7), we have

$$\eta_t, \varkappa_s \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}} \Rightarrow (\eta \wedge \varkappa)_{t \wedge s} \in \vee q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}. \tag{8}$$

Therefore, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ is an $(\in, \in \vee q)$-$AFSU$ of $\aleph$.

(2) First, let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be an $(\in, \in \vee q)$-HAIFSPI of $\aleph.$

Case 1: Let $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) > 0.5$ for each $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}.$ Then

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5 = \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa).$$

If possible, let $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa).$

A real number t should be chosen such that

$$\begin{aligned} &\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) > t > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \\ \Rightarrow \quad &\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < t, \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) < t \\ \Rightarrow \quad &\eta_t \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}, \varkappa_t \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}. \end{aligned}$$

But, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) > t$

$$\Rightarrow \quad (\eta \wedge \varkappa)_t \notin \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} \text{ and } \hbar_{2_{\tilde{\mathfrak{H}}[`\rho]}}(\eta) + t > 2t$$
$$\Rightarrow \quad \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + t > 2(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)) > 2 \times 0.5 = 1$$
$$\Rightarrow \quad \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + t > 1,$$

which is a contradiction.

Therefore, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) = \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5.$

Case 2: Let $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5$ for each $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$. Then
$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) = 0.5.$$

If possible, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5 = 0.5.$ Then
$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \leq 0.5 \text{ and } \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \leq 0.5.$$

Therefore, $\eta_{0.5}, \varkappa_{0.5} \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$. But $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) > 0.5$, therefore, $(\eta)_{0.5} \notin \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + 0.5 > 0.5 + 0.5 = 1$, which is a contradiction.

Hence, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq 0.5 = \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5.$

Conversely, let
$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5. \tag{9}$$

Let $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$ be such that $\eta_t, \varkappa_s \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$.

Then $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \leq t$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \leq s$.

Therefore, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \leq t \vee s$.

By (9), we have $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq t \vee s \wedge 0.5$.

Now, if $t \vee s \geq 0.5$, then $t \vee s \vee 0.5 = t \vee s$. Therefore, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq t \vee s$. Thus
$$(\eta \wedge \varkappa)_{t \vee s} \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}. \tag{10}$$

Again, if $t \vee s < 0.5$, then $t \vee s \vee 0.5 = 0.5$. Therefore, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq t \vee s \vee 0.5 = 0$, i.e., $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + t \vee s < 0.5 + 0.5 = 1$. Thus
$$(\eta \wedge \varkappa)_{t \vee s} q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}. \tag{11}$$

From (10) and (11), we have
$$\eta_t, \varkappa_s \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} \Rightarrow (\eta \wedge \varkappa)_{t \vee s} \in \vee q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}. \tag{12}$$

Therefore, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$ is an $(\in,\in\vee q)$-HAFPI of $\aleph$.

Hence, (8) and (12) $\Rightarrow \tilde{H}[\rho']$ is an $(\in,\in\vee q)$-HAIFPI of $\aleph$. Thus, $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in\vee q)$-HAIFSPI of $\aleph$.

Theorem 4.9: *A HIFSS* $(\tilde{\mathfrak{H}},\mathfrak{A})$ *of* $\aleph$ *is an* $(\in,\in\vee q)$*-HAIFSPI of* $\aleph$ *and if* $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) > 0.5$, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < 0.5$ *for each* $\eta,\varkappa\in\aleph$ *and* $\rho'\in\mathfrak{A}$, *then* $(\tilde{\mathfrak{H}},\mathfrak{A})$ *is also an* $(\in,\in\vee q)$*-HAIFSPI of* $\aleph$.

Proof: Let $(\tilde{\mathfrak{H}},\mathfrak{A})$ be an $(\in,\in\vee q)$-HAIFSPI of $\aleph$. Then $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < 0.5$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) > 0.5$ for all $\eta,\varkappa\in\aleph$.

Let $\eta_t\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$, $\varkappa_s\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$. Then $t\le\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < 0.5$ and $s\le\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) <$ 0.5.

Therefore, $t\wedge s<0.5$. Also $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa) < 0.5$.

Thus, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t\wedge s<0.5+0.5=1$.

Since $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ is an $(\in,\in\wedge q)$-HAFPI of $\aleph$, we have $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\ge t\wedge s$ or $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa) + t\wedge s>1$. So, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\ge t\wedge s \Rightarrow (\eta\wedge\varkappa)_{t\wedge s}\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$. Therefore,

$$\eta_t\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}},\varkappa_s\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\Rightarrow(\eta\wedge\varkappa)_{t\wedge s}\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}. \tag{13}$$

Thus, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ is $(\in,\in)$-HAFPI of $\aleph$.

Again, let $\eta_t\in\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$, $\varkappa_s\in\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$. Then $0.5 < \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\le t$ and $0.5 < \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\le s$.

Therefore, $t\vee s>0.5$ and also $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)>0.5$.

Thus, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t\vee s>0.5+0.5=1$.

Since $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$ is an $(\in,\in\vee q)$-HAFPI of $\aleph$, we have $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\le t\vee s$ or $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa) + t\vee s < 1$. So, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\le t\vee s\Rightarrow(\eta\wedge\varkappa)_{t\vee s}\in\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$. Therefore,

$$\eta_t\in\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}},\varkappa_s\in\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}\Rightarrow(\eta\wedge\varkappa)_{t\vee s}\in\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}. \tag{14}$$

Thus, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$ is an $(\in,\in)$-HAFPI of $\aleph$. Therefore, (13) and (14) $\Rightarrow\tilde{H}[\rho']$ is an $(\in,\in)$-HAIFPI of $\aleph$. Hence, $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in)$-HAIFSPI of $\aleph$.

Theorem 4.10: *A HIFSS* $(\tilde{\mathfrak{H}},\mathfrak{A})$ *of* $\aleph$ *is an* $(\in,\in\vee q)$*-HAIFSPI of* $\aleph$ *if and only if the sets* $(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t = \{\eta\,|\,\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\ge t$, *where* $t\in(0,0.5)$, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(0)\ge t\}$ *and*

$(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s = \{\eta \mid \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < s,$ *where* $s \in (0.5,1],\ \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(0) < s\}$ *are prime ideal of* $\aleph$.

Proof: Assume $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in\vee q)$-HAIFSPI of $\aleph$. Clearly,

$$0 \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t, 0 \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s \text{ [Since } \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(0) \geq t, \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(0) \leq s].$$

Let $\eta,\varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$ be such that $\eta,\varkappa \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$ where $t \in (0,0.5)$. Therefore, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t$, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \geq t$.

Now by Theorem 4.8, we have

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5 > t \wedge t \wedge 0.5 = t$$

$$\Rightarrow \qquad \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq t \Rightarrow \eta \wedge \varkappa \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t.$$

Therefore, $\eta,\varkappa \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t \Rightarrow \eta \wedge \varkappa \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$.

Hence, $(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$ is a prime ideal of $\aleph$.

Again, let $\eta,\varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$ be such that $\eta,\varkappa \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s$, where $s \in (0.5,1]$. Therefore, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < s, \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) < s$.

Now by Theorem 4.8, we have

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5 < s \vee s \vee 0.5 = s$$

$$\Rightarrow \quad \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) < s$$

$$\Rightarrow \quad \eta \wedge \varkappa \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s.$$

Therefore, $\eta,\varkappa \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s \Rightarrow \eta \wedge \varkappa \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s$.

Hence, $(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s$ is a prime ideal of $\aleph$.

Conversely, $(\tilde{\mathfrak{H}},\mathfrak{A})$ be a HIFSS of $\aleph$ and the sets $(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t = \{\eta \mid \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t\}$, where $t \in (0,0.5)\}$ and $(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s = \{\eta \mid \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) < s\}$, where $s \in (0.5,1]\}$ are prime prime ideal of $\aleph$.

To prove $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in\wedge q)$-hesitant intuitionistic fuzzy soft of $\aleph$.

Suppose $\tilde{H}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph \text{ and } \rho' \in \mathfrak{A}\}$ is not an $(\in,\in\wedge q)$-hesitant intuitionistic fuzzy soft prime ideal of $\aleph$. Then there exist $a,b \in \aleph$ such that at least one of $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) < \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b) \wedge 0.5$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b) \vee 0.5$ hold.

Suppose $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) < \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b) \wedge 0.5$ holds.
Let $t = \{\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) + (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b) \wedge 0.5)\} / 2.$ Then $t \in (0.5, 1]$ and

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) < t < \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b) \wedge 0.5 \tag{15}$$

$\Rightarrow \ \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) > t, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b) > t$

$\Rightarrow \ a \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t, b \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$

$\Rightarrow \ a \wedge b \in (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$ [Since $(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$ is a prime ideal of $\aleph$].

Therefore, $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) > t,$ which is a contradiction. Hence,

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5. \tag{16}$$

Next let $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b) \vee 0.5$ holds.
Let $s = \{\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) + (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b) \vee 0.5)\} / 2.$ Then $s \in (0.5, 1]$ and

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) > s > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b) \vee 0.5 \tag{17}$$

$\Rightarrow \ \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a) < s, \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b) < s$

$\Rightarrow \ a \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s, b \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s$

$\Rightarrow \ a \wedge b \in (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s$ [Since $(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_s$ is prime ideal of $\aleph$].

Therefore, $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a \wedge b) < s,$ which is a contradiction. Hence,

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \leq \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5. \tag{18}$$

Therefore, (16) and (18) $\Rightarrow$ $\tilde{\mathfrak{H}}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph$ and $\rho' \in \mathfrak{A}\}$ is an $(\in, \in \vee q)$-HAIFPI of $\aleph$. Hence, $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in \vee q)$-HAIFSPI of $\aleph$.

Definition 4.11: Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a HIFSS of $\aleph$ and $t \in (0,1].$ Then let

1. $(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t = \{\eta \mid \eta_t \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\} = \{\eta \mid \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \geq t\},$
2. $\langle \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}} \rangle_t = \{\eta \mid \eta_t q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\} = \{\eta \mid \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + t > 1\},$
3. $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t = \{\eta \mid \eta_t \in \vee q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\} = \{\eta \mid \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}} \geq t$ or $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + t > 1\},$

where $(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$ is called t level set of $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$, $\langle \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}} \rangle_t$ is called q level set of $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ and $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ is called $\in \vee q$ level set of $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$.

Clearly,

1. $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t = \langle \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}} \rangle_t \cup (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t$ and
2. $(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_t = \{\eta \mid \eta_t \in \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}\} = \{\eta \mid \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \le t\}$,
3. $\langle \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} \rangle_t = \{\eta \mid \eta_t q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}\} = \{\eta \mid \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} + t < 1\}$,
4. $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t = \{\eta \mid \eta_t \in \vee q \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}\} = \{\eta \mid \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} \le t \text{ or } \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + t < 1\}$,

where $(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_t$ is called t level set of $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$, $\langle \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} \rangle_t$ is called q level set of $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$ and $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t$ is called $\in \vee q$ level set of $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}$. Clearly, $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t = \langle \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}} \rangle_t \cup (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_t$.

Theorem 4.12*: Let* $(\tilde{\mathfrak{H}}, \mathfrak{A})$ *be a HIFSS of* $\aleph$. *Then* $(\tilde{\mathfrak{H}}, \mathfrak{A})$ *is an* $(\in, \in \vee q)$*-HAIFSPI of* $\aleph$ *if and only if* $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ *and* $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t$ *is a prime ideal of* $\aleph$ *for each* $t \in (0,1]$. *We call* $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ *and* $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t$ *as* $\in \vee q$ *level prime ideal of* $\aleph$.

Proof: Assume that $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in \vee q)$-HAIFSPI of $\aleph$, to prove $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ and $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t$ is a prime ideal of $\aleph$.

Let $\eta, \varkappa \in [\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ for $t \in (0,1]$. Then $\eta_t, \varkappa_t \in \vee q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$,
i.e., $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \ge t$ or $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) + t > 1$ and $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \ge t$ or $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) + t > 1$.
Since $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}$ is an $(\in, \in \vee q)$-HAFPI of $\aleph$, we have
$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \ge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5$ for each $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$.
Now we have the following cases.

Case 1: $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \ge t, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \ge t$, let $t > 0.5$. Then

$$\begin{aligned} & \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \ge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5 = t \wedge t \wedge 0.5 = 0.5 \\ \Rightarrow\ & \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \ge 0.5 \Rightarrow \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) + t > 0.5 + 0.5 = 1 \\ \Rightarrow\ & (\eta \wedge \varkappa)_t q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}. \end{aligned}$$

Again if $t \le 0.5$, then

$$\begin{aligned} & \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \ge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5 \ge t \wedge t \wedge 0.5 = t, \\ & \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \ge t \Rightarrow (\eta \wedge \varkappa)_t \in \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}. \end{aligned}$$

Hence, $(\eta \wedge \varkappa)_t \in \wedge q \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}} \Rightarrow (\eta \wedge \varkappa)_t \in [\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$.

Case 2: $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \ge t, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) + t > 1$, let $t > 0.5$. Then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5>t\wedge(1-t)\wedge 0.5=1-t,$$

$$\Rightarrow\quad \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)>1-t\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t>1\Rightarrow(\eta\wedge\varkappa)_t q\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$$

Again if $t\leq 0.5$, then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5\geq t\wedge(1-t)\wedge 0.5=t,$$

$$\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq t\Rightarrow(\eta\wedge\varkappa)_t\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$$

Hence, $(\eta\wedge\varkappa)_t\in\wedge q\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\Rightarrow(\eta\wedge\varkappa)_t\in[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$.

Case 3: $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)+t>1, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\geq t$, let $t>0.5$. Then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5>(1-t)\wedge t\wedge 0.5=1-t,$$

$$\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)>1-t\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t>1\Rightarrow(\eta\wedge\varkappa)_t q\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$$

Again if $t\leq 0.5$, then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5=(1-t)\wedge t\wedge 0.5=t,$$

$$\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq t\Rightarrow(\eta\wedge\varkappa)_t\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$$

Hence, $(\eta\wedge\varkappa)_t\in\wedge q\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\Rightarrow(\eta\wedge\varkappa)_t\in[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$.

Case 4: $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)+t>1, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)+t>1$, let $t>0.5$. Then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5>(1-t)\wedge(1-t)\wedge 0.5=1-t,$$

$$\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)>1-t\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)+t>1\Rightarrow(\eta\wedge\varkappa)_t q\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$$

Again if $t\leq 0.5$, then

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5=(1-t)\wedge(1-t)\wedge 0.5=0.5\geq t,$$

$$\Rightarrow\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq t\Rightarrow(\eta\wedge\varkappa)_t\in\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}.$$

Hence, $(\eta\wedge\varkappa)_t\in\wedge q\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}\Rightarrow(\eta\wedge\varkappa)_t\in[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$.

Hence, from above four cases $\eta,\varkappa\in[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t\Rightarrow(\eta\wedge\varkappa)_t\in[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$.

Hence, $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ is a prime ideal of $\aleph$. Similarly, we can prove $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t$ is a prime ideal of $\aleph$.

Conversely, let $(\tilde{\mathfrak{H}},\mathfrak{A})$ be an HAIFSPI in $\aleph$ such that $[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ and $[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t$ is a prime ideal of $\aleph$ for each $t\in(0,1]$, to prove $(\tilde{\mathfrak{H}},\mathfrak{A})$ is an $(\in,\in\vee q)$-HAIFSPI of $\aleph$. Suppose $(\tilde{\mathfrak{H}},\mathfrak{A})$ is not an $(\in,\in\vee q)$-hesitant intuitionistic fuzzy soft prime ideal of $\aleph$, then there exist $a,b\in\aleph$ such that at least one of $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b) < \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5$ and $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b) > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b) \vee 0.5$ hold.

Suppose $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b) < \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b) \wedge 0.5$ is true, then decide on $t\in(0,1]$ such that

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b) < t < \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b)\wedge 0.5 \tag{19}$$

Then $\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a) > t, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(b) > t \Rightarrow \eta,\varkappa\in(\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}})_t\subset[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t$ which is a prime ideal.

Therefore, $(a\wedge b)\in[\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}]_t \Rightarrow \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b)\geq t \text{ or } \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b)+t>1$ which contradict (19).

Again, if $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b) > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a)\vee\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b)\vee 0.5$ is true, then decide on $t\in(0,1]$, such that

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b) > t > \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a)\vee\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b)\vee 0.5 \tag{20}$$

Then $\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a) < t, \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(b) < t \Rightarrow a,b\in(\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}})_t\subset[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t$ which is a prime ideal.

Therefore, $(a\wedge b)\in[\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}]_t \Rightarrow \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b)\leq t \text{ or } \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(a\wedge b)+t<1$ which contradict (20).

Hence,

$$\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\geq\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\wedge\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\wedge 0.5$$

and

$$\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta\wedge\varkappa)\leq\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)\vee\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)\vee 0.5$$

for each $\eta,\varkappa\in\aleph$ and $\rho'\in\mathfrak{A}$.

Therefore, $\tilde{\mathfrak{H}}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph \text{ and } \rho' \in \mathfrak{A}\}$ is an $(\in, \in \vee q)$-HAIFPI of $\aleph$. Hence, $(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in \vee q)$-HAIFSPI of $\aleph$.

On HIFSSs, we define two operators as follows: $\oplus(\tilde{\mathfrak{H}}, \mathfrak{A})$ and $\otimes(\tilde{\mathfrak{H}}, \mathfrak{A})$.

Definition 4.13: Let $(\tilde{\mathfrak{H}}, \mathfrak{A})$ be a HIFSS defined on $\aleph$. If $\tilde{\mathfrak{H}}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph \text{ and } \rho' \in \mathfrak{A}\}$ be a HIFS defined on $\aleph$, then operators $\oplus\tilde{H}[\rho']$ and $\otimes\tilde{H}[\rho']$ are defined as follows:

$$\oplus\tilde{H}[\rho'] = \{(\eta, \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta), \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \mid \eta \in \aleph \text{ and } \rho' \in \mathfrak{A}\}$$

and

$$\otimes\tilde{H}[\rho'] = \{(\eta, \overline{\tau}_{\tilde{H}[`\rho]}(\eta), \hbar_{2_{\tilde{\mathfrak{H}}[`\rho]}}(\eta)) \mid \eta \in \aleph \text{ and } \rho' \in \mathfrak{A}\},$$

respectively.

Theorem 4.14: *Let* $(\tilde{\mathfrak{H}}, \mathfrak{A})$ *is an* $(\in, \in \wedge q)$*-HAIFSPI of a* BG*-algebra* $\aleph$. *Then*

1. $\oplus(\tilde{\mathfrak{H}}, \mathfrak{A})$ *is an* $(\in, \in \wedge q)$*-HAIFSPI of* $\aleph$,
2. $\otimes(\tilde{\mathfrak{H}}, \mathfrak{A})$ *is an* $(\in, \in \wedge q)$*-HAIFSPI of* $\aleph$.

Proof: For (1), it is sufficient to demonstrate that $\overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}$ satisfies

$$\overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(0) \le \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee 0.5 \text{ and } \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \le \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5.$$

We have

$$\overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(0) = 1 - \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(0) \ge 1 - (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge 0.5) \le \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee 0.5.$$

Let $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$. Then

$$\begin{aligned}
\overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) &= 1 - \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \\
&\ge 1 - (\hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5) \\
&= (1 - \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \vee (1 - \hbar_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)) \vee (1 - 0.5) \\
\overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) &\le \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \overline{\hbar}_{1_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5.
\end{aligned}$$

Therefore, $\oplus\tilde{H}[\rho']$ is an $(\in, \in \wedge q)$-HAIFPI of $\aleph$.

Hence, $\oplus(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in \wedge q)$-HAIFSPI of $\aleph$.

For (2), it is sufficient to demonstrate that $\overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}$ satisfies $\overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(0) \geq \overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge 0.5$ and $\overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \geq \overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5.$

We have

$$\overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(0) = 1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(0) \leq 1 - (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee 0.5) \geq \overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge 0.5.$$

Let $\eta, \varkappa \in \aleph$ and $\rho' \in \mathfrak{A}$. Then

$$\begin{aligned}
\overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) &= 1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) \\
&\leq 1 - (\hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \vee \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \vee 0.5) \\
&= (1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta)) \wedge (1 - \hbar_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa)) \wedge (1 - 0.5) \\
\overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta \wedge \varkappa) &\geq \overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\eta) \wedge \overline{\hbar}_{2_{\tilde{\mathfrak{H}}[\rho']}}(\varkappa) \wedge 0.5.
\end{aligned}$$

Therefore, $\otimes\tilde{\mathfrak{H}}[\rho']$ is an $(\in, \in \wedge q)$-HAIFPI of $\aleph$. Hence, $\otimes(\tilde{\mathfrak{H}}, \mathfrak{A})$ is an $(\in, \in \wedge q)$-HAIFSPI of $\aleph$.

Conclusion

In this paper, we have introduced HAIFSCIs with examples and proved that every HAIFSCI is a HAIFSI. Also, we defined HAIFSPI and proved that HAIFSPI is a HAIFSI of commutative BCK-algebra. Furthermore, the necessity and possibility operator of HAIFSPIs is a HAIFSPI. Finally, the concept of $(\in, \in \vee q)$-IFPI of BCK-algebras are introduced and several properties are investigated.

Acknowledgements

The authors appreciate the anonymous reviewers for their informative remarks.

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

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Received May, 2022